ON COLORINGS OF GRAPHS WITHOUT SHORT CYCLES

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The following theorem is proved: Let \( n, k \) be natural numbers, \( n \geq 3 \), let \( A \) be a set and \( \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r \) different decompositions of \( A \) into at most \( n \) classes. Then there exists an \( n \)-chromatic graph \( G = (V(G), E(G)) \) such that \( A \subseteq V(G) \), \( G \) does not contain cycles of length \( \leq k \) and \( G \) has just \( r \) colorings \( \mathcal{B}_1, \ldots, \mathcal{B}_r \) by \( n \) colours such that \( \mathcal{A}_i = \mathcal{B}_i \mid A, i = 1, \ldots, r \).

From this theorem follows immediately existence of uniquely colorable graphs without short cycles. Further, characterizations of subgraphs of critical graphs may be given.

0. Introduction

In this paper we deal with finite undirected graphs without loops and multiple edges.

In 1968, Lovász [3] proved the following theorem:

**Theorem.** Let \( r, k \) be natural numbers. Then there exists a graph \( G \) such that its chromatic number \( \chi(G) \) is equal to \( r \) and \( G \) does not contain cycles of length \( \leq k \).

Extending earlier work of Harary et al. [1], Greenwell and Lovász [2] and Nešetřil [5] we prove that this graph can be chosen uniquely colorable. (Independently, the same result was recently obtained by P. Erdős by non-constructive means.) In fact a stronger statement can be proven:

**Theorem 1.** Let \( r, k \) be natural numbers, \( r \geq 3 \). Let \( A \) be a finite set, \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) different decompositions of \( A \) into at most \( r \) classes. Then there exists an \( r \)-chromatic graph \( G = (V(G), E(G)) \) such that \( A \subseteq V(G) \), \( G \) does not contain cycles of length \( \leq k \) and \( G \) has just \( n \) colorings \( \mathcal{B}_1, \ldots, \mathcal{B}_n \) by \( r \) colours such that \( \mathcal{A}_i = \mathcal{B}_i \mid A, i = 1, \ldots, n \).

The proof of Theorem 1 is the content of Section 2. In Section 3, we deal with critical graphs. As corollaries of Theorem 1 we can characterize all induced subgraphs of colorable critical graphs (Theorems 2, 3) (see also [2]) and all induced subgraphs of uniquely colorable critical graphs (Theorems 4, 5).

Results of this paper were communicated at the Second Prague Symposium on Graph Theory and its Applications, 1974 [4].
1. Preliminaries

A graph $G$ is a pair $(V(G), E(G))$, where $V(G)$ is a finite set and $E(G)$ is a set of two-element subsets of $V(G)$ (i.e. $E(G) \subseteq P_2(V(G))$).

We note that $G = \langle V(G), E(G) \rangle$ is called an induced subgraph of $H = \langle V(H), E(H) \rangle$ if $V(G) \subseteq V(H)$ and $E(G)$ consists of all edges of $E(H)$ having both endvertices in $V(G)$. We shall denote this relationship by $G = H \mid V(G)$.

For $r \in \mathbb{N}$ we set $R = \{1, 2, \ldots, r\}$ and $K_r = \langle R, P_2(R) \rangle$; $K_r$ is the complete graph with $r$ vertices.

For a graph $G = \langle V(G), E(G) \rangle$, $x, y \in V(G)$ we denote the distance between $x$ and $y$ in $G$ by $d_G(x, y)$ and the degree of $x$ in $G$ by $d_G(x)$. Let $H = \langle V(H), E(H) \rangle$ be a graph. Denote by $\text{Hom}(G, H)$ the set of all homomorphisms from $G$ into $H$ i.e. all those mappings $f : V(G) \to V(H)$ for which $\{f(x), f(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$. We denote the chromatic number of $G$ by $\chi(G)$. It is obvious that $\chi(G) = \min \{r \mid \text{Hom}(G, K_r) \neq \emptyset \}$.

Let $G = \langle V(G), E(G) \rangle$ be a graph. We will call the decomposition $\mathcal{A} = \{A_1, \ldots, A_r\}$ of the set $V(G)$ an $r$-coloring of the graph $G$ when the mapping $f : V(G) \to R$, $f(A_i) \in \{i\}$ for $i = 1, \ldots, r$ is a homomorphism from $G$ to $K_r$.

Let $G = \langle V(G), E(G) \rangle$ be a graph, $Y \subseteq V(G)$. The decomposition $\mathcal{B} = \{B_1, \ldots, B_r\}$ of $Y$ is called a relative $r$-coloring of $Y$, if there exists an $r$-coloring $\mathcal{A} = \{A_1, \ldots, A_r\}$ of $G$ such that $B_i = A_i \cap Y$ for $i = 1, \ldots, r$.

A graph is called uniquely $r$-colorable, if it has exactly one $r$-coloring.

We now give two simple constructions which shall be used later.

**Definition.** Let $G = \langle V(G), E(G) \rangle$ be a graph. A relation $\sigma \subseteq [V(G)]^2$ is called an orientation of $G$ if the following three conditions hold:

1. $(x, y) \in \sigma \Rightarrow (y, x) \notin \sigma$,
2. $(x, y) \in \sigma \Rightarrow \{x, y\} \in E(G)$,
3. $(x, y) \in \sigma \Rightarrow \text{either } (x, y) \in \sigma \text{ or } (y, x) \in \sigma$.

(exactly one arrow is associated to every edge).

**Construction 1.** Let $G_1 = \langle V(G_1), E(G_1) \rangle$, $G_2 = \langle V(G_2), E(G_2) \rangle$ be graphs, $\sigma$ an orientation of $G_1$, $a, b$ distinct elements of $V(G_2)$, $\{a, b\} \notin E(G_2)$. Then we denote by $G_1^r (G_2^b) = \langle V(H), E(H) \rangle$ the graph defined as follows:

$$V(H) = V(G_1) \cup \sigma \times V(G_2) - \{a, b\}$$

$\{x, y\} \in E(H) \Leftrightarrow \text{either } x = \langle s, x' \rangle, y = \langle s, y' \rangle, s \in \sigma,$

$x', y' \in V(G_2) - \{a, b\}, \{x', y'\} \in E(G_2)$

or $x \in V(G_1), y = \langle s, y' \rangle, s = \langle x, z \rangle, \{a, y'\} \in E(G_2)$

or $y \in V(G_1), x = \langle s, x' \rangle, s = \langle z, y \rangle, \{x', b\} \in E(G_2)$.

(This means that we replace every edge of $G_1$ with a copy of the graph $G_2$ oriented so that the endpoints of the edge are identified with $a, b$.)
Construction 2. Let $G_1 = (V(G_1), E(G_1)), G_2 = (V(G_2), E(G_2))$ be graphs and with $A \subset V(G_1), B \subset V(G_2)$, let $p : A \to B$ be a 1-1 mapping. Let $G_1 \mid A, G_2 \mid B$ be the empty graphs. We shall note $G_1 \sqcap_p G_2 = (V(H), E(H))$ the graph defined by:

$$\{x, y\} \in E(H) \iff \text{either } x \in V(G_1), y \in V(G_1), \{x, y\} \in E(G_1)$$

or

$$x \in V(G_2) - p(A), y \in V(G_2) - p(A), \{x, y\} \in E(G_2)$$

or

$$x \in A, y \in V(G_2) - p(A), \{p(x), y\} \in E(G_2)$$

(this is the well-known amalgam construction for graphs).

2. Main theorem

Lemma 1. Let $U$ be a finite set, $|U| = n$. Put $C = \{1, 2, 3\}^U = \{f : U \to \{1, 2, 3\}\}$. Let $\rho$ be an equivalence on $C$ with just three equivalence classes such that: $(f, g) \in \rho$ implies there exists a $u \in U$ with $f(u) = g(u)$. Then there exists an $a \in U$ such that $f(a) = g(a)$ for every $f, g, (f, g) \in \rho$.

Proof. Let $\rho$ be an equivalence with the property given in Lemma 1. We shall use the following notation: if the sets $U_1, U_2, U_3$ form the decomposition of $U$ we write $[U_1, U_2, U_3]$ for the mapping $[U_1, U_2, U_3] : U \to \{1, 2, 3\}$ with $[U_1, U_2, U_3] (U_i) \in \{i\}, i = 1, 2, 3$. Evidently the constant mappings $[U, \emptyset, \emptyset], [\emptyset, U, \emptyset], [\emptyset, \emptyset, U]$ lie in different equivalence classes of $\rho$. Let $C_1 = \rho[U, \emptyset, \emptyset], C_2 = \rho[\emptyset, U, \emptyset], C_3 = \rho[\emptyset, \emptyset, U]$ be the equivalence classes of $\rho$. Let us define the set $\mathcal{F} \subset \exp U$ by $V \in \mathcal{F} \iff [V, U - V, \emptyset] \in C_1$. First we prove some statements denoted by I–VI.

I. $V \in \mathcal{F} \iff U - V \notin \mathcal{F}$.

Proof. The mappings $f_1 = [V, U - V, \emptyset], f_2 = [U - V, V, \emptyset]$ lie in different equivalence classes because $f_1(u) \neq f_2(u)$ for every $u \in U$. Evidently $f_1 \notin C_3$ and $f_2 \notin C_3$. So there are only two possibilities:

(a) $f_1 \in C_1, f_2 \in C_2$ and $V \in \mathcal{F}, U - V \notin \mathcal{F}$.

(b) $f_1 \in C_2, f_2 \in C_1$ and $U - V \in \mathcal{F}, V \notin \mathcal{F}$.

II. Let $V \in \mathcal{F}$, then

1. $[U - V, V, \emptyset] \in C_2$;
2. $[U - V, \emptyset, V] \in C_3$;
3. $[V, \emptyset, U - V] \in C_1$;
4. $[\emptyset, V, U - V] \in C_2$;
5. $[\emptyset, U - V, V] \in C_3$. 

Proof. 

Proof. (1) follows from I.

(2) The mapping \([V, U-V, \emptyset]\) is an element of \(C_1, [V, U-V, \emptyset](u) \neq [U-V, \emptyset, V](u)\) for every \(u \in U\). Then \([U-V, \emptyset, V] \notin C_1\) and consequently \([U-V, \emptyset, V] \in C_3\).

(3) follows from (2) by I.

(4), (5) can be proven analogously.

III. Let \(V \in \mathcal{F}, A \subset U-V\). Then \([V, A, U-V-A] \in C_1\).

Proof. Let \(f=[V, A, U-V-A]\) and \(f \in C_2\). Put \(g=[U-V, V, \emptyset]\). Obviously \(g \in C_2, g(u) \neq f(u)\) for every \(u \in U\) and therefore \(f \notin C_2\). Analogously we can prove \(f \notin C_3\), hence \(f \in C_1\).

IV. Let \(V_1, V_2, V_3 \in \mathcal{F}, V_1 \cup V_2 \cup V_3 = U\). Then \(V_1 \cap V_2 \cap V_3 \neq \emptyset\).

Proof. To obtain a contradiction we assume that \(V_1, V_2, V_3 \subset \mathcal{F}, V_1 \cup V_2 \cup V_3 = U\) and \(V_1 \cap V_2 \cap V_3 = \emptyset\). We define the mapping \(f=[V_1-V_2, V_2-V_3, V_3-V_1]\) and the mappings \(g_1=[V_2, U-V_2, \emptyset], g_2=[\emptyset, V_3, U-V_3], g_3=[U-V_1 \emptyset, V_1]\). From II we have \(g_1 \in C_1, g_2 \in C_2, g_3 \in C_3\). Further \(f(u) \neq g_i(u), i = 1, 2, 3\) holds for every \(u \in U\). Hence \(f \notin C_i, i = 1, 2, 3\) which is a contradiction.

V. Let \(V_1, V_2 \in \mathcal{F}\). Then \(V_1 \cap V_2 \in \mathcal{F}\).

Proof. Let \(V_1, V_2 \in \mathcal{F}, V_1 \cap V_2 \notin \mathcal{F}\). Then if we put \(V_3 = U-(V_1 \cap V_2) \in \mathcal{F}\) we have \(V_1 \cup V_2 \cup V_3 = U\) and \(V_1 \cap V_2 \cap V_3 = \emptyset\), a contradiction.

VI. Let \(V_1 \in \mathcal{F}, V_2 \supset V_1\). Then \(V_2 \in \mathcal{F}\).

Proof. Let \(V_1 \in \mathcal{F}, V_2 \supset V_1, V_2 \notin \mathcal{F}\). Then \(U-V_2 \in \mathcal{F}, V_1 \cap (U-V_2) = \emptyset\), a contradiction.

Proof of Lemma 1. \(\mathcal{F}\) is a finite ultrafilter, hence it has one-point intersection. Let \(\bigcap_{V \in \mathcal{F}} V = \{a\}\). Let \(f \in C, f(a) = 1\). Then \(f^{-1}(1) \in \mathcal{F}\) (by VI) and from III \(f \in C_1\) follows.

Analogously we can prove \(f(a) = 2 \Rightarrow f \in C_2, f(a) = 3 \Rightarrow f \in C_3\). This proves Lemma 1.

Remark. For an infinite set \(U\) Lemma 1 is not true. In the same way we can prove that \(\mathcal{F}\) is an ultrafilter, but an infinite ultrafilter can have the empty intersection. Equivalences with properties as in Lemma 1 are in 1-1 correspondence with ultrafilters on the set \(U\) (see [2]).
Lemma 2. Let $U$ be a finite set, $|U| = n$, $r \in \mathbb{N}$, $r \geq 3$. Put $C_r = \{1, 2, \ldots, r\}^U = \{f \mid f : U \to \mathbb{R}\}$. Let $\rho$ be an equivalence on $C_r$ with just $r$ equivalence classes with the property $(f, g) \in \rho$ implies there exists $a \in U$, $f(u) = g(u)$. Then there exists an $a \in U$ such that $f(a) = g(a)$ holds whenever $(f, g) \in \rho$.

Proof. We prove Lemma 2 by induction.

For $r = 3$ Lemma 2 holds (by Lemma 1).

Let $r \geq 4$; given that the statement holds for $r - 1$ we prove it for $r$.

Evidently the constant mappings $f_i \in C_r$, $f_i(U) \subset \{i\}$, $i = 1, \ldots, r$ lie in different equivalence classes. Let $S_i = \rho[f_i]$. We put $C'_{r-1} = \{f \in C_r, f(U) \subset \{1, \ldots, r-1\}\}$. Evidently $S_i \cap S_r = \emptyset$, so that the equivalence $\rho' = \rho \mid C'_{r-1}$ on $C'_{r-1}$ satisfies the conditions given in Lemma 2. By the induction hypothesis there exists an $a \in U$ such that $f \in C_{r-1}$, $f(a) = i$ implies $f \in C'_{r-1} \cap S_i$. Let $f \in C_r$, $f(a) = r$. We prove that $f \in S_r$.

For a contradiction let $f \in S_r$, $i \neq r$. We can choose the mapping $g \in C'_{r-1} \cap S_r$ such that $g(u) \neq f(u)$ for every $u \in U$ (for example $g(u) = f(u) + 1 \pmod{r - 1}$ for $u \in U - \{a\}$, $g(a) = i$). This is a contradiction.

Analogously we can prove $f \in C_r$, $f(a) = i \in \{1, \ldots, r - 1\}$ implies $f \in S_i$. This proves Lemma 2.

Remark. For $r = 2$ the statement fails to hold. For example

\[ |U| = 3, \quad f \in S_1 \iff |f^{-1}(1)| \geq 2, \]
\[ f \in S_2 \iff |f^{-1}(2)| \geq 2. \]

The corresponding equivalence obeys the conditions of Lemma 2 but its statement is not true.

Lemma 3. Let $r \geq 3$, $n \in \mathbb{N}$. Let us define the graph $G_1 = \langle V(G_1), E(G_1) \rangle$:

\[ V(G_1) = \mathbb{R}^n, \quad x, y \in V(G_1), \quad x = \langle x_1, \ldots, x_n \rangle, \quad y = \langle y_1, \ldots, y_n \rangle \]

then

\[ \{x, y\} \in E(G_1) \iff x_i \neq y_i \text{ for every } i = 1, \ldots, n. \]

Then (I) The projection mappings $p_i : V(G_1) \to \mathbb{R}$, $p_i(x_1, \ldots, x_n) = x_i$ are homomorphisms $G_1 \to K_r$ for $i = 1, \ldots, n$.

(II) Let $f : G_1 \to K_r$ be a homomorphism. Then there exists \( i \in \{1, \ldots, n\} \) and a permutation $\pi : \mathbb{R} \to \mathbb{R}$ such that $f = \pi \circ p_i$ (so the projection mappings induce the only $r$-colorings of $G_1$).
Proof. (I) is evident.

(II) Given a homomorphism \( f : G_1 \to K_r \). Let us define the equivalence \( \rho \) on \( \mathbb{R}^n \) by \( (x, y) \in \rho \iff f(x) = f(y) \). Then the statement (II) is a translation of Lemma 2 into this context.

Remark. The statement of Lemma 3 is proved in [2] also in a different way.

Lemma 4. Let \( r \in \mathbb{N}, k \in \mathbb{N} \). Then there exists a graph \( G_2 = \langle V(G_2), E(G_2) \rangle \) and distinct vertices \( a, b \in V(G_2) \) that satisfy:

1. \( \chi(G_2) = r \);
2. \( G_2 \) does not contain cycles of length \( \leq k \);
3. \( f \in \text{Hom}(G_2, K_r) \Rightarrow f(a) = f(b) \).

Proof. There exists a graph \( G = \langle V(G), E(G) \rangle \) such that \( \chi(G) = r + 1 \) and \( G \) does not contain cycles of length \( \leq k \) (see [3]). Let \( G \) be a graph satisfying these two properties and having minimal number of edges. Let \( a, b \in V(G), \{a, b\} \in E(G) \) be an arbitrary edge of \( G \). Then for the graph \( G_2 = \langle V(G), E(G) - \{a, b\} \rangle \) \( \chi(G_2) = r \) and also \( f \in \text{Hom}(G_2, K_r) \Rightarrow f(a) = f(b) \) holds. This completes the proof.

Lemma 5. Let \( r \in \mathbb{N}, k \in \mathbb{N} \). Then there exists a graph \( G_3 = \langle V(G_3), E(G_3) \rangle \) and two distinct vertices \( a, b \in V(G_3) \) that satisfy the conditions (1)-(3) of Lemma 4 and further

4. \( v_{G_3}(a, b) \geq k \).

Proof. Let \( G_2 = \langle V(G_2), E(G_2) \rangle, G_2' = \langle V(G_2'), E(G_2') \rangle, a, b \in V(G_2), a', b' \in V(G_2') \) be two graphs with properties (1)-(3) of Lemma 4, and \( p \) the mapping \( p : \{b\} \to \{a'\} \). Denote \( G = G_2 \cup G_2' \). Then \( G \) still has properties (1)-(3) and \( v_{G_3}(a, b') = 2 \cdot v_{G_2}(a, b') \) holds. By repeating this construction \((\log_2 k)\)-times we can require condition (4) to hold.

Lemma 6. Let \( r \in \mathbb{N}, k \in \mathbb{N} \). Then there exists a graph \( G_4 = \langle V(G_4), E(G_4) \rangle \) and two distinct vertices \( a, b \in V(G_4) \) such that

1. \( \chi(G_4) = r \);
2. \( G_4 \) does not contain cycles of length \( \leq k \);
3. \( f \in \text{Hom}(G_4, K_r) \Rightarrow f(a) \neq f(b) \);
4. \( v_{G_4}(a, b) \geq k \).

Proof. Let \( G_3 = \langle V(G_3), E(G_3) \rangle, a, b \in V(G_3) \) satisfy the conditions of Lemma 5. Choose \( c \notin V(G_3) \). Then the graph \( G_4 = \langle V(G_4), E(G_4) \rangle \) defined by \( V(G_4) = V(G_3) \cup \{c\}, E(G_4) = E(G_3) \cup \{\{b, c\}\} \) realizes the existential statement of Lemma 6.
Lemma 7. Let \( r \in \mathbb{N}, r \geq 3, k \in \mathbb{N}, n \in \mathbb{N}, m \in \mathbb{N}, M = \{1, \ldots, m\} \). Then there exists a graph \( G_r = \langle V(G_r), E(G_r) \rangle \) and a set \( C = \mathbb{R}^n \times M, C \subset V(G_r) \) such that:

1. \( \chi(G_r) = r \)
2. \( G_r \) does not contain cycles of length \( \leq k \)
3. For every \( i \in \{1, \ldots, n\} \) there exists a homomorphism \( f: G_r \to K_r \) such that \( f((r_1, \ldots, r_n, m')) = r_i \) for every \( (r_1, \ldots, r_n, m') \in \mathbb{R}^n \times M \)
4. Let \( f \in \text{Hom}(G_r, K_r) \). Then there exists an \( i \in \{1, \ldots, n\} \) and a permutation \( \pi: \mathbb{R} \to \mathbb{R} \) such that \( \pi \circ f((r_1, \ldots, r_n, m')) = r_i \) for every \( (r_1, \ldots, r_n, m') \in \mathbb{R}^n \times M \) (i.e. the projection mappings induce the only relative \( r \)-colorings of \( C \) in \( G \))
5. \( x, y \in \mathbb{R}^n \times M, x \neq y \Rightarrow d_{G_r}(x, y) \geq k \)

Proof. Let \( G_1 = \langle V(G_1), E(G_1) \rangle, G_3 = \langle V(G_3), E(G_3) \rangle \), and \( G_4 = \langle V(G_4), E(G_4) \rangle \) be graphs with the properties of Lemma 3, Lemma 5 and Lemma 6, respectively. We shall preserve the notation used in the cited lemmas.

Choose \( \sigma \) an arbitrary orientation of the graph \( G_1 \). Denote \( \tilde{G}_1 = G_1^{\sigma}(G_4^{\sigma}) \).

Denote \( G = \langle V(G), E(G) \rangle \) the graph defined by \( V(G) = \mathbb{R}^n \times M = C \), if \( x, y \in V(G), x \neq y, x = (r_1, \ldots, r_n, m_1), y = (r_1, \ldots, r_n, m_2) \), then \( \{x, y\} \in E(G) \Leftrightarrow r_i = r_i' \) for \( i = 1, \ldots, n \).

For \( \sigma' \) arbitrary orientation of \( G \) denote further \( \tilde{G}_2 = G^{\sigma'}(G_3^{\sigma'}) \).

Let \( p \) be the mapping \( p: V(G_1) \to V(G) \) defined by \( p((r_1, \ldots, r_n)) = (r_1, \ldots, r_n, 1) \). Then the graph \( G_5 = \tilde{G}_1 \cup_{p} \tilde{G}_2 \) satisfies the properties of Lemma 7. This follows easily from the construction and from the properties of the graphs \( G_1, G_3, G_4 \).

Lemma 8. Let \( r \in \mathbb{N}, k \in \mathbb{N}, n \in \mathbb{N}, s \in \mathbb{N}, m \in \mathbb{N} \) with \( m > r \cdot (2r)^k \). Let \( M = \{1, \ldots, m\}, S = \{1, \ldots, s\}, R = \{1, \ldots, r\} \). Then there exists a graph \( G_6 = \langle V(G_6), E(G_6) \rangle \) where \( V(G_6) = S \times \mathbb{R}^n \times M \) such that:

1. \( d_{G_6}(x, y) = 1 \) and \( 0 \neq r_i - r_i' = r_i - r_i'(\text{mod } r) \) for every \( i, j \in \{1, \ldots, n\} \).
2. If \( x = (s_1, r_1, \ldots, r_n, m_1), s_1 > 2, 1 < r_i < r-1, \)\( \text{for exactly one element } y \text{ in } V(G_6) \) such that \( y = (s_{i-1}, r_i, \ldots, r_n, m(i)) \) and \( \{x, y\} \in E(G_6) \).
3. \( d_{G_6}(x) \leq 2r \) for every \( x \in V(G_6) \).
4. \( G_6 \) does not contain cycles of length \( \leq k \).

Proof. We order the set \( V(G_6) \) lexicographically: if \( x, y \in V(G_6), x = (s_1, r_1, \ldots, r_n, m_1), y = (s_2, r_1', \ldots, r_n', m_2) \), then \( x < y \) if either \( s_1 < s_2 \) or \( s_1 = s_2 \) and there exists \( i \in \{1, \ldots, n\} \) such that \( r_i = r_i' \) for every \( j < i \) and \( r_i < r_i' \) or \( s_1 = s_2 \) and \( r_i = r_i' \) for \( i = 1, \ldots, n \) and \( m_1 < m_2 \).

Denote \( U_x = \{y \in V(G_6), y < x\} \) for \( x \in V(G_6) \). We shall construct the graph \( G_6 \) inductively: For \( x = (2, 1, \ldots, 1, 1) \) let \( G_6 \arrow U_x \) be the empty graph. Let \( x \in V(G_6), x = (s_1, r_1, \ldots, r_n, m_1) \). Given a graph \( G_x = \langle U_x, E(G_x) \rangle \) satisfying conditions (1)-(4) (where we write \( G_x, U_x, E(G_x) \) instead of \( G_6, V(G_6), E(G_6) \)) we want
to construct a graph with properties (1)-(4) on the set \( U_x \cup \{ x \} \). For this it is enough to find a point \( y(i) \),

\[
y(i) = (s_1 - 1, r_1 + i, r_2 + i, \ldots, r_n + i, m(i)),
\]

(the addition is mod \( r \)) for each \( i = 1, \ldots, r-1 \), such that the properties (3), (4) would hold after addition of edges \( \{ x, y(i) \} \), \( i = 1, \ldots, r-1 \) to the set \( E(G_u) \). Let \( f \in \{ 1, \ldots, r-1 \} \). Suppose we have found points \( y(i), 1 \leq i < j \) such that the graph \( G_x = (U_x \cup \{ x \}, E(G_x')) \) where

\[
E(G_x') = E(G_x) \cup \{ \{ x, y(i) \}, 1 \leq i < j \}
\]
satisfies conditions (3), (4). We choose the point \( y(j) \) as follows: Condition (4) will hold after addition of the edge \( \{ x, y(j) \} \) iff \( d_{G_x}(x, y(j)) \geq k \). But

\[
| \{ y \in U_x, d_{G_x}(x, y) < k \} | \leq (2r)^k.
\]

Hence \( y(j) \) may be chosen from the remaining \( m - (2r)^k \) points. We prove that there exists at least one \( y(j) \) from these \( m - (2r)^k \) points with degree \( d_{G_x}(y(j)) < 2r \). Suppose the contrary. Then

\[
S = \sum_{m' \in M} d_{G_x}(r(s_1 - 1, r_1 + j, \ldots, r_n + j, m')) \geq 2r(m - (2r)^k).
\]

On the other hand it follows from conditions (1) and (2) that \( S \leq 2m(r - 1) \). As \( m > r(2r)^k \) we have

\[
2m(r - 1) = 2mr - 2m < 2mr - 2r(2r)^k = 2r(m - (2r)^k).
\]

This gives a contradiction. Hence we can find the point \( y(j) \in U_x \) such that condition (1)-(4) will hold after adding the edge \( \{ x, y(j) \} \). This proves Lemma 8.

Lemma 9. The graph \( G_6 = (V(G_6), E(G_6)) \) constructed in Lemma 8 has these properties:

(I) \( \chi(G_6) \leq r \).

(II) Let \( 1 \leq i \leq n \). Let \( f: G_6 \to K_r \) be a homomorphism satisfying \( f((1, r_1, \ldots, r_n, m')) = r_i \) for every \( r_1, \ldots, r_n, m' \in M \). Then \( f((s', r_1, \ldots, r_n, m')) = r_i \) for every \( s' \in S, r_1, \ldots, r_n \in R, m' \in M \).

Proof. (I) The projection mappings \( p_i: S \times R^n \times M \to R, p_i((s', r_1, \ldots, r_n, m')) = r_i \) are homomorphisms for every \( i = 1, \ldots, n \). (We even have \( \chi(G_6) = 2 \) because \( G_6 \) does not contain odd cycles—we shall not use this property.)

(II) Let \( i \in \{ 1, \ldots, n \}, f: G_6 \to K_r \) be a homomorphism satisfying
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\[ f((1, r_1, \ldots, r_n, m')) = r_i \text{ for every } r_1, r_2, \ldots, r_n \in R, m' \in M. \]

Let \( x = \langle s_1, r_1, \ldots, r_n, m_1 \rangle \) be the least element of \( V(G_0) \) (in the ordering defined in Lemma 8) such that \( f(x) \neq r_i \). Put \( a = f(x) - r_i (\text{mod } r) \). Obviously \( s_i \geq 2 \), hence there exists an element \( y = \langle s_1 - 1, r_1 + a, \ldots, r_n + a, m_2' \rangle \) such that \( \{x, y\} \in E(G_0) \). By the assumption \( f(y) = r_i + a = f(x) \), which is a contradiction.

**Theorem 1.** Let \( r \in \mathbb{N}, r \geq 3, k \in \mathbb{N}, n \in \mathbb{N} \). Let \( A \) be a finite set, \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) different decompositions of \( A \) each with \( r \) classes. Then there exists a graph \( G = \langle V(G), E(G) \rangle \), \( V(G) \ni A \) such that:

1. \( \chi(G) = r \)
2. \( G \) does not contain cycles of length \( \leq k \)
3. If \( x, y \in A, x \neq y \), then \( v_{\mathcal{A}_i}(x, y) \geq k \)
4. \( G \) has just \( n \) \( r \)-colorings \( \mathcal{B}_1, \ldots, \mathcal{B}_n \) such that appropriately numbered satisfy \( \mathcal{A}_i = \mathcal{B}_i \mid A, i = 1, \ldots, n \).

**Proof.** Let a set \( A \) and its decompositions \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) each with \( r \) classes be given. Let \( \mathcal{A}_i = \{\mathcal{A}_{i,1}, \ldots, \mathcal{A}_{i,n}\}, i = 1, \ldots, n \). For \( x \in A, i \in \{1, \ldots, n\} \) denote \( a_i(x) = j \) if \( x \in A_{i,j} \). Let \( G_5 = \langle V(G_5), E(G_5) \rangle \) and \( G_6 = \langle V(G_6), E(G_6) \rangle \) be graphs with the properties of Lemma 7 and Lemma 8, respectively. We shall use the notation of Lemmas 7,8 (in particular, \( r, k, n, m \) are the same numbers in both lemmas). Choose \( s \geq k \cdot r \cdot |V(G_5)|, m \geq |A| \cdot (2r)^k \). Let \( p \) be the mapping \( p : C \rightarrow V(G_6) \) defined by \( p((r_1, \ldots, r_n, m')) = \langle 1, r_1, \ldots, r_n, m' \rangle \). Set \( G' = G_5 \sqcup G_6 = \langle V(G'), E(G') \rangle \). The set \( V(G_6) \cup C \) has in the graph \( G' \) just \( n \) relative \( r \)-colorings \( \mathcal{B}_1, \ldots, \mathcal{B}_n \):

\[ \mathcal{B}_i = \{B_{i,1}', \ldots, B_{i,r}'\}, \quad B_{i,j}' = \langle s_1, r_1, \ldots, r_n, m_1 \rangle, \quad r_i = j \]

for \( i = 1, \ldots, n, j = 1, \ldots, r \).

(this follows from the properties of \( G_5, G_6 \)).

For every relative \( r \)-coloring \( \mathcal{B}_i', i = 1, \ldots, n \) let us fix an extension \( \mathcal{B}_i \) of \( \mathcal{B}_i' \) which is an \( r \)-coloring of \( G' \). Certainly there exists such a extension, but it need not be unique. Let
\[ \mathcal{B}_i = \{B_{i,1}, \ldots, B_{i,r}\}, \quad B_{i,j} \supset B_{i,j}' \text{ for } i = 1, \ldots, n, j = 1, \ldots, r. \]

Set \( D = [V(G_5) - C] \times \{1, \ldots, r - 1\} \). Let
\[ D = \{d_1, \ldots, d_q\}, \quad d_t = \langle x_t, i_t \rangle \text{ for } t = 1, \ldots, q. \]

Denote \( G = \langle V(G'), E(G) \rangle \) the graph containing just all edges of \( G' \) and all edges of the form
\[ \{x_t, \langle k \cdot t, b_1(x_t) + i_t, \ldots, b_n(x_t) + i_t, 1 \rangle\}, \quad t = 1, \ldots, q \]

(the addition is (mod \( r \))).
The graph $G$ has just $n$ $r$-colorings $\mathcal{B}_1, \ldots, \mathcal{B}_n$. Choose a 1-1 mapping $f: A \rightarrow C$ such that $x \in A \Rightarrow f(x) = \{a_1(x), \ldots, a_n(x), m(x)\}$ and $x, y \in A, x \neq y \Rightarrow v_{G_1}(f(x), f(y)) \geq k$. If we identify the points $x \in A$ with their images $f(x) \in C$, the graph $G$ satisfies all conditions of Theorem 1.

Theorem 1 yields two corollaries:

**Corollary 1.** Let $r \in \mathbb{N}, r \geq 3, k \in \mathbb{N}$. Let $H = \langle V(H), E(H) \rangle$ be a graph not containing cycles of length $\leq k$ with its chromatic number $\leq r$. Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be some $r$-colorings of $H$. Then there exists a graph $G = \langle V(G), E(G) \rangle$ containing $H$ as an induced subgraph such that:

1. $\chi(G) = r$,
2. $G$ does not contain cycles of length $\leq k$,
3. $G$ has just $n$ $r$-colorings $\mathcal{B}_1, \ldots, \mathcal{B}_n$ and $\mathcal{B}_i \mid V(H) = \mathcal{A}_i, i = 1, \ldots, n$ holds.

**Corollary 2.** Let $r \in \mathbb{N}, k \in \mathbb{N}$. Let $H = \langle V(H), E(H) \rangle$ be a graph with chromatic number $\chi(H) \leq r$ and without cycles of length $\leq k$. Then there exists a uniquely $r$-colorable graph $G = \langle V(G), E(G) \rangle$ without cycles of length $\leq k$ that contains $H$ as an induced subgraph.

**Proof.** For $r \geq 3$ this follows from Corollary 1. Let $r = 2$. A graph is 2-colorable iff it does not contain odd cycles. A graph is uniquely 2-colorable iff it is 2-colorable and connected, so that the proof is trivial.

**Remark.** Corollary 1 can also be written in a simpler form: Let $r \in \mathbb{N}, r \geq 3$. Let $H = \langle V(H), E(H) \rangle$ be an arbitrary graph with $\chi(H) \leq r$. Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be some $r$-colorings of $H$. Then there exists an $r$-colorable graph $G = \langle V(G), E(G) \rangle$ containing $H$ as an induced subgraph which has just $n$ $r$-colorings $\mathcal{B}_1, \ldots, \mathcal{B}_n$ such that $\mathcal{B}_i \mid V(H) = \mathcal{A}_i, i = 1, \ldots, n$.

The proof of this statement is much easier. Instead of the graph $G'$ in the proof of Theorem 1 we can take only the graph $G_1$ from Lemma 1 and proceed similarly.

3. **Applications to critical graphs**

Using Theorem 1 we can characterize the subgraphs of the critical graphs:

**Notation.** If $G = \langle V(G), E(G) \rangle$ is a graph, $\{x, y\} \in E(G)$, we write

$$G_{x,y} = \langle V(G), E(G) \rangle - \{\{x, y\}\}.$$

**Definition.** A graph $G = \langle V(G), E(G) \rangle$ is called $r$-critical, if $\chi(G) = r$ and $\chi(G_{x,y}) = r - 1$ for every edge $\{x, y\} \in E(G)$. 
Theorem 2. Let $r \in \mathbb{N}, r \geq 4, k \in \mathbb{N}$. A graph $G = \langle V(G), E(G) \rangle$ is an induced subgraph of an $r$-critical graph without cycles of length $\leq k$ iff $G$ does not contain cycles of length $\leq k$ and there exists a homomorphism $f_{x,y} : G_{x,y} \to K_{r-1}$ with $f(x) = f(y)$ for every edge $\{x, y\} \in E(G)$.

Proof. (I) Let $G = \langle V(G), E(G) \rangle$ be an induced subgraph of the $r$-critical graph $H = \langle V(H), E(H) \rangle$. Then necessarily there exists a homomorphism $\bar{f} : H_{x,y} \to K_{r-1}$ satisfying $f(x) = f(y)$ for every edge $\{x, y\} \in E(G)$. Hence $\bar{f} | V(G) : G_{x,y} \to K_{r-1}$ is the desirable homomorphism.

(II) Let $G = \langle V(G), E(G) \rangle$ have the properties from Theorem 2. For every edge $\{x, y\} \in E(G)$ fix an $(r - 1)$-coloring of $G_{x,y}$ such that $x, y$ belong to the same class of decomposition $\mathcal{A}_{x,y}$. According to Theorem 1 there exists a graph $H = \langle V(H), E(H) \rangle, V(H) \supseteq V(G)$ without short cycles having just $|E(G)|$ $(r-1)$-colorings $\mathcal{B}_{x,y}, \{x, y\} \in E(G)$ extending $\mathcal{A}_{x,y}$. Then the graph $H' = \langle V(H), E(H) \cup E(G) \rangle$ has obviously chromatic number $r$. $G$ is an induced subgraph of $H'$ and $H'$ does not contain cycles of length $\leq k$ (see Theorem 1). Let $\bar{H} = \langle V(H), E(\bar{H}) \rangle$ be some $r$-critical subgraph of $H'$. It is enough to prove $E(G) \subseteq E(\bar{H})$. Suppose instead that there exists $\{x, y\} \in E(G), \{x, y\} \notin E(\bar{H})$. Then obviously $\mathcal{B}_{x,y}$ is an $(r-1)$-coloring of $\bar{H}$, hence $\chi(\bar{H}) \leq r - 1$, a contradiction.

Remark. For $r = 3$ Theorem 3 does not hold. As an example consider any tree with a point of degree $\geq 3$.

Theorem 2 can be stated also in simpler form (proved in [2]):

Theorem 3. Let $r \in \mathbb{N}, r \geq 4$. A graph $G = \langle V(G), E(G) \rangle$ is an induced subgraph of some $r$-critical graph iff there exists a homomorphism $f_{x,y} : G_{x,y} \to K_{r-1}$ satisfying $f(x) = f(y)$ for every edge $\{x, y\} \in E(G)$.

Proof. See the remark following Theorem 1.

Analogous statements are true for uniquely colorable graphs:

Definition. A graph $G = \langle V(G), E(G) \rangle$ is said to be uniquely $r$-critical iff $G$ is uniquely $r$-colorable and $G_{x,y}$ fails to be uniquely $r$-colorable for every edge $\{x, y\} \in E(G)$.

Theorem 4. Let $r \in \mathbb{N}, k \in \mathbb{N}$. A graph $G = \langle V(G), E(G) \rangle$ is an induced subgraph of some uniquely $r$-critical graph without cycles of length $\leq k$ iff $\chi(G) \leq r, G$ does not contain cycles of length $\leq k$ and for every edge $\{x, y\} \in E(G)$ there exists a homomorphism $f_{x,y} : G_{x,y} \to K_r$ satisfying $f(x) = f(y)$.

Proof. Let $G = \langle V(G), E(G) \rangle$ be an induced subgraph of a uniquely $r$-critical graph $H = \langle V(H), E(H) \rangle, \{x, y\} \in E(G)$. Then there exists a homomorphism
\( \tilde{f}_{x,y} : H_{x,y} \rightarrow K_r \) satisfying \( f(x) = f(y) \). The restriction \( \tilde{f}_{x,y} \mid V(G) : G_{x,y} \rightarrow K_r \) is the desirable homomorphism.

(II (A)) Let \( r = 2 \). The graph is uniquely 2-colorable iff it is connected without odd cycles. A graph is uniquely 2-critical iff it is a tree. A graph is a subgraph of some uniquely 2-critical graph iff its components are trees. Hence Theorem 4 holds for \( r = 2 \).

(II (B)) Let \( r \geq 3 \), let \( G = (V(G), E(G)) \) have the properties of Theorem 4. Fix \( \mathcal{A} \) some \( r \)-coloring of \( G \) and \( \mathcal{A}_{x,y} \) an \( r \)-coloring of \( G \) for every edge \( \{x, y\} \in E(G) \) such that \( x, y \) belong to the same class of decomposition of \( \mathcal{A}_{x,y} \). By Theorem 1 there exists a graph \( H = (V(H), E(H)) \), \( V(H) = V(G) \) with just \( |E(G)| + 1 \) \( r \)-colorings \( \mathcal{B}, \mathcal{B}_{x,y}, \{x, y\} \in E(G) \) extending \( \mathcal{A}, \mathcal{A}_{x,y} \) respectively. Then the graph \( H' = (V(H), E(H) \cup E(G)) \) is uniquely \( r \)-colorable, because it has only one \( r \)-coloring \( \mathcal{B} \). Obviously \( G \) is an induced subgraph of \( H' \) and \( H' \) does not contain short cycles (see Theorem 1). Let \( \tilde{H} = (V(H), E(\tilde{H})) \) be an arbitrary uniquely \( r \)-critical subgraph of \( H' \). It is enough to prove \( E(G) \subseteq E(\tilde{H}) \). Let \( \{x, y\} \in E(G), \{x, y\} \notin E(\tilde{H}) \). Then \( \tilde{H} \) has at least two different \( r \)-colorings \( \mathcal{B}, \mathcal{B}_{x,y} \) and it is not uniquely \( r \)-colorable. Hence \( E(G) \subseteq E(\tilde{H}) \) and \( G \) is an induced subgraph of \( \tilde{H} \).

And analogously to Theorem 3:

**Theorem 5.** A graph \( G = (V(G), E(G)) \) is an induced subgraph of some uniquely \( r \)-critical graph iff \( \chi(G) \leq r \) and there exists a homomorphism \( f_{x,y} : G_{x,y} \rightarrow K_r \) satisfying \( f(x) = f(y) \) for every edge \( \{x, y\} \in E(G) \).

**References**


