Singular Poisson–Kähler geometry of Scorza varieties and their secant varieties

Johannes Huebschmann

Université des Sciences et Technologies de Lille, UFR de Mathématiques, CNRS-UMR 8524, F-59 655 Villeneuve d’Ascq, France

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Abstract

Each Scorza variety and its secant varieties in the ambient projective space are identified, in the realm of singular Poisson–Kähler geometry, in terms of projectivizations of holomorphic nilpotent orbits in suitable Lie algebras of hermitian type, the holomorphic nilpotent orbits, in turn, being affine varieties. The ambient projective space acquires an exotic Kähler structure, the closed stratum being the Scorza variety and the closures of the higher strata its secant varieties. In this fashion, the secant varieties become exotic projective varieties. In the rank 3 case, the four regular Scorza varieties coincide with the four critical Severi varieties. In the standard cases, the Scorza varieties and their secant varieties arise also via Kähler reduction. An interpretation in terms of constrained mechanical systems is included.

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1. Introduction

Let \( m \geq 2 \). A Severi variety is a non-singular variety \( X \) in complex projective \( m \)-space \( \mathbb{P}^m \mathbb{C} \) having the property that, for some point \( O \notin X \), the projection from \( X \) to \( \mathbb{P}^{m-1} \mathbb{C} \) is a closed immersion, cf. [6,
Ex. 3.11, p. 316, [15,23]. Let $X$ be a Severi variety and $n$ the dimension of $X$. The critical cases are when $m = \frac{1}{2}n + 2$. Zak [23] proved that only the following four critical cases occur:

(1.1) $X = \mathbb{P}^2 \mathbb{C} \subseteq \mathbb{P}^5 \mathbb{C}$ (Veronese embedding);
(1.2) $X = \mathbb{P}^2 \mathbb{C} \times \mathbb{P}^2 \mathbb{C} \subseteq \mathbb{P}^8 \mathbb{C}$ (Segre embedding);
(1.3) $X = G_2(\mathbb{C}^6) = U(6)/(U(2) \times U(4)) \subseteq \mathbb{P}^{14} \mathbb{C}$ (Plücker embedding);
(1.4) $X = \text{Ad}(\mathcal{E}(\mathbb{C}^{n-7}))/(\text{SO}(10, \mathbb{R}) \cdot \text{SO}(2, \mathbb{R})) \subseteq \mathbb{P}^{26} \mathbb{C}$.

These varieties arise from the projective planes over the four real normed division algebras (reals, complex numbers, quaternions, octonions) by complexification, cf. e.g. [2].

In the book [24], Zak introduced and classified Scorza varieties. These generalize the critical Severi varieties. Recall that, given a non-singular variety $X$ in $\mathbb{P}^m \mathbb{C}$, for $0 \leq k \leq m$, the $k$th secant variety $S^k(X)$ is the projective variety in $\mathbb{P}^m \mathbb{C}$ which arises as the closure of the union of all $k$-dimensional projective spaces in $\mathbb{P}^m \mathbb{C}$ that contain $k + 1$ independent points of $X$. Then $S^0(X) = X$ and $S^1(X)$ is the ordinary secant variety, referred to as well as chordal variety. For $k \geq 2$, a $k$-Scorza variety is defined to be a non-singular complex projective variety of maximal dimension among those varieties whose $(k - 1)$-secant variety is not the entire ambient projective space; a precise description of that notion of maximality will be recalled in Section 4 below.

The Severi variety (1.4) is a very special 2-Scorza variety. Let $k \geq 2$. We now list the other $k$-Scorza varieties, cf. Theorem 5.6 in Chapter 6 of [24]:

(1.1.k) $X = \mathbb{P}^k \mathbb{C} \subseteq \mathbb{P}^{\frac{4(k+1)}{3}} \mathbb{C}$ (Veronese embedding);
(1.2.k.r) $X = \mathbb{P}^k \mathbb{C} \times \mathbb{P}^k \mathbb{C} \subseteq \mathbb{P}^{4k+2} \mathbb{C}$ (Segre embedding);
(1.2.k.n) $X_2 = \mathbb{P}^k \mathbb{C} \times \mathbb{P}^{k+1} \mathbb{C} \subseteq \mathbb{P}^{4k^2+3k+1} \mathbb{C}$ (Segre embedding);
(1.3.k.r) $X = G_2(\mathbb{C}^{2k+1}) \subseteq \mathbb{P}^{4k^2+4k+3} \mathbb{C}$ (Plücker embedding);
(1.3.k.n) $X = G_2(\mathbb{C}^{2k+3}) \subseteq \mathbb{P}^{2k^2+5k+2} \mathbb{C}$ (Plücker embedding).

For reasons that will become clear below we will refer to the Scorza varieties (1.1.k), (1.2.k.r), (1.3.k.r), and (1.4) as regular and to the remaining ones as non-regular. The critical Severi varieties are exactly the regular 2-Scorza varieties.

When the chordal variety of a non-singular projective variety $Q$ in $\mathbb{P}^5 \mathbb{C}$ is a hypersurface (and not the entire ambient space) the projection from a generic point gives an embedding in $\mathbb{P}^5 \mathbb{C}$. A classical result of Severi [22] says that the Veronese surface is the only surface (not contained in a hyperplane) in $\mathbb{P}^5 \mathbb{C}$ with this property. This is the origin of the terminology “Severi variety”. In [2], the chordal varieties are written as $Z_n(C)$ ($n = 0, 1, 2, 3$). The terminology “Scorza variety” has been introduced in [24], to honor Scorza’s pioneering work on linear normalization of varieties of small codimension [19,20].

The purpose of the present paper is to exhibit an interesting geometric feature of Scorza varieties, in particular of the critical Severi varieties, in the world of singular Poisson–Kähler geometry, to be given as Theorem 1.5 below: There are exactly four simple regular rank 3 hermitian Lie algebras over the reals; these result from the euclidean Jordan algebras of hermitian $(3 \times 3)$-matrices over the four real normed division algebras by the superstructure construction; and each critical Severi variety arises from the minimal holomorphic nilpotent orbit in such a Lie algebra. The resulting geometric insight into the four critical Severi varieties will be made explicit in Addendum 1.7 below. For higher rank, that is,
when the rank $r$ (say) is at least equal to 4, there are exactly three simple regular rank $r$ hermitian Lie algebras over the reals; these result from the euclidean Jordan algebras of hermitian $(r \times r)$-matrices over the three associative real normed division algebras by the superstructure construction. Another result of Zak's [24], combined with Theorem 3.3.11 in [11], entails that each $k$-Scorza variety arises from the minimal holomorphic nilpotent orbit in a simple hermitian Lie algebra of rank $k + 1$ ($k \geq 2$); furthermore, the regular Scorza varieties arise in this fashion from regular simple hermitian Lie algebras and hence, cf. Remark 5.10 and Theorem 5.11 in Chapter 6 of [24], from simple complex Jordan algebras, and the non-regular Scorza varieties arise from non-regular simple hermitian Lie algebras and hence from simple positive definite hermitian Jordan triple systems. In particular, every simple complex Jordan algebra of rank at least 3 occurs here, indeed, the classification of regular Scorza varieties parallels that of simple complex Jordan algebras and hence that of tube domains. However not every positive definite hermitian Jordan triple system gives rise to a Scorza variety; see Remark 5.7 below for details.

We will spell out the resulting geometric insight into general Scorza varieties, in the realm of singular Poisson–Kähler geometry, in Theorem 1.5 below. To explain this geometric insight, we need some preparation. Let $Y$ be a stratified space. A complex analytic stratified Kähler structure on $Y$ in the sense of [11] is a stratified symplectic structure together with a compatible complex analytic structure which, on each stratum, combine to a Kähler structure; when at least two strata are involved, we refer to an exotic Kähler structure. A complete definition will be reproduced in Section 5 below. Suffice it to mention at this stage that the structure includes a real Poisson algebra of continuous functions on $Y$ which, on each stratum, restricts to an ordinary smooth symplectic Poisson algebra. This Poisson structure is independent of the complex analytic structure.

**Theorem 1.5.** Let $k \geq 2$, let $X$ be a $k$-Scorza variety, and let $\mathbb{P}^m \mathbb{C}$ be the ambient complex projective space. Then this projective space carries an exotic normal Kähler structure with the following properties:

1. The closures of the strata constitute an ascending sequence

\begin{equation}
Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_{k+1} = \mathbb{P}^m \mathbb{C}
\end{equation}

of normal Kähler spaces where the closed stratum $Q_1$ coincides with the given Scorza variety and where, complex algebraically, for $1 \leq \rho \leq k$, $Q_{\rho}$ is a projective determinantal variety in $Q_{k+1} = \mathbb{P}^m \mathbb{C}$; in particular, in the regular case, $Q_k$ is a projective degree $k + 1$ hypersurface.

2. For $1 \leq \rho \leq k$, $Q_{\rho+1}$ is the $\rho$th secant variety $S^{\rho}(Q_1)$ of $Q_1$ in $Q_{k+1} = \mathbb{P}^m \mathbb{C}$.

3. For $2 \leq \rho \leq k + 1$, $Q_{\rho-1}$ is the singular locus of $Q_{\rho}$, in the sense of stratified Kähler spaces.

4. The exotic Kähler structure on $\mathbb{P}^m \mathbb{C}$ restricts to an ordinary Kähler structure on $Q_1$ inducing, perhaps up to rescaling, the standard hermitian symmetric space structure.

We now spell out the result explicitly in the regular rank 3 case, which is somewhat special. This case corresponds to the critical Severi varieties and includes the exceptional case of the Severi variety (1.4) arising from the octonions (see below).

**Addendum 1.7.** For $m = 5, 8, 14, 26$, the complex projective space $\mathbb{P}^m \mathbb{C}$ carries an exotic normal Kähler structure with the following properties:

1. The closures of the strata constitute an ascending sequence

\begin{equation}
Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_{k+1} = \mathbb{P}^m \mathbb{C}
\end{equation}
of normal Kähler spaces where, complex algebraically, \( Q_1 \) is a (critical) Severi variety and \( Q_2 \) a projective cubic hypersurface, the chordal variety of \( Q_1 \).

(2) The singular locus of \( Q_3 \), in the sense of stratified Kähler spaces, is the hypersurface \( Q_2 \), and that of \( Q_2 \) (still in the sense of stratified Kähler spaces) is the non-singular variety \( Q_1 \); furthermore, \( Q_1 \) is as well the complex algebraic singular locus of \( Q_2 \).

(3) The exotic Kähler structure on \( \mathbb{P}^m \mathbb{C} \) restricts to an ordinary Kähler structure on \( Q_1 \) inducing, perhaps up to rescaling, the standard hermitian symmetric space structure.

In case (1.4), the term “determinantal variety” refers to Freudenthal’s determinant, see Sections 8 and 10 of [11].

For a compact Kähler manifold \( N \) which is complex analytically a projective variety, with reference to the Fubini–Study metric, the Kodaira embedding will not in general be symplectic; the above theorem shows that there are interesting situations where the ambient complex projective space carries an exotic Kähler structure which, via the Kodaira embedding, restricts to the Kähler structure on \( N \). More generally, for a projective variety \( N \) with singularities, the correct question to ask is whether, via the Kodaira embedding, \( N \) inherits, from a suitable exotic Kähler structure on the ambient complex projective space, a (stratified) Kähler structure. In [12] we have developed a Kähler quantization scheme for not necessarily smooth stratified Kähler spaces, including examples of Kähler quantization on projective varieties with stratified Kähler structure. This procedure applies to the circumstances of the above theorem. It is worthwhile noting that ignoring the lower strata means working on a non-compact space, and this then leads to inconsistencies in the sense that the principle that quantization commutes with reduction is violated. See [12, (4.12)] for details.

An interesting feature of the situations isolated in Theorem 1.5 is that the real Poisson algebra on \( \mathbb{P}^m \mathbb{C} \) (beware: it contains more functions than just the ordinary smooth ones) detects the lower strata \( Q_s \setminus Q_{s-1} \) (\( 1 \leq s \leq k+1 \)) where \( Q_0 \) is understood to be empty) by means of the rank of the Poisson structure, independently of the complex analytic structures. This notion of rank can be made precise by means of the Lie–Rinehart structure related with a general not necessarily smooth Poisson structure which we introduced in [8], see also [13].

More geometric consequences will be explained later. In particular, in Section 6 below we will show how the ascending sequence (1.6) (and hence the ascending sequence (1.6.SEVERI) for the classical cases (1.1), (1.2), (1.3)) results from Kähler reduction which, in particular, exhibits, for \( 1 \leq s \leq k+1 \), each constituent \( Q_s \) complex analytically as a G(ometric)I(nvariant)T(heory)quotient. For \( s = 1 \), this construction includes Zak’s Veronese mapping [24], see Remark 6.8 below. When the GIT-quotient is related with the corresponding symplectic quotient via the nowadays familiar Kempf–Ness observation, Theorem 1.5 provides geometric insight into the singular structure of the corresponding symplectic quotients. As for the exceptional case (1.4), we do not know whether the ascending sequence (1.6.SEVERI) may be obtained from a smooth Kähler manifold via Kähler reduction.

The underlying real spaces for the affine situation from which the sequence (1.6) in case (1.1.k) arises by projectivization result from symplectic reduction of the phase space of \( n = k+1 \) particles moving in \( \mathbb{R}^n \), with reference to total angular momentum zero. In [16] it has been shown that, in this case, the \( \text{Sp}(n, \mathbb{R}) \)-momentum mapping identifies the reduced space, as a stratified symplectic space, with the closure of a certain nilpotent orbit in \( \mathfrak{sp}(n, \mathbb{R}) \) and, likewise, the lower strata are similar nilpotent orbits
which correspond to the reduced systems of \( n \) particles moving in \( \mathbb{R}^k \) for \( k < n \). The requisite stratified symplectic Poisson algebra on the reduced spaces is that constructed in [1]. Comparing the result in [16] just quoted with the Kempf–Ness observation in geometric invariant theory I noticed that the nilpotent orbits isolated in [16] are precisely those, for the special case of \( \mathfrak{sp}(n, \mathbb{R}) \) \( (n \geq 1) \), which I identified as holomorphic nilpotent orbits in [11]. This was the starting point of the entire research program the present paper is part of.

For a fixed rank \( k + 1 \), in the two other (classical) cases, the constituents of the ascending sequences (1.6) admit presumably a similar interpretation in terms of certain constrained systems in mechanics; see Remark 6.7 below for details. Here the complex analytic structure does not seem to have a direct mechanical significance; it helps understanding the kinematical description. For issues related with quantization, the complex analytic structure has its intrinsic significance, though, cf. [9,10,12,13].

2. Lie algebras of hermitian type

Following [18, p. 54], we define a (semisimple) Lie algebra of hermitian type to be a pair \((g, z)\) which consists of a real semisimple Lie algebra \( g \) with a Cartan decomposition \( g = \mathfrak{k} \oplus \mathfrak{p} \) and a central element \( z \) of \( \mathfrak{k} \), referred to as an \( H \)-element, such that \( J_z = \text{ad}(z)|_\mathfrak{p} \) is a (necessarily \( K \)-invariant) complex structure on \( \mathfrak{p} \). Slightly more generally, a \emph{reductive} Lie algebra of hermitian type is a reductive Lie algebra \( g \) together with an element \( z \in \mathfrak{g} \) whose constituent \( z' \) (say) in the semisimple part \([g, g]\) of \( g \) is an \( H \)-element for \([g, g]\) [18, p. 92]. Below we will sometimes refer to \( g \) alone (without explicit choice of \( H \)-element \( z \)) as a \emph{hermitian} Lie algebra. For a real semisimple Lie algebra \( g \), with Cartan decomposition \( g = \mathfrak{k} \oplus \mathfrak{p} \), we write \( G \) for an appropriate Lie group having \( g = \text{Lie}(G) \) (matrix realization or adjoint realization; both will do) and \( K \) for the (compact) connected subgroup of \( G \) with \( \text{Lie}(K) = \mathfrak{k} \); the requirement that \((g, z)\) be of hermitian type is equivalent to \( G/K \) being a (non-compact) hermitian symmetric space with complex structure induced by \( z \).

A real semisimple hermitian Lie algebra \( g \) decomposes as \( g = g_0 \oplus g_1 \oplus \cdots \oplus g_k \) where \( g_0 \) is the maximal compact semisimple ideal and where \( g_1, \ldots, g_k \) are non-compact and simple. For a non-compact simple Lie algebra \( g \) with Cartan decomposition \( g = \mathfrak{k} \oplus \mathfrak{p} \), the \( \mathfrak{k} \)-action on \( \mathfrak{p} \) coming from the adjoint representation of \( g \) is faithful and irreducible whence the center of \( \mathfrak{k} \) then is at most one-dimensional; indeed \( g \) has an \( H \)-element turning it into a Lie algebra of hermitian type if and only if the center of \( \mathfrak{k} \) has dimension one. In view of E. Cartan’s infinitesimal classification of irreducible hermitian symmetric spaces, the Lie algebras \( \mathfrak{su}(p, q) \) \( (A_r, n \geq 1, \text{ where } n + 1 = p + q) \), \( \mathfrak{so}(2, 2n - 1) \) \( (B_r, n \geq 2) \), \( \mathfrak{sp}(n, \mathbb{R}) \) \( (C_n, n \geq 2) \), \( \mathfrak{so}(2, 2n - 2) \) \( (D_{n,1}, n > 2) \), \( \mathfrak{so}^*(2n) \) \( (D_{n,2}, n > 2) \) together with the real forms \( \mathfrak{e}_6(-14) \) and \( \mathfrak{e}_7(-25) \) of \( \mathfrak{e}_6 \) and \( \mathfrak{e}_7 \), respectively, constitute a complete list of simple hermitian Lie algebras.

We refer to \((g, z)\) as \emph{regular} when the relative root system is of type \( C_r \), \( r \geq 1 \); see [11, Proposition 3.3.2] for details. The natural number \( r \) is then the \emph{real} rank of \( g \). Thus \( \mathfrak{sp}(1, \mathbb{R}) \cong \mathfrak{su}(1, 1) \cong \mathfrak{so}(2, 1) \) and \( \mathfrak{so}^*(4) \) are the only regular rank 1 simple hermitian Lie algebras; the \( \mathfrak{sp}(p, 2) \)’s \( (p \geq 3) \) are the only regular rank 2 simple hermitian Lie algebras; in particular, \( \mathfrak{sp}(2, \mathbb{R}) \cong \mathfrak{so}(2, 3) \), \( \mathfrak{su}(2, 2) \cong \mathfrak{so}(2, 4) \), \( \mathfrak{so}^*(8) \cong \mathfrak{so}(2, 6) \), \( \mathfrak{sp}(3, \mathbb{R}) \), \( \mathfrak{su}(3, 3) \), \( \mathfrak{so}^*(12) \) and \( \mathfrak{e}_7(-25) \) are the only regular rank 3 simple hermitian Lie algebras; and, for \( n \geq 4 \), \( \mathfrak{sp}(n, \mathbb{R}) \), \( \mathfrak{su}(n, n) \), and \( \mathfrak{so}^*(4n) \) are the only regular rank \( n \) simple hermitian Lie algebras. Moreover, \( \mathfrak{so}^*(10) \), \( \mathfrak{e}_6(-14) \), and the \( \mathfrak{su}(p, 2) \)’s for \( p > 2 \) are the only non-regular rank 2 simple hermitian Lie algebras; and, for \( n \geq 3 \), the \( \mathfrak{su}(p, n) \)’s for \( p > n \) and \( \mathfrak{so}^*(4n + 2) \) constitute a complete list of the non-regular rank \( n \) simple hermitian Lie algebras.
3. Jordan algebras and hermitian Jordan triple systems

The symmetric constituent \( p \) of the Cartan decomposition \( \mathfrak{g} = \mathfrak{t} \oplus p \) of a Lie algebra \((\mathfrak{g}, z)\) of hermitian type inherits a hermitian Jordan triple system structure which, in case \( \mathfrak{g} \) is simple and regular, comes down to a simple complex Jordan algebra. We now explain briefly the relevant pieces of structure which we shall need; see Section 7 of [11] for more details and notation.

As usual, denote by \( \mathbb{O} \) the octonions or Cayley numbers. For \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \) and \( n \geq 1 \), let \( \mathcal{H}_n(\mathbb{K}) \) be the euclidean Jordan algebra of hermitian \((n \times n)\)-matrices over \( \mathbb{K} \) (\( \mathcal{H}_n(\mathbb{R}) = \mathbb{S}^2_\mathbb{R}(\mathbb{R}^n) \)), with Jordan product \( \circ \) given by \( x \circ y = \frac{1}{2}(xy + yx) \) (\( x, y \in \mathcal{H}_n(\mathbb{K}) \)) where \( n \leq 3 \) when \( \mathbb{K} = \mathbb{O} \). See [4,17,18] for notation and details. Here \( \mathcal{H}_1(\mathbb{R}) \cong \mathcal{H}_1(\mathbb{C}) \cong \mathcal{H}_1(\mathbb{H}) \cong \mathcal{H}_1(\mathbb{O}) \cong \mathbb{R}, \mathcal{H}_2(\mathbb{O}) \) is isomorphic to the euclidean Jordan algebra \( J(1,9) \) arising from the Lorentz form of type \((1,9)\) on \( \mathbb{R}^{10} \), and \( \mathcal{H}_3(\mathbb{O}) \) is the real exceptional rank 3 Jordan algebra of dimension 27. Complexification yields the complex simple Jordan algebras \( \mathbb{S}^2_\mathbb{C}(\mathbb{C}^n), \mathbb{M}_{n,p}(\mathbb{C}), \mathcal{A}^2(\mathbb{C}^{2n}), \mathcal{H}_2(\mathbb{O}_\mathbb{C}) = \mathcal{H}_3(\mathbb{O}) \oplus \mathbb{C} \), and it is well known that, when \( n \geq 3 \), these exhaust the simple complex Jordan algebras of rank \( \geq 3 \); see e.g. [4].

Let \((\mathfrak{g}, z)\) be a simple Lie algebra of hermitian type of rank \( r \geq 3 \). The Cartan decomposition \( \mathfrak{g} = \mathfrak{t} \oplus p \) has the following form where the decomposition \((C.2)\) is spelled out for the reductive hermitian Lie algebra \( \mathfrak{u}(q, p) \) instead of its simple brother \( \mathfrak{su}(q, p) \):

\[
\begin{align*}
(C.1) & \quad \mathfrak{sp}(n, \mathbb{R}) = \mathfrak{u}(n) \oplus \mathbb{S}^2_\mathbb{C}(\mathbb{C}^n) \ (r = n \geq 3); \\
(C.2) & \quad \mathfrak{u}(q, p) = (\mathfrak{u}(q) \oplus \mathfrak{u}(p)) \oplus \mathbb{M}_{q,p}(\mathbb{C}) \ (p \geq q = r \geq 3); \\
(C.3) & \quad \mathfrak{so}^\ast(2n) = \mathfrak{u}(n) \oplus \mathcal{A}^2(\mathbb{C}^n) \ (n \geq 6, r = [n/2]); \\
(C.4) & \quad \mathcal{E}_{7(-25)} = (\mathcal{E}_{6(-78)} \oplus \mathbb{R}) \oplus \mathcal{H}_3(\mathbb{O}_\mathbb{C}), \mathcal{E}_{6(-78)} being the compact form of \mathcal{E}_6 (only for r = 3).
\end{align*}
\]

Let \( K \) be the compact constituent in the Cartan decomposition of the adjoint group \( G \) of \( \mathfrak{g} \), and let \( K^\mathbb{C} \) be its complexification. The resulting (unitary) \( K \)-representation on the complex vector space \( p \) extends to a \( K^\mathbb{C} \)-representation on \( p \) which turns the latter into a pre-homogeneous space which we refer to as regular whenever the corresponding Lie algebra of hermitian type is regular. Explicitly, these representations have the following forms:

\[
\begin{align*}
(R.1) & \quad The symmetric square of the standard \( GL(n, \mathbb{C}) \)-representation on \( \mathbb{C}^n \), given by \( x \cdot S = x S x^t \), for \( x \in GL(n, \mathbb{C}) \) and \( S \in \mathbb{S}^2_\mathbb{C}(\mathbb{C}^n) \); it is always regular. \\
(R.2) & \quad The standard representation of \( GL(q, \mathbb{C}) \times GL(p, \mathbb{C}) \) on \( \mathbb{M}_{q,p}(\mathbb{C}) \) given by \( (x, y) \cdot M = x M y^t \), \( x \in GL(p, \mathbb{C}) \), \( y \in GL(q, \mathbb{C}) \), \( M \in \mathbb{M}_{q,p}(\mathbb{C}) \); it is regular if and only if \( p = q \). \\
(R.3) & \quad The exterior square of the standard \( GL(n, \mathbb{C}) \)-representation on \( \mathbb{C}^n \); it is regular if and only if \( n \) is even. \\
(R.4) & \quad The classical representation of \( E_6(\mathbb{C}) \), extended by a central copy of \( \mathbb{C}^* \); it is regular and has complex dimension 27. This representation was studied already by E. Cartan.
\end{align*}
\]

The complexification \( \mathfrak{g}^\mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{t}^\mathbb{C} \oplus \mathfrak{p}^− \), where \( \mathfrak{p}^+ \) and \( \mathfrak{p}^− \) refer to the holomorphic and antiholomorphic constituents of \( \mathfrak{p}^\mathbb{C} \), respectively, inherits a real symmetric Lie algebra structure, the requisite involution being complex conjugation, and this structure, in turn, determines the structure of a simple positive definite hermitian Jordan triple system (JTS) on \( \mathfrak{p}^+ \). Indeed, every positive definite hermitian
JTS arises in this fashion, cf. [18, Proposition II.3.3 on p. 56]. Under this correspondence, the regular simple Lie algebras of hermitian type correspond to positive definite hermitian JTS’s which arise from a Jordan algebra (with unit element). Furthermore, the action on \( p^+ \), of the compact constituent \( K \) in the Cartan decomposition of the adjoint group \( G \) of \( g \), extends to an action of the complexification \( K^C \) on \( p^+ \) and turns the latter into a pre-homogeneous space for \( K^C \). The obvious isomorphism of complex vector spaces between \( p \) and \( p^+ \) identifies the pre-homogeneous space structures. In this fashion, \( p \) acquires a JTS structure. More details may be found in Sections 7 and 8 of [11].

4. Jordan rank and Severi, Scorza, and secant varieties

Let \( g \) still be a simple hermitian Lie algebra, with Cartan decomposition \( g = k \oplus p \), and let \( r \) be the real rank of \( g \). For \( 1 \leq s \leq r \), let \( O_s \subseteq p \) be the subspace of Jordan rank \( s \). This notion of Jordan rank is discussed in Section 7 of [11]. In the regular (i.e., Jordan algebra) case, the Jordan rank of a non-zero element is the number of non-zero eigenvalues in its spectral decomposition, with multiplicities counted [4, p. 77]. For \( g = sp(r, \mathbb{R}) \) and \( g = su(p, r) \) \(( p \geq r \geq 1)\), the Jordan rank amounts to the ordinary rank of a matrix, where \( S^2_{\mathbb{C}}[\mathbb{C}^r] \) is identified with the symmetric complex \((r \times r)\)-matrices. For \( g = so^*(2n) \) \(( r = [n/2] \geq 2)\), when we identify \( A^2[\mathbb{C}^n] \) with the skew-symmetric complex \((n \times n)\)-matrices, the Jordan rank amounts to one half the ordinary rank of a matrix. The resulting decomposition of \( p \) is a stratification whose strata coincide with the \( K^C \)-orbits, and the closures constitute an ascending sequence

\[
\{0\} \subseteq O_1 \subseteq O_2 \subseteq \cdots \subseteq O_r = p
\]

of complex affine varieties. Projectivization yields the ascending sequence

\[
O_1 \subseteq O_2 \subseteq \cdots \subseteq O_r = \mathbb{P}(p).
\]

As far as the complex analytic structures are concerned, this is the sequence (1.6) when \( r \geq 3 \), in particular the sequence (1.6.SEVERI) for \( r = 3 \) in the regular case.

In the Severi case, that is, under the circumstances of Theorem 1.5 for \( k = 2 \) in the regular case or, equivalently, under the circumstances of Addendum 1.7, by construction, \( Q_2 \) is a determinantal cubic, the requisite determinant over the octonions being that introduced by Freudenthal [5], and \( Q_1 \) is the corresponding Severi variety. In fact, \( Q_1 \) is the closed \( K^C \)-orbit in \( \mathbb{P}(p) \), and the homogeneous space descriptions (1.1)–(1.4) are immediate. For \( g = e_7(-25) \), the cubic \( Q_2 \) is the (projective) generic norm hypersurface, sometimes referred to in the literature as Freudenthal cubic; it has been studied already by E. Cartan, though. Jacobson has shown that this cubic is rational [14].

We now recall a description of Scorza varieties. Let \( X \subseteq \mathbb{P}^m \mathbb{C} \) be a non-singular complex projective variety which does not lie in a hyperplane. Let \( k_0(X) \) be the smallest natural number \( k \) such that \( S^k(X) = \mathbb{P}^m \mathbb{C} \). Given another projective variety \( Y \), let \( \delta(X, Y) = \dim X + \dim(Y) + 1 - \dim S(X, Y) \) be the defect of \( X \) and \( Y \) where \( S(X, Y) \) refers to the join of \( X \) and \( Y \), that is, to the closure of the union of the lines through a point of \( X \) and a point of \( Y \); following Zak [24], for \( 1 \leq i \leq k_0(X) \), let

\[
\delta_i = \delta_i(X) = \delta(X, S^{i-1}(X)) = \dim X + \dim S^{i-1}(X) + 1 - \dim S^i(X),
\]
and write $\delta = \delta_1$. Zak proved that $\delta_i \geq \delta_{i-1} + \delta$ whenever $2 \leq i \leq k_0(X)$ \cite[Chapter 5, Theorem 1.8]{24} and deduced that $k_0(X) \leq \frac{\dim X}{\delta}$. Rather than reproducing Zak’s definition of a Scorza variety, we recall that, by virtue of Proposition 1.2 in Chapter 6 of \cite{24}, a non-singular projective variety $X$ in $\mathbb{P}^m \mathbb{C}$ which does not lie in a hyperplane is a Scorza variety if and only if it satisfies (4.3) and (4.4) below:

(4.3) $k_0(X) = \lceil \frac{\dim X}{\delta} \rceil$;
(4.4) $\delta_i = i\delta$ for $1 \leq i \leq k_0(X)$.

With these preparations out of the way, we recall Zak’s classification, cf. Theorem 5.6 in Chapter 6 of \cite{24}: For the case where $\delta$ divides $\dim X$, given $k \geq 2$, the $k$-Scorza varieties $X$ are exactly the projectivizations of rank 1 strata in the simple complex Jordan algebras of rank $k + 1$, that is, the regular ones listed above as (1.1.k), (1.2.k.r), (1.3.k.r), together with the Severi variety (1.4) in case $k = 2$. The classification of regular Scorza varieties has been reworked in \cite{3}. When $\delta$ does not divide $\dim X$, for $k \geq 2$, there remain two families of $k$-Scorza varieties $X$ which are exactly those listed above as (1.2.k.n) and (1.3.k.n); these are the projectivizations of rank 1 strata in the corresponding rank $k + 1$ hermitian Jordan triple systems $M_{k+1,k+2}(\mathbb{C})$ and $\Lambda^2(\mathbb{C}^{2k+1})$, respectively. In particular, given an $n$-dimensional non-singular projective variety $X$, since $k_0(X) \leq n/\delta$, the requirement $k_0(X) = [n/\delta]$ is equivalent to $S^{[n/\delta]-1}X \neq \mathbb{P}^m \mathbb{C}$ and, when $X$ is a regular $k$-Scorza variety ($k \geq 2$), $S^{k-1}X$ is a hypersurface of degree $k + 1$. More details about this hypersurface will be given in Remark 5.6 below.

5. Holomorphic nilpotent orbits and the proof of Theorem 1.5

A detailed account of holomorphic nilpotent orbits is given in our paper \cite{11}. Here we recall only what we need for ease of exposition.

Let $(\mathfrak{g}, \mathfrak{z})$ be a semisimple Lie algebra of hermitian type, with Cartan decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$. We refer to an adjoint orbit $\mathcal{O} \subseteq \mathfrak{g}$ having the property that the projection map from $\mathfrak{g}$ to $\mathfrak{p}$, restricted to $\mathcal{O}$, is a diffeomorphism onto its image, as a pseudoholomorphic orbit. A pseudoholomorphic orbit $\mathcal{O}$ inherits a complex structure from the complex structure $J_\mathcal{O}$ on $\mathfrak{p}$, and this complex structure, combined with the Kostant–Kirillov–Souriau form on $\mathcal{O}$, viewed as a coadjoint orbit by means of (a positive multiple of) the Killing form, turns $\mathcal{O}$ into a (not necessarily positive) Kähler manifold.

We now choose a positive multiple of the Killing form. We say that a pseudoholomorphic orbit $\mathcal{O}$ is holomorphic provided the resulting Kähler structure on $\mathcal{O}$ is positive. The name “holomorphic” is intended to hint at the fact that the holomorphic discrete series representations of $G$ arise from holomorphic quantization on integral semisimple holomorphic orbits but, beware, the requisite complex structure (needed for the construction of the holomorphic discrete series representation) is not the one arising from projection to $\mathfrak{p}$.

Let $\mathcal{O}$ be a holomorphic nilpotent orbit, and let $C^\infty(\overline{\mathcal{O}})$ be the algebra of Whitney smooth functions on the ordinary topological closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ (not the Zariski closure) resulting from the embedding of $\overline{\mathcal{O}}$ into $\mathfrak{g}^*$. The Lie bracket on $\mathfrak{g}$ passes to a Poisson bracket $\{\cdot, \cdot\}$ on $C^\infty(\overline{\mathcal{O}})$, even though $\overline{\mathcal{O}}$ is not a smooth manifold. This Poisson bracket turns $\overline{\mathcal{O}}$ into a stratified symplectic space.

A complex analytic stratified Kähler structure on a stratified space $N$ is a stratified symplectic structure $(C^\infty(N), \{\cdot, \cdot\})$ ($N$ is not necessarily smooth and $C^\infty(N)$ not necessarily an algebra of ordinary smooth functions) together with a complex analytic structure which, on each stratum, “combines” with
the symplectic structure on that stratum to a Kähler structure, in the following sense: The stratification underlying the stratified symplectic structure is a refinement of the complex analytic stratification whence each stratum is a complex manifold; each holomorphic function is a smooth function in $C^\infty(N, \mathbb{C})$; and on each stratum, the Poisson structure is symplectic in such a way that the symplectic structure combines with the complex structure to a Kähler structure. See Section 2 of [11] for details. The structure may be described in terms of a stratified Kähler polarization [11] which induces the Kähler polarizations on the strata and encapsulates the mutual positions of these polarizations on the strata. A complex polarization can no longer be thought of as being given by the $(0, 1)$-vectors of a complex structure, though. When the complex analytic structure is normal we simply refer to a normal Kähler structure. We now recall a few facts from [11].

(5.1) Given a holomorphic nilpotent orbit $O$, the closure $\overline{O}$ is a union of finitely many holomorphic nilpotent orbits. Moreover, the diffeomorphism from $O$ onto its image in $p$ extends to a homeomorphism from the closure $\overline{O}$ onto its image in $p$, this homeomorphism turns $\overline{O}$ into a complex affine variety, and the complex analytic structure, in turn, combines with the Poisson structure $(C^\infty(\overline{O}), \{\cdot, \cdot\})$ to a normal (complex analytic stratified) Kähler structure. See Theorem 3.2.1 in [11].

Let $r$ be the real rank of $g$. There are $r + 1$ holomorphic nilpotent orbits $O_0, \ldots, O_r$, and these are linearly ordered in such a way that

\begin{equation}
\{0\} = O_0 \subseteq \overline{O}_1 \subseteq \cdots \subseteq \overline{O}_r
\end{equation}

cf. (3.3.10) in [11]. Recall that the Cartan decomposition induces the decomposition $g^C = k^C \oplus p^+ \oplus p^-$ of the complexification $g^C$ of $g$, $p^+$ and $p^-$ being the holomorphic and antiholomorphic constituents, respectively, of $p^C$.

(5.3) The projection from $\overline{O}_s$ to $p$ is a homeomorphism onto $p$, and the $G$-orbit stratification of $\overline{O}_s$ passes to the $K^C$-orbit stratification of $p \cong p^+$. Thus, for $1 \leq s \leq r$, restricted to $O_s$, this homeomorphism is a $K$-equivariant diffeomorphism from $O_s$ onto its image in $p^+$, and this image is a $K^C$-orbit in $p^+$. See Theorem 3.3.11 in [11].

Remark 5.4. The holomorphic nilpotent orbits in a simple hermitian Lie algebra $g$ are precisely those which have the property that the projection to $p \cong p^+$ realizes the Kostant–Sekiguchi correspondence [21]. See Remark 3.3.13 in [11].

Proof of Theorem 1.5. The ascending sequence (5.2) of affine complex varieties determines the ascending sequence

\begin{equation}
Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_r = \mathbb{P}(p)
\end{equation}

of projective varieties where $\mathbb{P}(p)$ is the projective space on $p$ and where each $Q_s$ ($1 \leq s \leq r$) arises from $\overline{O}_s$ by projectivization. The stratified Kähler structures on the constituents of (5.2) pass to stratified Kähler structures on the constituents of (5.5), and all stratified Kähler structures in sight are normal. See Section 10 in [11] for details.
Exploiting the sequence (5.5) for each of the four simple regular rank 3 hermitian Lie algebras, from the description of Severi varieties in terms of the simple regular rank 3 hermitian Lie algebras given in Section 4, we conclude that Theorem 1.5 holds for \( k = 2 \) in the four regular cases or, equivalently, that the statements in Addendum 1.7 hold. In particular, the discussion in Section 8 (Appendix) of [2] confirms that, under these circumstances, \( Q_2 \) is indeed the chordal variety of the Severi variety \( Q_1 \), the exceptional case (1.4) being included. Likewise, for arbitrary rank \( r \geq 3 \), exploiting the sequence (5.5) for each of the simple rank \( r \) hermitian Lie algebras, from the description of \((r-1)\)-Scorza varieties in terms of the simple regular hermitian Lie algebras of rank \( r \) given in Section 4, we deduce that Theorem 1.5 holds for arbitrary rank \( r \geq 3 \) as well, the non-regular cases being included. In particular, in the remaining cases, which are all classical, that is, the corresponding Jordan algebras or JTS’s are matrix Jordan algebras or matrix JTS’s over the associative division algebras, any of the corresponding Scorza varieties is the corresponding subspace \( Q_1 \) arising from Jordan rank 1 matrices by projectivization and, for \( 1 \leq k \leq r - 1 \), the secant variety \( S^k(Q_1) \) arises likewise from the closure of the corresponding stratum of Jordan rank \( k + 1 \); indeed, this stratum consists of ordinary matrices of Jordan rank \( k + 1 \), and this observation entails the secant properties.

**Remark 5.6.** In each of the four regular rank 3 cases (critical Severi varieties), the cubic polynomial which determines the hypersurface \( Q_2 \) is the fundamental relative invariant (Bernstein–Sato polynomial) of the corresponding (irreducible regular) pre-homogeneous space, cf. Theorem 7.1 in [11] for the cases (1.1)–(1.3) and Theorem 8.4.1 in [11] for the case (1.4); in the latter case, the cubic polynomial is the generic norm or Freudenthal’s generalized determinant mentioned in Section 4. Likewise, still in view of Theorem 7.1 in [11], for higher rank \( r \geq 4 \), for each of the three regular \((r-1)\)-Scorza varieties \( Q_1 \), the degree \( r \) polynomial which determines the hypersurface \( Q_{r-1} = S^{r-2}Q_1 \) is the fundamental relative invariant (Bernstein–Sato polynomial) of the corresponding (irreducible regular) pre-homogeneous space.

**Remark 5.7.** Let \( p \geq q \geq 3 \). The hermitian Jordan triple system \( M_{q,p}(\mathbb{C}) \) has rank \( q \) and gives rise to a sequence

\[
\mathbb{P}^{p-1}\mathbb{C} \times \mathbb{P}^{q-1}\mathbb{C} = Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_q = \mathbb{P}^{pq-1}\mathbb{C}
\]

of stratified Kähler spaces of the kind (1.6) which includes the Segre embedding of \( Q_1 = \mathbb{P}^{p-1}\mathbb{C} \times \mathbb{P}^{q-1}\mathbb{C} \) into \( \mathbb{P}^{pq-1}\mathbb{C} \). In this case, \( \delta = 2 \),

\[
k_0(Q_1) + \left\lfloor \frac{p-q}{2} \right\rfloor = \left\lceil \frac{\dim Q_1}{\delta} \right\rceil,
\]

and the requirement (4.4) holds (with \( Q_1 \) substituted for \( X \)), whence \( Q_1 \) is a \((q-1)\)-Scorza variety if and only if \( p = q \) or \( p = q + 1 \); indeed, these varieties are then exactly those which belong to the family (1.2.k.n). Even though, for \( p \geq q + 2 \), \( \mathbb{P}^{p-1}\mathbb{C} \times \mathbb{P}^{q-1}\mathbb{C} \) is not a Scorza variety, the assertions of Theorem 1.5 still hold for this case (except the statement referring explicitly to a Scorza variety), and the sequence (5.7.1) still enjoys similar secant properties as the sequence (1.6) for an ordinary Scorza variety. The classification of Scorza varieties together with this additional class of varieties is exactly parallel to the classification of positive definite hermitian JTS’s and hence to the classification of irreducible hermitian symmetric spaces of non-compact type (which goes back to E. Cartan), cf. e.g. [18], and therefore as well to the classification of irreducible bounded symmetric domains.
Remark 5.8. By the same procedure as for higher rank, for $q \geq 3$, the regular rank 2 simple hermitian Lie algebra $\mathfrak{so}(2, q)$ leads to the standard quadric in $\mathbb{P}^{q-1} \mathbb{C}$ (a conic when $q = 3$), cf. Sections 6 and 10 of [11]. Since $\mathfrak{sp}(2, \mathbb{R}) \cong \mathfrak{so}(2, 3)$, $\mathfrak{su}(2, 2) \cong \mathfrak{so}(2, 4)$, $\mathfrak{so}^*(8) \cong \mathfrak{so}(2, 6)$, every regular simple rank 2 hermitian Lie algebra occurs here. Likewise, the non-regular rank 2 simple hermitian Lie algebras $\mathfrak{su}(p, 2)$ ($p \geq 3$), $\mathfrak{so}^*(10)$, and $\mathfrak{so}^*(14)$ lead to the projective varieties, respectively, $\mathbb{P}^{p-1} \mathbb{C} \times \mathbb{P}^1 \mathbb{C} \subseteq \mathbb{P}^{2p-1} \mathbb{C}$ (Segre embedding), $G_2 \mathbb{C}^5 \subseteq \mathbb{P}^9 \mathbb{C}$ (Plücker embedding), and $Q \subseteq \mathbb{P}^{15} \mathbb{C}$, where $Q$ arises from the subspace of pure spinors in $\mathbb{p}^+ \cong D_4^+$ (positive half-spin representation of complex dimension 16) by projectivization; see Sections 6 and 10 in [11]. These varieties can be interpreted as the limiting case of 1-Scorza varieties; in particular, with $k = 1$, the assertions in Theorem 1.5 still hold for these varieties and, in the regular case, these varieties are quadratic.

6. The classical cases via Kähler reduction

Given a Hodge manifold $N$, endowed with an appropriate group of symmetries and momentum mapping, reduction carries it to a complex analytic stratified Kähler space $N^\text{red}$ which is as well a projective variety [11, Section 4] and the question arises whether a complex projective space into which $N^\text{red}$ embeds carries an exotic structure which, via the (Kodaira) embedding, restricts to the complex analytic stratified Kähler structure on $N^\text{red}$. We will now show that, for the classical cases, each constituent $Q$, of the ascending sequence (1.6) ($1 \leq s \leq r = k + 1 \geq 3$) is of this kind. The construction to be given below includes, in particular, Zak’s Veronese mapping [24], cf. Remark 6.8 below.

Let $s \geq 1$, $d \geq 1$, let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and consider the standard (right) $\mathbb{K}$-vector space $\mathbb{K}^s$ of dimension $s$, endowed with a (non-degenerate) positive definite hermitian form $(\cdot, \cdot)$; further, let $V = \mathbb{K}^d$, endowed with a skew form $\mathcal{B}$ and compatible complex structure $J_V$ such that associating $\mathcal{B}(u, J_V v)$ to $u, v \in V$ yields a positive definite hermitian form on $V$. Moreover, let $H = H(s) = U(V^s, (\cdot, \cdot))$, $\mathfrak{h} = \text{Lie}(H)$, $G = U(V, \mathcal{B})$, $\mathfrak{g} = \text{Lie}(G)$, and denote the split rank of $G = U(V, \mathcal{B})$ by $r$. More explicitly:

\begin{align*}
(6.1) \quad & \mathbb{K} = \mathbb{R}, \ V = \mathbb{R}^d, \ d = 2\ell, \ H = O(s), \ G = \text{Sp}(\ell, \mathbb{R}), \ r = \ell; \\
(6.2) \quad & \mathbb{K} = \mathbb{C}, \ V = \mathbb{C}^d, \ d = p + q, \ H = U(s), \ G = U(p, q), \ p \geq q, \ r = q; \\
(6.3) \quad & \mathbb{K} = \mathbb{H}, \ V = \mathbb{H}^d, \ H = U(s, \mathbb{H}) = \text{Sp}(s), \ G = O^+(2d), \ r = [\frac{d}{2}].
\end{align*}

Let $W = W(s) = \text{Hom}_\mathbb{K}(\mathbb{K}^s, V)$. The 2-forms $(\cdot, \cdot)$ and $\mathcal{B}$ induce a symplectic structure $\omega_W$ on $W$, and the groups $H$ and $G$ act on $W$ in an obvious fashion: Given $x \in H$, $\alpha \in \text{Hom}_\mathbb{K}(\mathbb{K}^s, V)$, $y \in G$, the action is given by the assignment to $(x, y, \alpha)$ of $y x \alpha^{-1}$. These actions preserve the symplectic structure $\omega_W$ and are hamiltonian: Given $\alpha : \mathbb{K}^s \to V$, define $\alpha^\sharp : V \to \mathbb{K}^s$ by

\[(\alpha^\sharp u, v) = \mathcal{B}(u, \alpha v), \quad u \in V, \ v \in \mathbb{K}^s.\]

Under the identifications of $\mathfrak{h}$ and $\mathfrak{g}$ with their duals by means of the half-trace pairings, the momentum mappings $\mu_H$ and $\mu_G$ for the $H$- and $G$-actions on $W$ are given by

\[\mu_H : W \to \mathfrak{h}, \quad \mu_H(\alpha) = -\alpha^\sharp \alpha : \mathbb{K}^s \to \mathbb{K}^s,\]

\[\mu_G : W \to \mathfrak{g}, \quad \mu_G(\alpha) = \alpha \alpha^\sharp : V \to V.\]
respectively; these are the Hilbert maps of invariant theory for the $H$- and $G$-actions. Moreover, the groups $G$ and $H$ constitute a reductive dual pair in $\Sp(W, \omega_W)$ [7]. $J_V$ lies in $\mathfrak{g}$, and $(\mathfrak{g}, \frac{1}{2} J_V)$ is a simple (or reductive with simple semisimple constituent) Lie algebra of hermitian type. For $\phi$ complex structure. See Section 5 of [11] for details.

Since the $G$- and $H$-actions centralize each other, the $G$-momentum mapping $\mu_G$ induces a map $\overline{\mu}_G$ from the $H$-reduced space $W(s)^{\text{red}} = \mu_H^{-1}(0)/H$ into $\mathfrak{g}$. We now recall the following facts from Theorem 5.3.3 in [11].

(6.4) (1) The induced map $\overline{\mu}_G$ from the $H$-reduced space $W(s)^{\text{red}}$ into $\mathfrak{g}$ is a proper embedding of $W(s)^{\text{red}}$ into $\mathfrak{g}$ which, for $s \leq r$, induces an isomorphism of normal Kähler spaces from $W(s)^{\text{red}}$ onto the closure $\overline{O}_s$ of the holomorphic nilpotent orbit $O_s$.

(2) For $2 \leq s \leq r$, the injection of $W(s - 1)$ into $W(s)$ induces an injection of $W(s - 1)^{\text{red}}$ into $W(s)^{\text{red}}$ which, under the identifications with the closures of holomorphic nilpotent orbits, amounts to the inclusion $\overline{O}_{s - 1} \subseteq \overline{O}_s$.

(3) For $s > r$, the obvious injection of $W(s - 1)$ into $W(s)$ induces an isomorphism of $W(s - 1)^{\text{red}}$ onto $W(s)^{\text{red}}$.

(4) For $s \geq 1$, under the projection from $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ to $\mathfrak{p}$, followed by the identification of the latter with $\mathfrak{p}^+$, the image of the reduced space $W^{\text{red}}$ in $\mathfrak{g}$ is identified with the affine complex categorical quotient $W_J//H^C$, realized in the complex vector space $\mathfrak{p}^+$.

The projective space $\mathbb{P}[W]$ being endowed with a positive multiple of the Fubini–Study metric, the momentum mapping passes to a momentum mapping $\mu_H : \mathbb{P}[W] \to \mathfrak{h}$. For $s = r$, Kähler reduction yields the projective space $\mathbb{P}[\mathfrak{p}]$ on $\mathfrak{p}$, endowed with an exotic Kähler structure; likewise, for $1 \leq s \leq r$, Kähler reduction yields a normal Kähler space $Q_s$ together with an ascending sequence

(6.5) $Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_s$

of normal Kähler spaces which are, in fact, the closures of the strata of $Q_s$; the statement (6.4) above implies that this sequence amounts to the sequence (5.5), truncated at $Q_s$. Moreover, the embeddings of the $Q_s$’s into $Q_s = \mathbb{P}[\mathfrak{p}]$ are Kodaira embeddings. In view of (6.4) above, the normal Kähler structure coming from Kähler reduction coincides with the structure coming from projectivization of the closure of the corresponding holomorphic nilpotent orbit. See Section 10 of [11] for details.

Let $r \geq 3$ and $s = r$; with $d = 2r$, the sequence (6.5) then recovers the ascending sequence (1.6) for each of the three classical rank $r$ cases, where the parameter $n$ in (1.6) is now written as $r$. In particular, for $r = 3$, we obtain the sequence (1.6.SEVERI) for the three regular classical cases. For illustration, we describe briefly the various constituents in this particular case where the numbering (6.6.*) corresponds to the numbering (1.*) for $s = 1, 2, 3$: The sequence $\mathbb{P}[W(1)] \subseteq \mathbb{P}[W(2)] \subseteq \mathbb{P}[W]$ has the form:

(6.6.1) $\mathbb{P}^3 \subseteq \mathbb{P}^5 \subseteq \mathbb{P}^8 \subseteq \mathbb{P}^8$;
(6.6.2) $\mathbb{P}^5 \subseteq \mathbb{P}^{11} \subseteq \mathbb{P}^{17}$;
(6.6.3) $\mathbb{P}^{11} \subseteq \mathbb{P}^{23} \subseteq \mathbb{P}^{35}$. 
The sequence (1.6.SEVERI) has the form:

\[(6.6.1') \ X = \mathbb{P}^2 \subseteq Q^4 \subseteq \mathbb{P}^5 \cong \mathbb{P}^H//\text{O}(3, \mathbb{C});\]
\[(6.6.2') \ X = \mathbb{P}^2 \times \mathbb{P}^2 \subseteq Q^7 \subseteq \mathbb{P}^8 \cong \mathbb{P}^{17}/\text{GL}(3, \mathbb{C});\]
\[(6.6.3') \ X = G_2(\mathbb{C}^6) \subseteq Q^{13} \subseteq \mathbb{P}^{14} \cong \mathbb{P}^{35}/\text{Sp}(3, \mathbb{C}).\]

As GIT-quotients, the Severi varieties \(X\) and the cubics \(Q^s\) may be written out in the following fashion:

\[(6.6.1'') \ X = \mathbb{P}^2//\text{O}(1, \mathbb{C}) \cong \mathbb{P}^3, \quad Q^4 = \mathbb{P}^5//\text{O}(2, \mathbb{C});\]
\[(6.6.2'') \ X = \mathbb{P}^2 \times \mathbb{P}^2//\text{GL}(1, \mathbb{C}), \quad Q^7 = \mathbb{P}^{11}//\text{GL}(2, \mathbb{C});\]
\[(6.6.3'') \ X = G_2(\mathbb{C}^6) \cong \mathbb{P}^{11}//\text{Sp}(1, \mathbb{C}), \quad Q^{13} = \mathbb{P}^{23}//\text{Sp}(2, \mathbb{C}).\]

Here, for \(1 \leq s \leq 3\), the actions of the groups \(O(s, \mathbb{C}), \text{GL}(s, \mathbb{C})\), and \(\text{Sp}(s, \mathbb{C})\) on the corresponding projective spaces arise from the actions of the groups \(H\) on \(W(s)\) listed in (6.1)–(6.3) above. The sequences (6.6.1)–(6.6.3) may be viewed as resolutions of singularities (in the sense of stratified Kähler spaces) for the corresponding sequences (1.6.SEVERI). The disjoint union

\[\mathbb{P}[W] = H^C\mathbb{P}[W(1)] \cup H^C(\mathbb{P}[W(2)] \setminus \mathbb{P}[W(1)]) \cup (\mathbb{P}[W] \setminus H^C\mathbb{P}[W(2)])\]

is the \(H^C\)-orbit type decomposition of \(\mathbb{P}[W]\) in each case; here \(H^C\) denotes the complexification of \(H\).

In particular, the cubic \(Q_2\) (written as \(Q^4\), \(Q^7\), \(Q^{13}\) according to the case considered where the superscript indicates the complex dimension) arises by Kähler reduction, applied to the projective space \(\mathbb{P}[W(2)]\). This is an instance of the situation referred to at the beginning of this section. Furthermore, the Severi variety \(Q_1\) arises by Kähler reduction, applied to the projective space \(\mathbb{P}[W(1)]\) on \(W(1) (= V)\).

**Remark 6.7.** In the special case where \((g, h) = (\mathfrak{sp}(\ell, \mathbb{R}), \mathfrak{so}(s, \mathbb{R}))\), the space \(W\) may be viewed as the unreduced phase space of \(\ell\) particles in \(\mathbb{R}^s\), and \(\mu_H\) is the angular momentum mapping; cf. [16]. Thus the normal Kähler space \(\overline{Q}_s\) arises as the reduced phase space of a system of \(\ell\) particles in \(\mathbb{R}^s\) with total angular momentum zero. Likewise, the compact normal Kähler space \(Q_s\), which, complex analytically, is a projective variety, arises as the reduced phase space of a system of \(\ell\) harmonic oscillators in \(\mathbb{R}^s\) with total angular momentum zero and constant energy. Here the energy is encoded in the stratified symplectic Poisson structure; changing the energy amounts to rescaling the Poisson structure. The constituents \(Q_s\) \((1 \leq s \leq \ell)\) of the ascending sequence (5.5) have the following interpretation: The top stratum \(Q_s \setminus Q_{s-1}\) consists of configurations in general position, that is, \(\ell\) harmonic oscillators in \(\mathbb{R}^s\) with total angular momentum zero such that the positions and momenta do not lie in a cotangent bundle \(T^*\mathbb{R}^s\) for some \(\mathbb{R}^s \subseteq \mathbb{R}^\ell\) with \(s < \ell\). For \(1 \leq s < \ell\), the stratum \(Q_s \setminus Q_{s-1}\) consists of configurations of \(\ell\) harmonic oscillators in \(\mathbb{R}^s\) with total angular momentum zero such that the positions and momenta lie in the cotangent bundle \(T^*\mathbb{R}^s\) for some \(\mathbb{R}^s \subseteq \mathbb{R}^s\) but not in a cotangent bundle of the kind \(T^*\mathbb{R}^{s-1}\), whatever \(\mathbb{R}^{s-1} \subseteq \mathbb{R}^s\); here the convention is that \(Q_0\) is empty. Under these circumstances, the complex analytic structure does not have an immediate mechanical significance but helps understanding the geometry of the reduced phase space. The complex analytic structure is important for issues related with quantization, cf. [12]. Though this is not relevant here, we note that, for \(s > \ell\), the obvious map from \(Q_s\) to \(Q_s\) is an isomorphism of stratified Kähler spaces, and no new geometrical phenomenon occurs. Since the other dual pairs lie in some \(\mathfrak{sp}(n, \mathbb{R})\), it is likely that in the other cases (corresponding to (1.2) and (1.3)), the
constituents $Q_s$ ($1 \leq s \leq r$) of the ascending sequence (5.5) admit as well interpretations in terms of suitable constrained mechanical systems.

**Remark 6.8.** For $s = 1$, the composite of the momentum mapping $\mu_G : W \rightarrow \mathfrak{g}$ with the orthogonal projection to the symmetric part $\mathfrak{p}$ of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is exactly the Veronese mapping introduced in Zak’s book [24].

**Remark 6.9.** The exceptional Severi variety (1.4) is notably absent here. The question whether this variety and the corresponding ambient spaces $Q_2$ and $Q_3$ arise from Kähler reduction in the same way as the varieties (1.1)–(1.3) and the corresponding ambient spaces or, more generally, the varieties (1.1.$k$)–(1.3.$k$.n) and the constituents of the corresponding sequences of the kind (1.6), is related with that of existence of a dual pair $(G, H)$ with $G = E_7(-25)$ and $H$ compact but apparently there is no such dual pair. It would be extremely interesting to develop an alternative construction which yields the ascending sequence (1.6.SEVERI) for the exceptional Severi variety by Kähler reduction applied to a suitable Kähler manifold, perhaps more complicated than just complex projective space.

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