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Sampling Eulerian orientations of triangular lattice graphs ☆

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ABSTRACT

We consider the problem of sampling from the uniform distribution on the set of Eulerian orientations of subgraphs of the triangular lattice. Although Mihail and Winkler (1989) showed that this can be achieved in polynomial time for any graph, the algorithm studied here is more natural in the context of planar Eulerian graphs. We analyse the mixing time of a Markov chain on the Eulerian orientations of a planar graph which moves between orientations by reversing the edges of directed faces. Using path coupling and the comparison method we obtain a polynomial upper bound on the mixing time of this chain for any solid subgraph of the triangular lattice. By considering the conductance of the chain we show that there exist non-solid subgraphs (subgraphs with holes) for which the chain will always take an exponential amount of time to converge. Finally, we show that the problem of counting Eulerian orientations remains #P-complete when restricted to planar graphs (Mihail and Winkler had already established this for general graphs).

1. Introduction

Let $G = (V, E)$ be an Eulerian graph; that is, a graph with all vertices of even degree. An Eulerian orientation of G is an orientation of the edges of G such that for every vertex v, the number of edges oriented towards v is equal to the number oriented away from v: $\deg_{\text{in}}(v) = \deg_{\text{out}}(v)$. It is well known that the problem of finding an Eulerian orientation of an Eulerian graph can be solved efficiently. In this paper we consider the problem of sampling the set of Eulerian orientations of a planar Eulerian graph; that is, the problem of generating an Eulerian orientation from a distribution that is close to uniform. Most of the results obtained here deal with the special case of the triangular lattice. We focus on the *Markov chain Monte Carlo* method, a standard approach to random sampling of combinatorial structures. The Markov chain we study is the most natural chain whose state space is the set of Eulerian orientations of a planar graph. To move from one Eulerian orientation to another the chain randomly selects a face of the graph. If the edges of this face form a directed cycle in the original Eulerian orientation the chain reverses the orientation of its edges. We will hereafter refer to this chain as the *face-reversal* chain.

Markov chain simulation is generally only useful when we know the chain is *rapidly mixing*; that is, when the number of steps required to get within variation distance ϵ of the stationary distribution is bounded from above by a polynomial in the size of G and ϵ^{-1}. We will show, in Sections 4 and 5, that the face-reversal chain is rapidly mixing on solid subgraphs of the triangular lattice.

The problems of sampling and counting are closely related; indeed, almost all approximate counting algorithms rely on the existence of an efficient sampling algorithm. The #P-completeness of counting Eulerian orientations of a graph was established by Mihail and Winkler [21], thus motivating the study of rapidly mixing Markov chains for this problem. We will show, in Section 7, that this problem remains #P-complete when restricted to planar graphs.

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Mihail and Winkler [21] demonstrated that the problem of sampling from the Eulerian orientations of any graph can be reduced to the problem of sampling a perfect matching of a specially constructed bipartite graph. Sampling a perfect matching of this bipartite graph can be achieved in polynomial time using a Markov chain shown to be rapidly mixing by Jerrum and Sinclair [17] (see also [4,18]). Nevertheless, studying the face-reversal chain is still of interest as it is the more natural approach in the context of planar graphs. In particular, in this paper we show that the face-reversal chain provides a more efficient sampling algorithm for the case of solid subgraphs of the triangular lattice, which correspond to configurations of the 20-vertex ice model studied by statistical physicists [2].

The problem of counting Eulerian orientations of several planar lattices has been studied in statistical physics as it corresponds to evaluating the partition function Z_{ICE} of certain *ice models* [3,10]. In particular, Baxter [2] has found an asymptotic estimate for the number of Eulerian orientations of a grid-like section of the triangular lattice. More recently, Felsner and Zickfeld [13] have obtained upper and lower bounds on the number of Eulerian orientations of any planar map in the more general context of α-orientations.

We now discuss previous work on the problem of sampling Eulerian orientations for special cases of planar graphs. Luby et al. [19] and Goldberg et al. [14] both considered the face-reversal chain on Eulerian orientations of the square lattice in the cases of a particular fixed boundary condition and free boundary conditions, respectively. Both these proofs followed a similar pattern. First they extended the chain with extra transitions, and a coupling argument was used to find a bound on the mixing time of this extended chain. Finally, the comparison technique of Diaconis and Saloff-Coste [8] was applied to infer rapid mixing of the face-reversal chain. Fehrenbach and Rüschendorf [11] studied the mixing time of the face-reversal chain on the set of Eulerian orientations of the triangular lattice. Following the approach of [14,19] they defined an extension of the face-reversal chain (a different extension to the one we will use), and attempted to use the path coupling technique of [7] to show that it is rapidly mixing. However, the path coupling described therein does not contract as claimed; in fact, no one-step coupling of their particular chain will contract.

In this paper we follow a similar approach to [14,19] to show that the face-reversal chain is rapidly mixing on the set of Eulerian orientations of *solid* subgraphs of the triangular lattice. In addition to this positive result, we exhibit an infinite family of subgraphs of the triangular lattice for which this chain is not rapidly mixing. Finally, we include a proof that the problem of counting Eulerian orientations remains #P-complete when restricted to planar graphs. This provides some justification for studying the mixing time of the face-reversal Markov chain; however, we were not able to prove #P-completeness for the special case of the triangular lattice.

The remainder of the paper is laid out as follows: In Section 2 we define the notation we use for Markov chains and Eulerian orientations, along with an alternative way to view Eulerian orientations (due to Felsner [12]) which proves useful for our analysis. We also describe the machinery that we will use to prove results about the mixing time of the chains we study. Section 3 contains a description of two Markov chains, the face-reversal Markov chain that is the chief interest of this paper, and an extension of this chain which we call the *tower-moves* chain. Section 4 contains the details of the path-coupling argument we use to show that this extended chain mixes rapidly for solid subgraphs of the triangular lattice. This is then used to infer rapid mixing of the face-reversal chain via the comparison method of Diaconis and Saloff-Coste [8] in Section 5. In Section 6 we show that there exist subgraphs of the triangular lattice *with holes* for which the face-reversal chain will always take an exponential amount of time to converge. Finally, Section 7 contains the proof that exact counting of Eulerian orientations remains hard when restricted to planar graphs.

2. Background

In this section we summarise the techniques and theory used in our analysis.

A discrete-time Markov chain with transition probability matrix P defined on a finite state space Ω is called *ergodic* if it is both *aperiodic* and *irreducible*. Any ergodic Markov chain has a unique stationary distribution π. We say a Markov chain is *time-reversible* if it satisfies the *detailed balance condition*

$$\pi(x)P(x,y) = \pi(y)P(x,y), \quad \text{for all } x, y \in \Omega.$$

Moreover, if P is symmetric then π is uniform over Ω. Given two probability distributions, ρ and μ, on Ω the *variation distance* is defined as

$$\|\rho - \mu\|_{\mathrm{TV}} = \sup_{A \subset \Omega} \bigl|\rho(A) - \mu(A)\bigr|.$$

The mixing time is a measure of the number of steps taken by a Markov chain to get close to its stationary distribution. This is defined as

$$\tau(\epsilon) = \sup_{x \in \Omega} \min\bigl\{t\colon\ \bigl\|P^t(x,\cdot) - \pi\bigr\|_{\mathrm{TV}} \leqslant \epsilon\bigr\}.$$

A Markov chain is said to be rapidly mixing if $\tau(\epsilon)$ is bounded above by some polynomial in the size of the elements of Ω and in ϵ^{-1}. For example, in this paper we use the number of faces to measure the size of the problem.

Coupling is a standard technique for proving upper bounds on the mixing time of Markov chains [1,7]. A coupling of a chain $\mathcal{M}$ is a stochastic process $(X_t, Y_t)_{t \geqslant 0}$ on $\Omega \times \Omega$ such that each of the marginal distributions of $(X_t)_{t \geqslant 0}$ and $(Y_t)_{t \geqslant 0}$

are a faithful copy of the original Markov chain. To bound the mixing time via coupling we use the coupling inequality [1], which states that the variation distance between π and the distribution at time t is bounded above by the probability of any coupling coalescing by time t, i.e.

$$\|P^t - \pi\|_{\mathrm{TV}} \leqslant \sup_{X_0, Y_0} \Pr[X_t \neq Y_t].$$

Therefore, in order to obtain a polynomial bound on the mixing time it suffices to construct a coupling which will have coalesced (with high probability) after a polynomial number of steps. This can be proven by showing that the coupling causes all pairs of states to "move together" under some measure of distance.

We will use a simplified variation of coupling, due to Bubley and Dyer [7], known as *path coupling*. This involves defining a coupling (X_t, Y_t) by considering a *path* $X_t = Z_0, Z_1, \ldots, Z_r = Y_t$ between X_t and Y_t where each (Z_i, Z_{i+1}) is a pair of states adjacent in the Markov chain, and the path is a shortest path between X and Y. This allows us to restrict our attention to the pairs of states which are adjacent in the chain, as shown by the following theorem:

Theorem 1. *(See Bubley and Dyer [7].) Let $\mathcal{M}$ be an ergodic Markov chain with state space Ω and let δ be an integer valued metric defined on $\Omega \times \Omega$ which takes values in $\{0, \ldots, D\}$. Let S be a subset of $\Omega \times \Omega$ such that for all $(X, Y) \in \Omega \times \Omega$ there exists a path*

$$X = Z_0, Z_1, \ldots, Z_r = Y$$

between X and Y such that $(Z_i, Z_{i+1}) \in S$ for $0 \leqslant i < r$ and

$$\sum_{i=0}^{r-1} \delta(Z_i, Z_{i+1}) = \delta(X, Y).$$

Now suppose (X_t, Y_t) is a coupling of $\mathcal{M}$ defined on S. If there exists $\beta \leqslant 1$ such that for all $(X, Y) \in S$

$$\mathbb{E}\big[\delta(X_{t+1}, Y_{t+1}) | (X_t, Y_t) = (X, Y)\big] \leqslant \beta\delta(X_t, Y_t)$$

then this coupling can be extended to a coupling (X_t, Y_t) defined on the whole of $\Omega \times \Omega$ such that

$$\mathbb{E}\big[\delta(X_{t+1}, Y_{t+1})\big] \leqslant \beta\delta(X_t, Y_t).$$

Moreover, if $\beta < 1$ then $\tau(\epsilon) \leqslant \frac{\log(D\epsilon^{-1})}{1-\beta}$.

If $\beta = 1$ in Theorem 1 then in order to use standard path coupling techniques it must be shown that the variance of the distance between *any* two states after one step of the coupling can be bounded away from 0 [7]. However, a recent result has removed this condition:

Theorem 2. *(See Bordewich and Dyer [5].) Suppose we have a path coupling $\mathcal{C} = (X_t, Y_t)$ for an ergodic Markov chain $\mathcal{M}$ with distance metric $\delta : \Omega \times \Omega \to [0 \ldots D]$. Let $S \subset \Omega \times \Omega$ be the set of pairs of states at distance 1 and let p denote the minimum transition probability for S. We define a new chain, the* lazy chain *$\mathcal{M}^\star$, with probability transition matrix*

$$P^\star(X, Y) = \begin{cases} \frac{P(X,Y)+p}{1+p} & (X = Y), \\ \frac{P(X,Y)}{1+p} & \textit{otherwise}. \end{cases}$$

If $\beta \leqslant 1$ for the coupling $\mathcal{C}$ then the mixing time of the chain $\mathcal{M}^\star$ is $\tau(\epsilon) \leqslant \lceil p^{-1} e D^2 \rceil \lceil \log \epsilon^{-1} \rceil$. Moreover, if $\tau^\star(\epsilon)$ denotes the random time $\mathrm{Bin}(\tau(\epsilon), (1+p)^{-1})$*, where* Bin *denotes the binomial distribution, then the distribution of $X_{\tau^\star(\epsilon)}$ is within ϵ of the stationary distribution of the original chain $\mathcal{M}$.*

The second part of the above theorem means that, for all practical purposes, we can consider the mixing time of the chain $\mathcal{M}$ to be the same as the mixing time of $\mathcal{M}^\star$.

To prove a lower bound on the mixing time of a time-reversible Markov chain we rely on the notion of conductance. For any non-empty set $S \subset \Omega$ the conductance $\Phi(S)$ of S is defined to be

$$\Phi(S) = \frac{\sum_{x \in S, y \in \Omega \setminus S} \pi(x) P(x, y)}{\pi(S)}.$$

It is well known that conductance is inversely related to the mixing time (see e.g. [20,23]);

$$\tau(e^{-1}) \geqslant \frac{1}{2 \min_{0 < \pi(S) < 1/2} \Phi(S)}.$$

Thus, if n is some measure of the size of the elements of Ω then by finding an upper bound on the conductance of a Markov chain that is exponentially small in n we can conclude that the chain is not rapidly mixing. Conventionally, a chain is said to be *torpidly mixing* when it is known that it is not rapidly mixing.

Defining the boundary of a set S as

$$\partial S = \{x \in S\colon\ P(x, y) > 0 \text{ for some } y \in \Omega \setminus S\}$$

we get

$$\Phi(S) \leqslant \pi(\partial S)/\pi(S).$$

Hence, to show that a chain is torpidly mixing it suffices to find a set for which this last expression is bounded above by some exponentially small function. This is encapsulated in the following theorem, taken from [20]:

Theorem 3. *If, for some $S \subset \Omega$ satisfying $0 < |S| \leqslant |\Omega|/2$, the ratio $\pi(\partial S)/\pi(S)$ is exponentially small in the size of the elements of Ω, then the Markov chain is torpidly mixing.*

We will use the above theorem in Section 6 to show that there exist subgraphs of the triangular lattice for which the face-reversal chain is torpidly mixing.

Let $G = (V, E)$ be a planar graph and let $\mathcal{F}(G)$ denote the set of bounded faces in some planar embedding of G. We will use f to denote the number of elements in $\mathcal{F}(G)$ and $\mathrm{EO}(G)$ to denote the set of Eulerian orientations of G. For any simple cycle $C \subset E$, $\mathrm{Int}(C)$ is defined to be the set of faces on the interior of C. A face is said to be directed in an orientation of G if its boundary edges form a directed cycle. Felsner [12] showed that it is possible to convert any Eulerian orientation of G into another by performing a sequence of reversals of the edges of directed faces. Furthermore, a partial order was defined on $\mathrm{EO}(G)$ by letting $X \prec Y$ if Y can be obtained from X by performing a sequence of reversals of clockwise directed faces. This order has a unique maximum (resp. minimum) element: the unique Eulerian orientation with no clockwise (resp. counter-clockwise) cycles. Felsner proved this order forms a finite distributive lattice by giving a bijection between the Eulerian orientations of a planar graph and a set of functions of the form $\mathrm{EO}(G) \to \mathbb{N}$ called α-potentials. To define these functions Felsner used a partial order on $\mathcal{F}(G)$: $\sigma \prec_{\mathcal{F}} \eta$ if σ and η share an edge and that edge is counter-clockwise on σ in the minimum orientation. The α-potentials are then defined as the set of functions $\wp_X : \mathcal{F}(G) \to \mathbb{N}$ such that:

$$\sigma \text{ and } \eta \text{ share an edge } \Rightarrow \left|\wp_X(\sigma) - \wp_X(\eta)\right| \leqslant 1, \tag{1}$$

$$\sigma \text{ is on the boundary } \Rightarrow \wp_X(\sigma) \leqslant 1, \tag{2}$$

$$\sigma \prec_{\mathcal{F}} \eta \Rightarrow \wp_X(\sigma) \leqslant \wp_X(\eta). \tag{3}$$

The bijection is given as follows: let $\wp_X(\sigma)$ equal the number of times σ is reversed on any shortest path from $X_{\min}$ to X.

In this paper we use Felsner's results to show that the face-reversal Markov chain is irreducible and to define a natural distance metric on the set of Eulerian orientations of a planar graph. For the rest of this paper we will refer to this lattice as the *Felsner lattice* of a planar graph G, and denote it by $\mathrm{Fels}(G)$.

We now return to the (infinite) triangular lattice, a structure studied by members of the statistical physics community, e.g. [2,3,10]. This is the infinite graph with the following description:

$$V = \{v_{i,j}\colon\ i, j \in \mathbb{Z}\},$$

$$E = \big\{\{v_{i,j}, v_{i+1,j}\}, \{v_{i,j}, v_{i,j+1}\}, \{v_{i,j}, v_{i+1,j-1}\}\colon\ i, j \in \mathbb{Z}\big\}.$$

This graph can be embedded in the plane as shown in Fig. 1. In this paper we focus on two *finite* subgraphs of this lattice:

1. A *solid subgraph* is a graph which can be defined by specifying a simple cycle in the triangular lattice as the boundary and taking everything on its interior.

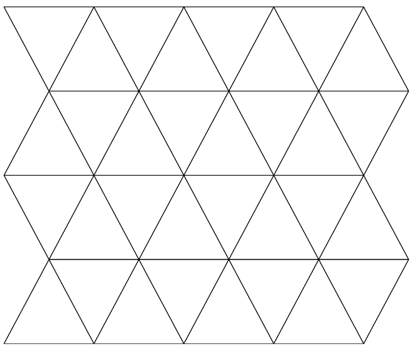

Fig. 1. A section of the triangular lattice.

2. A *subgraph with holes* is any subgraph of the triangular lattice which is not solid.

Sections 4 and 5 are devoted to showing that the face-reversal chain mixes rapidly on the set of Eulerian orientations of any Eulerian solid subgraph. In Section 6 we show that this is not true in the case of subgraphs with holes. We prove this negative result by exhibiting an infinite family of Eulerian subgraphs with holes on which the face-reversal chain is torpidly mixing.

3. The Markov chains

We now define the face-reversal Markov chain on the set of Eulerian orientations of any Eulerian planar graph. We use X and X' to denote the states of the chain before and after each step.

One step of the chain $\mathcal{M}$

1. With probability $1/2$, set $X' = X$.
2. With the remaining probability
   a. Choose $\kappa \in \mathcal{F}(G)$ u.a.r.
   b. If κ is directed then obtain X' from X by reversing the orientation of all the edges in κ.
   c. Otherwise, set $X' = X$.

Irreducibility of this chain follows from the fact that the underlying graph of the chain is the cover graph of $\mathrm{Fels}(G)$. The holding probability of $1/2$ guarantees *aperiodicity*, whence the chain is ergodic and converges to a unique stationary distribution π. Moreover, the chain is symmetric so π is the uniform distribution on the set of Eulerian orientations of G.

In order to apply the path coupling theorem we will need to extend this chain with extra moves in the style of [14,19]. These "tower moves" allow us to couple with $\beta \leqslant 1$, something which does not seem to be possible for the basic "face-reversal" chain with the metric we use (to be defined in Section 4). We now define a tower in X.

Definition 4. Let G be an Eulerian planar graph, X an Eulerian orientation of G, and γ a face of G. We say γ is *almost-directed* in X if all but one of the edges of γ have a common direction. We call the edge with the disagreeing direction the *blocking edge* of γ. For faces σ and η, which are, respectively, almost-directed and directed in the orientation X, we say there is a *tower* starting at σ and ending at η if there is a sequence of adjacent faces $\sigma = \gamma_1, \dots, \gamma_h = \eta$ such that

- γ_i is almost directed in X, and the blocking edge of γ_i is the one shared with γ_{i+1} for $1 \leqslant i \leqslant h-1$.
- γ_h is directed in X.

We say h is the *height* of the tower.

Observe that the definition of a tower implies that $C = \bigoplus_{1\leqslant i\leqslant h} E(\gamma_i)$ is a directed cycle in X. We say that a tower is clockwise (resp. counter-clockwise) in an orientation if this cycle is clockwise (resp. counter-clockwise) in the orientation. We call γ_h and γ_1 the top and bottom of the tower, respectively, and refer to the right and left sides of the towers in terms of a walk from the bottom to the top. It follows that in a clockwise tower the internal edges are all directed from the right to the left, and vice-versa for counter-clockwise towers.

Let $X \in \mathrm{EO}(G)$ and let $\sigma \in \mathcal{F}(G)$ be almost directed in X. If there is a tower in X starting at σ we can find it by walking along the faces of G, starting at σ and choosing the face sharing the blocking edge with the current face at each step. If at any point we reach a directed face then we have found a tower. On the other hand, we can be certain there is no tower starting at σ if we reach a face γ_i satisfying one of the following: γ_i is undirected but not almost directed; the blocking edge of γ_i lies on the boundary of the graph; the blocking edge of γ_i is the same edge as the blocking edge of γ_{i-1}. To see that the process of forming a tower terminates (i.e. does not wrap around on itself) note that reaching some already explored face implies the existence of a directed cut of blocking edges, something which is impossible in an Eulerian orientation. In particular, no vertex will be incident with more then 3 faces in any tower in an Eulerian orientation of a subgraph of the triangular lattice.

We now define the extended *tower-moves* chain. The definition includes an undetermined probability p_T which will be fixed later.

One step of the Markov chain $\mathcal{M}_T$

1. With probability $1/2$, set $X' = X$.
2. With the remaining probability
   a. Choose $\kappa \in \mathcal{F}(G)$ u.a.r.
   b. If κ is directed then obtain X' from X by reversing the orientation of all the edges in κ.

c. Otherwise, if there is a tower $T = (\gamma_i)_{1 \leqslant i \leqslant h}$ with $\gamma_1 = \kappa$ then let $C = \bigoplus_{1 \leqslant i \leqslant h} \gamma_i$. With probability p_T obtain X' from X by reversing all the edges of C.
d. Otherwise, set $X' = X$.

This type of chain has been used to extend the face reversal chain in the past, see [14,19]. The ergodicity of this chain is inherited from the ergodicity of $\mathcal{M}$, since every transition in $\mathcal{M}$ is also a transition in $\mathcal{M}_T$. As long as p_T is chosen so that the probability of reversing a tower is independent of whether it is a clockwise or a counter-clockwise tower, $\mathcal{M}_T$ converges to the uniform distribution. To see this suppose X can be obtained from Y by reversing a clockwise tower T. But then we can obtain Y from X by reversing a counter-clockwise tower containing the same faces as T, so $P(X, Y) = P(Y, X)$. Hence, the stationary distribution of $\mathcal{M}_T$ is also the uniform distribution on the set of Eulerian orientations of G.

4. Rapid mixing of $\mathcal{M}_T$ on solid subgraphs

In this section we use the path coupling technique of Bubley and Dyer [7] to show that the Markov chain $\mathcal{M}_T$ is rapidly mixing on $\mathrm{EO}(G)$ when G is a solid subgraph of the triangular lattice and p_T is chosen appropriately. This is not the first result regarding the mixing time of this type of chain. $\mathcal{M}_T$ has been shown to be rapidly mixing on the square lattice (using different p_T values) for the case of a particular fixed boundary condition [19] and for the case of a free boundary [14]. Fehrenbach and Rüschendorf [11] attempted to give a proof of rapid mixing for a related Markov chain in which only towers of height 2 are used. However, the path coupling defined for the chain in [11] does not contract as claimed. In fact, it is possible to show that no one-step path coupling can prove rapid mixing for the chain in [11] (without resorting to a more complicated metric, see e.g. [6]).

Note that rapid mixing proofs for this chain are dependent on the correct choice of the probabilities p_T. For example, the proof of [14] sets p_T to $1/4h$ if the tower runs along the boundary, and $1/2h$ otherwise, where h is the height of the tower T. We now state the main result of this section but defer the proof until we have presented some useful lemmas.

Theorem 5. *For any solid subgraph of the triangular lattice $\mathcal{M}_T^\star$, the lazy version of the Markov chain $\mathcal{M}_T$, is rapidly mixing with*

$$\tau_{\mathcal{M}_T^\star}(\epsilon) \in O\left(f^4 \log \epsilon^{-1}\right),$$

when $p_T = 1/3h$ for all towers T of height h.

Consequentially, the chain $\mathcal{M}_T$ can be considered to mix in this time as well, for all practical purposes (see [5]).

In our application of path coupling to proving rapid mixing of $\mathcal{M}_T$ we will need to define a metric on the set of Eulerian orientations of a planar graph G. We first define $\phi(G)$ to be the covering graph of $\mathrm{Fels}(G)$. With this definition, $\phi(G)$ is a connected graph with edge set

$$\big\{\{X, Y\}\colon\ X \text{ can be obtained from } Y \text{ by reversing a single face}\big\}.$$

We now define our metric: $\delta(X, Y)$ is the length of a shortest path from X to Y in $\phi(G)$. In particular $\delta(X, Y) = 1$ if X and Y differ on a single face. Let $X_{\min}$ and $X_{\max}$ denote the unique minimum and maximum elements of $\mathrm{Fels}(G)$. The fact that $\mathrm{Fels}(G)$ is a distributive lattice implies $D = \delta(X_{\min}, X_{\max})$, since we can always find a path of length D between two states X and Y either by going down from X to $X_{\min}$ and then up to Y, or by taking a similar path through $X_{\max}$.

Lemma 6. *Let G be any solid subgraph of the triangular lattice with f bounded faces. Then the maximum distance between any pair of Eulerian orientations in $\phi(G)$ is $O(f^{3/2})$.*

Proof. We write $\wp_{\max}$ for $\wp_{X_{\max}}$. Then, from the definition of the bijection between Eulerian orientations and α-potentials given in Section 2 we can conclude that the distance between the maximum and minimum Eulerian orientations is $\sum_{\gamma \in \mathcal{F}(G)} \wp_{\max}(\gamma)$.

Conditions (1) and (2) of the definition of α-potentials imply that $\wp_{\max}(\gamma)$ is exactly the minimum number of edges crossed by a path in the dual graph of G from γ to the boundary. Let G_k be the smallest graph which contains a face γ with $\wp_{\max}(\gamma) = k$. We can construct G_k inductively, starting with $G_1 = K_3$. To extend G_k to G_{k+1} we add a triangular face onto each edge of the boundary of G_k. See Fig. 2 for an example of what these look like. A simple inductive argument shows that the number of faces added at each step is $3k$. This implies $|\mathcal{F}(G_k)| \in \Theta(k^2)$, so $\wp_{\max}(\gamma) \in O(\sqrt{f})$ for any face γ. □

In the following let X, Y denote two Eulerian orientations of G which are adjacent in $\phi(G)$, let γ be the face on which they disagree, and let $N(\gamma)$ denote the set of faces which share an edge with γ. Our path coupling for $\mathcal{M}_T$ couples the transitions in X and Y as follows:

- with probability $1/2 - 1/2f$, leave both X and Y unchanged;
- with probability $1/2f$ reverse γ in X and leave Y unchanged;
- with probability $1/2f$ reverse γ in Y and leave X unchanged;

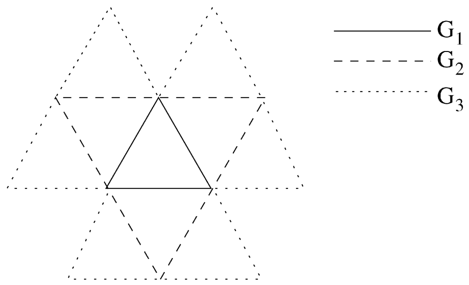

Fig. 2. G_1, G_2, and G_3 laid over each other.

- for each $\kappa \neq \gamma$, with probability $1/2f$ apply step 2 of $\mathcal{M}_T$ to κ both in X and Y.

In order to apply the path coupling theorem we need to show that the expected distance between the two Eulerian orientations does not increase after a single step of the coupling. To do this we need to consider which choice of faces will cause the distance to increase, which choices will leave the distance unchanged, and which choices will cause the distance to decrease. We say the move at κ involves $\eta \in N(\gamma)$ if $\kappa = \eta$ or there is a tower starting at κ that contains η. The distance between the coupled states may increase if and only if $\kappa \neq \gamma$ but the move at κ involves some $\eta \in N(\gamma)$. Note that no move can involve more than one element of $N(\gamma)$, so we can treat each $\eta \in N(\gamma)$ separately. In the following analysis, for each $\eta \in N(\gamma)$, we use δ_η to denote the maximum total expected change to the distance between the two adjacent Eulerian orientations after a single step of $\mathcal{M}_T$ resulting from *moves involving the face* η.

Lemma 7. *Suppose $\eta \in N(\gamma)$ is a directed face in X or Y. Then $\mathbb{E}[\delta_\eta] = \frac{1}{3f}$.*

Proof. Assume w.l.o.g. that η is directed in X. Let $\kappa \in \mathcal{F}(G)$ such that selecting κ gives a move which involves η in at least one of the coupled chains. We have two cases to consider:

Case $\kappa = \eta$: Since η is a neighbour of γ, and η is directed in X, it follows that the blocking edge of η in Y is the edge shared with γ. Then $T = \{\eta, \gamma\}$ is a tower of height 2 in Y with a reversal probability of $\frac{1}{6}$, conditional on η being the chosen face. η is directed in X so the coupling amounts to reversing η in X and T in Y with probability $\frac{1}{6}$, and reversing η in X but leaving Y unchanged with probability $\frac{5}{6}$. The former results in coalescence, whereas the latter yields a pair of orientations which are distance 2 apart. Thus, since we choose to apply the chain to η with probability $\frac{1}{2f}$, the expected value of δ_η is

$$\frac{1}{2f}\left[(+1)\left(\frac{5}{6}\right) - 1\left(\frac{1}{6}\right)\right] = \frac{1}{3f}.$$

Case $\kappa \neq \eta$: Since γ and η are both directed in X it follows that there must be a tower T_1 starting at κ and ending at η in X that does not contain γ, and a tower $T_2 = T_1 \cup \{\gamma\}$ starting at κ in Y. Let h be the height of T_1, so T_1 is reversed in X with probability $\frac{1}{3h}$ and T_2 is reversed in Y with probability $\frac{1}{3(h+1)}$. Observe that if we reverse T_1 in X we obtain an orientation which is distance $h+1$ from Y, but if we also reverse T_2 in Y then we have the same orientation in both chains. Therefore, the expected value of δ_η is

$$\frac{1}{2f}\left[h\left(\frac{1}{3h} - \frac{1}{3(h+1)}\right) - \frac{1}{3(h+1)}\right] = 0.$$

Thus, the only face whose selection will result in a move which involves η and increases the distance between the coupled chains is η itself, so $\mathbb{E}[\delta_\eta] = 1/3f$. $\square$

Lemma 8. *If $\eta \in N(\gamma)$ is not directed in X or Y then $\mathbb{E}[\delta_\eta]$ is no more than $\frac{1}{3f}$.*

Proof. Observe that, since η is not directed in X or Y, the blocking edge of η will not be shared with γ in either copy and the blocking edge is different in both. Hence, no tower can include both η and γ. Assuming there is a tower containing η in at least one of the two orientations we have two cases to consider. Suppose X has a tower starting at some $\sigma \in N(\eta)\backslash\{\gamma\}$. Then there will be no tower containing η in Y as the tower construction algorithm described in Section 3 is guaranteed to reach a pair of consecutive faces sharing the same blocking edge (η and σ). Thus, when we are in this situation we can assume that one of the orientations will be unchanged after one step of the coupling. In the second case, when there is no tower starting at any $\sigma \in N(\eta)\backslash\{\gamma\}$ in X or Y, there may be a tower starting at η in either.

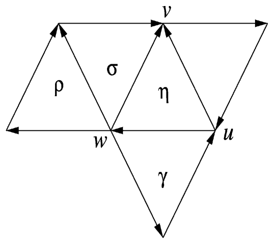

Fig. 3. Example from Lemma 8.

$\exists$ a tower starting at $\sigma \in N(\eta)\backslash\gamma$ in X or Y: Suppose (w.l.o.g.) X is the orientation with a tower starting at some $\sigma \in N(\eta)\backslash\gamma$. Since no move involving η is possible in Y we only need to bound the expected distance between X and all X' which can be obtained by making a move involving η in X.

We begin by showing that any tower containing η in X must start at η or a neighbour of η. To see this suppose we have a tower containing ρ, σ, and η where σ is a neighbour of η and $\rho \in N(\sigma)\backslash\{\eta\}$. Let u, v, w be the vertices of η, and suppose that the edges of η are oriented uv, uw, and wv. Then η shares $\{v, w\}$ with σ, and η shares $\{u, w\}$ with γ. Recall that no vertex can be incident with more then 3 faces in any tower. Therefore, v cannot belong to the edge shared between ρ and σ, so ρ must contain w (as illustrated in Fig. 3). To satisfy the definition of a tower ρ must have two edges oriented away from w. But this implies that there are 4 edges oriented away from w in X, a contradiction (e.g. see Fig. 3). An identical argument holds if the edges of η are oriented vu, uw, and vw. Hence, any tower containing η must start at σ or η.

Let h be the height of the tower starting at η. Then, in the coupling, the move in which η is the chosen face is made with probability $\frac{1}{2f} \cdot \frac{1}{3h}$, and the move in which σ is the chosen face (if it exists) is made with probability $\frac{1}{2f} \cdot \frac{1}{3(h+1)}$. Since these moves increase the distance by h and $h+1$, respectively, we have

$$\mathbb{E}[\delta_\eta] \leqslant \frac{h}{6fh} + \frac{h+1}{6f(h+1)} = \frac{1}{3f}.$$

$\nexists$ a tower starting at $\sigma \in N(\eta)\backslash\gamma$ in X or Y: In the worst case, we could have a tower starting at η in both orientations. Let T_1 and T_2 denote the towers in X and Y, respectively, and let h_1 and h_2 be the heights of each tower. If $h_1 \leqslant h_2$ then the coupling reverses the tower in X and Y with probability $\frac{1}{3h_2}$, and only reverses the tower in X with probability $\frac{1}{3h_1} - \frac{1}{3h_2}$. The first of these will give a pair orientations which are distance $h_1 + h_2 + 1$ apart, and the second gives a pair orientations which are $h_1 + 1$ apart. Hence,

$$\mathbb{E}[\delta_\eta] = \frac{1}{2f}\left[(h_1 + h_2)\frac{1}{3h_2} + h_1\left(\frac{1}{3h_1} - \frac{1}{3h_2}\right)\right] = \frac{1}{3f}.$$

The analysis is identical if $h_2 \leqslant h_1$.

Combining these two cases, we see that the expected value of δ_η is no more than $\frac{1}{3f}$. □

Note 1. It is not necessary to deal with the situation of any of the relevant faces lying on the boundary of G in the previous two lemmas as this will have no effect on the first and can only serve to reduce the value of δ_η in the second. Indeed, it is straightforward to check that the lemmas also hold in the case of any particular fixed boundary condition. In this case we are not strictly sampling Eulerian orientations, but a restricted case of α-orientations [12] of G.

We now apply Theorem 2 to obtain a bound on the mixing time.

Proof of Theorem 5. Consider the coupling defined earlier. With probability $1/f$ one of the two chains reverses γ causing the two chains to coalesce. Combining this fact with the results of Lemmas 7 and 8 we find that for all X and Y differing on the orientation of a single face

$$\mathbb{E}\big[\delta(X_{t+1}, Y_{t+1}) - \delta(X_t, Y_t) | (X_t, Y_t) = (X, Y)\big] \leqslant 3\frac{1}{3f} - \frac{1}{f} = 0.$$

It follows that $\beta = 1$ for the path coupling so we can apply Theorem 2 with $p = 1/2f$ and $D \in O(f^{3/2})$ (Lemma 6). Thus, we have a chain $\mathcal{M}_T^\star$ which mixes in time $\tau(\epsilon) \in O(f^4 \log \epsilon^{-1})$. □

5. Comparison of $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{M}_T}$

In this section we use the comparison method of Diaconis and Saloff-Coste [8] to infer a bound on the mixing time of $\mathcal{M}$ from the bound on the mixing time of $\mathcal{M}_T$ obtained in the previous section. We will use the formulation of the Diaconis and Saloff-Coste result from [22], restated here for convenience. Note that we are using $E(P)$ to denote the set of edges corresponding to moves between adjacent states in the Markov chain with transition matrix P.

Theorem 9. *(See [22, Proposition 4].) Suppose P and $\widetilde{P}$ are the transition matrices of two reversible Markov chains, $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, both with the state space Ω and stationary distribution π, and let $\pi_\star = \min_{x\in\Omega} \pi(x)$. For each pair $(u,v) \in E(\widetilde{P})$, define a path γ_{uv} which is a sequence of states $u = u_0, u_1, \ldots, u_k = v$ with $(u_i, u_{i+1}) \in E(P)$ for all i. For $(x,y) \in E(P)$, let $\Gamma(x,y) = \{(u,v) \in E(\widetilde{P})\colon (x,y) \in \gamma_{uv}\}$. Let*

$$A = \max_{(x,y)\in E(P)} \left\{ \frac{1}{\pi(x)P(x,y)} \sum_{(u,v)\in\Gamma(x,y)} |\gamma_{uv}|\pi(u)\widetilde{P}(u,v) \right\}.$$

Suppose that the second largest eigenvalue, λ_1, of $\widetilde{P}$ satisfies $\lambda_1 \geqslant 1/2$. Then for any $0 < \epsilon < 1$

$$\tau_{\mathcal{M}}(\epsilon) \in O\big(A\tau_{\tilde{\mathcal{M}}}(\epsilon)\log 1/\pi_\star\big).$$

To use the comparison method to bound the mixing time of the face-reversal chain we need to show that every move of $\mathcal{M}_T$ can be simulated by moves of the chain $\mathcal{M}$, and that we can do this without overloading any particular move of $\mathcal{M}$ in the sense described in the above theorem. Suppose $T = \{\gamma_1, \gamma_2, \ldots, \gamma_h\}$ is a tower in X and that Y is the orientation obtained by reversing T. Observe that by the definition of a tower γ_h is a directed cycle in X. Then we can perform a move on X in $\mathcal{M}$ to obtain a new Eulerian orientation X' in which γ_h has been reversed. But there is now a tower $T' = \{\gamma_1, \gamma_2, \ldots, \gamma_{h-1}\}$ in X' which can be reversed to obtain Y. Repeating this process until we reach Y gives a decomposition of the tower move into moves of the chain $\mathcal{M}$. We begin by bounding the size of any tower move:

Lemma 10. *Let G be a solid subgraph of the triangular lattice. Then the maximum height of a tower in an Eulerian orientation of G is $O(\sqrt{f})$.*

Proof. Let $T = (\gamma_1, \gamma_2, \ldots, \gamma_h)$ denote a tower in any Eulerian orientation of G. Assume w.l.o.g. that T is a clockwise tower. Recall that all the internal edges of a clockwise tower are directed towards the vertices on the left, so any Eulerian orientation of G must contain a set of $h-1$ edge-disjoint directed paths linking the left-side vertices to the right-side vertices. Each of these paths must go around the top or the bottom of the tower. But each path that goes around the bottom (resp. top) contributes 1 to the distance from γ_1 (resp. γ_h) to the boundary. Hence, by (1), (2) and (3), $\max(\wp_{\max}(\gamma_1), \wp_{\max}(\gamma_h)) \geqslant h/2$. But $\wp_{\max}(\gamma) \in O(\sqrt{f})$ for any γ (see proof of Lemma 6), whence $h \in O(\sqrt{f})$. $\square$

We are now ready to prove our rapid mixing result for $\mathcal{M}$.

Theorem 11. *Suppose G is a solid subgraph of the triangular lattice. Then the mixing time of the face-reversal Markov chain $\mathcal{M}$ on* $\mathrm{EO}(G)$ *satisfies*

$$\tau_{\mathcal{M}}(\epsilon) \in O(f^6 \log\epsilon^{-1}).$$

Proof. Note that the self-loop probability of 1/2 in the definition of a Markov chain ensures that the second-largest eigenvalue will be at least 1/2, so we can apply Theorem 9.

Let P and $\widetilde{P}$ denote the transition matrices of $\mathcal{M}$ and $\mathcal{M}_T^\star$ (the *lazy chain* obtained in the application of Theorem 2 to $\mathcal{M}_T$). For each pair of states (x,y) that differ on the orientation of exactly one face (i.e. each $(x,y) \in \ker(\mathcal{M})$), we define $\Gamma(x,y)$ to be the set of all transitions in $\mathcal{M}_T^\star$ containing the transition $t = (x,y)$ as a sub-move. For each such pair we have

$$A_{x,y} = \frac{1}{\pi(x)P(x,y)} \sum_{(u,v)\in\Gamma(x,y)} |\gamma_{uv}|\pi(u)\widetilde{P}(u,v), \tag{4}$$

$$= 2f \sum_{(u,v)\in\Gamma(x,y)} |\gamma_{uv}|\widetilde{P}(u,v), \tag{5}$$

$$\leqslant 1 + \frac{1}{3}\big(\big|\Gamma(x,y)\big| - 1\big), \tag{6}$$

where (5) is due to the fact that all transition probabilities in P are $\frac{1}{2f}$ and that π is uniform, and (6) is due to the fact that $\widetilde{P}(u,v) = \frac{1}{2f+1}$ if $(u,v) = (x,y)$, and $\widetilde{P}(u,v) = \frac{1}{3(2f+1)|\gamma_{uv}|}$ if (u,v) is the reversal of a tower.

Let σ be the face that is reversed in the transition $t = (x,y)$. We need to consider the different cases in which t can feature as part of the decomposition of a tower move $(u,v) \in E(\widetilde{P})$. Observe that there are three different directions in which a tower can pass through σ and contain t as a sub-move (one for each pair of edges of σ). Let γ and γ' denote the top and bottom of the maximal tower (in any Eulerian orientation) which passes through σ in the ith direction and whose encoding contains t. We use h_i to denote the height of this tower. Any other tower which passes through σ in this direction and whose encoding uses t must be subset of the faces of the maximal tower. Moreover the top (resp. bottom) must lie between γ (resp. γ') and σ. Hence, the number of tower moves that use a particular transition is $O(h_0^2) + O(h_1^2) + O(h_2^2)$. But $h_i \in O(\sqrt{f})$ whence $\Gamma(x,y)$ is in $O(f)$ for all $(x,y) \in \ker(\mathcal{M})$.

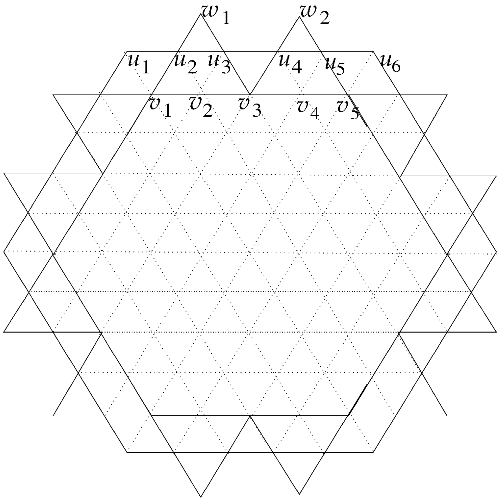

Fig. 4. The graph H_2.

Thus, we have $A_{x,y} \in O(f)$ for all $(x, y) \in \ker(\mathcal{M})$. The number of edges in G is no more than $3f$, so 2^{3f} provides an upper bound on the number of orientations of G, and so also on $1/\pi_\star$. Combining all this with Theorems 5 and 9 we get

$$\tau_{\mathcal{M}}(\epsilon) \in O\left(f^6 \log \epsilon^{-1}\right). \qquad \square$$

6. Subgraphs with holes

Given the small collection of positive results regarding the mixing time of the face-reversal chain $\mathcal{M}$ (Theorem 11 and [14,19]) and given that the reduction of [21]) allows us to sample from EO(G) in polynomial time for any graph, one might hope that $\mathcal{M}$ might be rapidly mixing on the set of Eulerian orientations of any planar graph. In fact, this is not true and in this section we exhibit an infinite family of subgraphs of the triangular lattice (with holes) for which $\mathcal{M}$ is torpidly mixing. Consider the family of graphs H_N, of which H_2 is shown in Fig. 4. Note that the dotted lines in Fig. 4 represent the parts of the triangular lattice which have been omitted. Formally, H_N is a graph with vertex set

$$V = \{v_i\colon\ 1 \leqslant i \leqslant 12N\} \cup \{u_i\colon\ 1 \leqslant i \leqslant 12N+6\} \cup \{w_i\colon\ 1 \leqslant i \leqslant 6N\}.$$

The edges of H_N consists of the disjoint union of three large cycles:

$$E_1 = (v_1, \ldots, v_{12N}, v_1),$$

$$E_2 = (u_1, \ldots, u_{12N+6}, u_1),$$

$$E_3 = (v_1, u_2, w_1, u_3, v_3, \ldots, u_{12N+6}, v_1)$$

e.g. see Fig. 4. It is the large face in the centre of each of these graphs that creates the bottleneck in the Markov chain we will use to show torpid mixing. We label this face C and its neighbours γ_i (for $1 \leqslant i \leqslant 6N$). The face that is adjacent to both γ_i and γ_{i+1} is labelled σ_i, and the face that is only adjacent to γ_i is labelled η_i.

Theorem 12. *The face-reversal chain $\mathcal{M}$ is torpidly mixing on* EO(H_N) *for large N.*

Proof. From Theorem 3 we know that $\mathcal{M}$ is torpidly mixing on a set of Eulerian orientations Ω if there exists some $S \subset \Omega$, with $0 < |S| \leqslant |\Omega|/2$, such that $|\partial S|/|S|$ is exponentially small in f.

Let S be the set of all Eulerian orientations X on H_N satisfying $\wp_X(C) \leqslant 1$. Note that the maximum value of $\wp_{\max}$ is 3. Hence, we can define a bijection between S and $\Omega \backslash S$ by mapping $\wp_X$ to $\wp_{\max} - \wp_X$ for each $X \in S$, so $|S| = |\Omega|/2$.

An Eulerian orientation X is an element of ∂S if and only if C is a counter-clockwise directed cycle in X and $\wp_X(C) = 1$. For this to occur we must have $\wp_X(\gamma_i) = 1$ for each γ_i. Hence, the number of Eulerian orientations satisfying this condition is exactly 2^{2k} since each of the γ_i and η_i can take potential value 0 or 1, where $k = 2N$.

C is the only directed cycle in $X_{\min}$ so $|S| = |S'| + 1$, where $S' = \{X \in \Omega\colon\ \wp_X(C) = 1\}$. We can partition S' as $\bigcup_{I \subset [k]} S_I$, where

$$S_I = \left\{X \in \Omega\colon\ \wp_X(C) = 1 \wedge \wp_X(\gamma_i) = 1 \Leftrightarrow i \in I\right\}.$$

We can find the size of each of the S_I by counting the number of potential functions which correspond to members of S_I. If $X \in S_I$ then there are two possible values for $\wp_X(\sigma_i)$ for each i with $\wp_X(\gamma_i) = 1$ and $\wp_X(\gamma_{i+1}) = 1$, and two possible values for $\wp_X(\eta_i)$ for each i with $\wp_X(\gamma_i) = 1$. All of the other σ_i and η_i must have potential value 0. Hence,

$$|S_I| = 2^{|I|+c(I)},$$

where $c(I)$ counts the number of circular successions in I. The number of j-subsets of $\{1, \dots, k\}$ containing m circular successions is given by the following expression[1]:

$$c(k, j, m) = \begin{cases} 0 & \text{if } j = 0,\ \ j > k, \text{or } m < 2j - k, \\ \frac{k}{j}\binom{j}{m}\binom{k-j-1}{j-m-1} & \text{otherwise.} \end{cases}$$

Then,

$$|S| = 1 + \sum_{j=0}^{k} 2^j \sum_{m=0}^{j} 2^m c(k, j, m) \tag{7}$$

$$= \sum_{j=1}^{k-1} \sum_{m=\max(0,2j-k)}^{j-1} \frac{k}{j}\binom{j}{m}\binom{k-j-1}{j-m-1} 2^{j+m} + 1 + 2^{2k} \tag{8}$$

$$> \sum_{j=1}^{k-1} \binom{j}{2j-k} 2^{3j-k} \tag{9}$$

$$> \binom{\lfloor \frac{16}{17}k \rfloor}{\lceil \frac{1}{17}k \rceil} 2^{\frac{31}{17}k-3} \quad \text{if } k \geqslant 17 \tag{10}$$

$$\geqslant 2^{(2+\frac{1}{17})k-3}. \tag{11}$$

The last line of this follows from the fact that $\binom{\lfloor \frac{16}{17}k \rfloor}{\lceil \frac{1}{17}k \rceil} \geqslant 2^{\frac{4k}{17}}$ when $k \geqslant 17$. Hence,

$$|\partial S|/|S| < 8 \cdot 2^{-\frac{1}{17}k} \in O(2^{-\frac{1}{51}f}). \qquad \square$$

7. Complexity of #PlanarEO

In this section we demonstrate a polynomial-time reduction from #EO to #PlanarEO:

Name. #EO
Input. A graph G
Output. The number of Eulerian orientations of G

Name. #PlanarEO
Input. A planar graph G
Output. The number of Eulerian orientations of G

This suffices to show that #PlanarEO is #P-complete since the #P-completeness of #EO is already known [21]. Our reduction uses a recursive gadget (suggested by Mark Jerrum [16]) and can be seen as an application of the so-called Fibonacci method of Vadhan [24].

Theorem 13. *There exists a polynomial time reduction from* #EO *to* #PlanarEO.

Proof. Let G be any non-planar graph for which we have an embedding in the plane with l crossings. We start by creating a planar graph G' by replacing each crossing $\{x, y\}$ and $\{u, v\}$ by a vertex s joined to each of u, v, x, y as in Fig. 5(a). Using our oracle for #PlanarEO we can count the number of Eulerian orientations of G' in polynomial time. Not all of these correspond to Eulerian orientations of G, e.g. $\{(u, s), (v, s)\}$ may be present in an Eulerian orientation of G'. We call the configurations of arcs at a crossover that correspond to Eulerian orientations of G *valid* configurations.

For each k we define H_k recursively as in Fig. 5(b) with H_0 given by Fig. 5(a). Let G_k be the graph obtained by replacing each crossing in G with H_k, so $G_0 = G'$.

Now let x_k (resp. y_k) denote the number of possible configurations of the edges of H_k satisfying the Eulerian condition which correspond to *valid* (resp. *invalid*) orientations of G'. These values satisfy

[1] This can be obtained by standard generating function calculations, see e.g. [15].

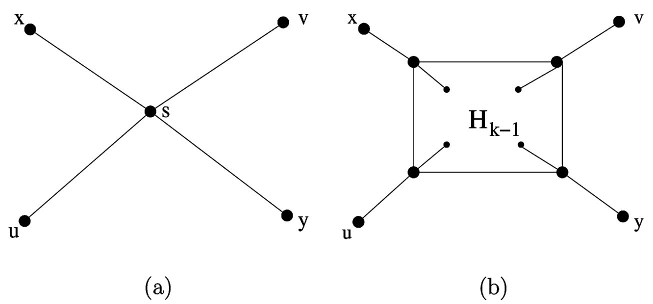

Fig. 5. The crossover box.

$$x_k = 4x_{k-1} + 2y_{k-1} \tag{12}$$

$$y_k = 4x_{k-1} + 3y_{k-1} \tag{13}$$

with $x_0 = y_0 = 1$. Now let N_i denote the number of Eulerian orientations of G' which have exactly i valid crossover boxes, so $N_l = \#\mathrm{EO}(G)$. Each Eulerian orientation of G' counted by N_i corresponds to exactly $x_k^i y_k^{l-i}$ Eulerian orientations of G_k, so we can write

$$\#\mathrm{Planar\,EO}(G_k) = \sum_{i=0}^{l} N_i x_k^i y_k^{l-i}.$$

Then calculating $\#\mathrm{Planar\,EO}(G_k)/y_k^l$ corresponds to evaluating the polynomial $p(z) = \sum_{i=0}^{l} N_i z^i$ at the point x_k/y_k. Since x_k/y_k is a non-repeating sequence (see [24, Lemma 6.2]) it suffices to evaluate $\#\mathrm{Planar\,EO}(G_k)/y_k^l$ for $k = 0 \ldots l+1$ to obtain enough information to recover the values of N_i by polynomial interpolation.

Letting n and m denote the number of vertices and edges in G we have $|V(G_k)| = n + (4k+1)l$ and $|E(G_k)| = m + (8k+2)l$. Since l is certainly bounded by a polynomial in n it follows that the whole reduction can be performed in time polynomial in n and m. □

8. Conclusions

We have shown that the mixing time of the face-reversal chain is $O(f^6)$ on the set of Eulerian orientations of any solid subgraph of the triangular lattice. This compares favourably with $O(f^7 \log^4(f))$, the best known bound on the sampling algorithm for general graphs [4,21]. On the other hand we have demonstrated that there exist subgraphs of the triangular lattice for which the face-reversal chain takes an exponential amount of time to converge. Given that the presence of a single very large face is the obstacle to rapid mixing in these graphs one might ask whether there exists some function $g(f)$ such that the face-reversal chain mixes rapidly whenever the number of edges in any face is bounded by $g(f)$.

We also showed that the problem of finding an exact value for $|\mathrm{EO}(G)|$ is #P-complete when G is a planar graph. This does not preclude a polynomial-time algorithm for evaluating $|\mathrm{EO}(G)|$ for more restricted classes of graphs, e.g. solid subgraphs of the triangular lattice. However, if such an algorithm cannot be found then our rapid mixing result could be used to construct a *fully polynomial randomised approximation scheme*, that is, an algorithm which can approximate the value of $|\mathrm{EO}(G)|$ to arbitrary precision ϵ and runs in time polynomial in f and ϵ^{-1}. We sketch such an algorithm below:

Let G be a solid subgraph of the triangular lattice and let γ be a face on the boundary of G. We construct G_1 by deleting the edges of γ from G and removing any isolated vertices. It is straightforward to check that $|\mathrm{EO}(G_1)|/|\mathrm{EO}(G)| \geqslant 1/4$. Hence, we can approximate this value to arbitrary precision with relatively few samples from $\mathrm{EO}(G)$. But G_1 is also a solid subgraph of the triangular lattice so we can repeat this procedure until we reach a final G_k for which we can easily find $|\mathrm{EO}(G_k)|$, e.g. K_3. Since the number of faces decreases by at least 2 at each step of this process it follows that $k \leqslant f/2$. Hence, we can estimate $|\mathrm{EO}(G)|$ to arbitrary precision ϵ in time polynomial in f and in ϵ^{-1}. A detailed description of this general method to convert efficient samplers into approximate counting algorithms can be found in [9].

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