On some categorical properties of uniform spaces of probability measures *

Vitaly V. Fedorchuk *, Yury V. Sadovnichy

Chair of General Topology and Geometry, Mechanics and Mathematics Faculty, Moscow State University,
Moscow 119899, Russia

Received 4 September 1996

Abstract

We deal with the functor $P_{\beta}^{\pi}: Unif \rightarrow Unif$ of uniform spaces of probability measures, defined by Sadovnichy (1994). We show that there is a unique natural transformation $T: S \circ P_{\beta}^{\pi} \rightarrow P \circ S$, where $S: Unif \rightarrow cUnif$ is the functor of Samuel compactification. In our first main result (Theorem 4.3) it is established that for a uniform space $(X, \mathcal{U})$ the component $T_{X}$ of this natural transformation $T$ is a homeomorphism iff $\mathcal{U}$ is a precompact uniformity. The second main result (Theorem 4.6) shows that there is no embedding $U: Tych \rightarrow Unif$ such that $P_{\beta}^{\pi} \circ U = U \circ P_{\beta}$.

Keywords: Probability measure; Uniform space; Pseudometric; Precompact space; Natural transformation; Samuel compactification

AMS classification: 46E27; 54E70

Introduction

For a Tychonoff space $X$ by $P_{\beta}(X)$ we denote the set of all probability measures on $X$ with compact supports, i.e.,

$$P_{\beta}(X) = \{\mu \in P(\beta X) : \mu(K) = 1 \text{ for some compact set } K \subset X\}.$$ 

This set $P_{\beta}(X)$ is equipped with the *-weak topology. $P_{\beta}$ is a covariant functor acting in the category $Tych$ of Tychonoff spaces. In [11–14] Sadovnichy lifted this functor onto the categories $Met r_b$ of bounded metric spaces and $Unif$ of uniform spaces, and investigated

* Corresponding author.

This paper was written while the authors were supported by the Russian Basic Research Foundation under grant 95 01-01273.
some properties of these liftings. We answer two questions arising in connection with these investigations.

Let $P^u_\beta$ be the lifting of $P_\beta$ onto the category $\text{Unif}$. We show (Proposition 4.2) that there is a unique natural transformation $T : S \circ P^u_\beta \to P \circ S$, where $S : \text{Unif} \to \text{cUnif}$ is the functor of Samuel compactification, and $P : \text{Comp} \to \text{Comp}$ is the probability measures functor. In our first main result (Theorem 4.3) it is established that for a uniform space $(X, \mathcal{U})$ the component $T_{\mathcal{U}}$ of this natural transformation $T$ is a homeomorphism iff $\mathcal{U}$ is a precompact uniformity. The second main result (Theorem 4.6) shows that there is no embedding (uniformization functor) $U : \text{Tygh} \to \text{Unif}$ such that $P^u_\beta \circ U = U \circ P_\beta$.

In Section 1 we recall all necessary notions and facts about pseudometrics and uniformities. More detailed information can be found in [4,9,10]. In Section 2 we give basic information about probability measures spaces and (pseudo)metrics on them. One can find additional information about spaces and functors of probability measures in [6,7]. The main result of Section 2 is Theorem 2.4. In this theorem sufficient conditions on a family of pseudometrics generating $*$-weak topology on $P_\beta(X)$ are given. Theorem 2.4 allows us to get a simple proof of Theorem 3.1 stating that for an arbitrary uniform space $(X, \mathcal{U})$ the uniformity $P_\beta(\mathcal{U})$ generates the $*$-weak topology. The main result of Section 3 (Theorem 3.11) establishes that the functor of square $\Pi^2$ is a subfunctor of $P^u_\beta$. This theorem plays a crucial role in Section 4 which contains the main results of the article.

1. Pseudometrics and uniformities

A pair $(X, \rho)$, where $X$ is a set and $\rho$ is a pseudometric on $X$, is said to be a pseudometric space. Every pseudometric space $(X, \rho)$ is equipped with topology $\tau_\rho$. An open base of this topology is formed by open $\varepsilon$-balls

$$O(x, \varepsilon) = \{y \in X : \rho(x, y) < \varepsilon\}, \quad x \in X, \varepsilon > 0.$$  

The topology $\tau_\rho$ is Hausdorff iff $\rho$ is a metric.

Let $(X, \tau)$ be a topological space and $\rho$ be a pseudometric on $X$. This pseudometric is called continuous if the mapping

$$\rho : (X \times X, \tau \times \tau) \to \mathbb{R}$$

is continuous. It is clear that $\rho$ is continuous iff the identity mapping $(X, \tau) \to (X, \tau_\rho)$ is continuous.

Let $(X, \rho)$ be a pseudometric space. We denote by $(\rho)$ a binary relation on $X$ which is defined in the following way:

$$x(\rho)y \iff \rho(x, y) = 0.$$  

Evidently, $(\rho)$ is an equivalence relation. The quotient set $X/(\rho)$ we denote by $X_\rho$, the quotient mapping $X \to X_\rho$ we denote by $\pi_\rho$. Let $\xi, \eta \in X_\rho$, $x, x' \in \pi_\rho^{-1}(\xi)$, $y, y' \in \pi_\rho^{-1}(\eta)$. Then it is easy to see that $\rho(x, y) = \rho(x', y')$. So we can define a
mapping $\hat{\rho}: X_\rho \times X_\rho \to \mathbb{R}$ by $\hat{\rho}(\xi, \eta) = \rho(x, y)$ for any $x \in \pi_\rho^{-1}(\xi)$ and $y \in \pi_\rho^{-1}(\eta)$. Clearly, $\hat{\rho}$ is a metric on $X_\rho$. The next statement is well known and trivial in proof.

**Proposition 1.1.** Let $X$ be a topological space and let $\rho$ be a continuous pseudometric on $X$. Then the metric $\hat{\rho}$ is continuous on $X_\rho$ with respect to the quotient topology.

By a uniformity on a set $X$ we mean a family $\mathcal{U}$ of symmetric entourages of the diagonal $\Delta_X \subset X \times X$ such that:

1. If $E_1, E_2 \in \mathcal{U}$, then $E_1 \cap E_2 \in \mathcal{U}$.
2. If $E \in \mathcal{U}$, then there is $E_1 \in \mathcal{U}$ such that $E_1 \circ E_1 \subset E$.
3. If $E \in \mathcal{U}$, $E \subset E_1$, and $E_1$ is symmetric, then $E_1 \in \mathcal{U}$.
4. $\bigcap \{E : E \in \mathcal{U}\} = \Delta_X$.

If $\mathcal{U}$ satisfies Conditions (1$\_U$)-(3$\_U$), then we say that $\mathcal{U}$ is a preuniformity. A family $\mathcal{B} \subset \mathcal{U}$ is said to be a base of a preuniformity $\mathcal{U}$, if for any $E \in \mathcal{U}$ there is $E_1 \in \mathcal{B}$ such that $E_1 \subset E$. If $\mathcal{B}$ is a base of preuniformity $\mathcal{U}$, then

$$\mathcal{U} = \{E \subset X \times X : E = E^{-1} \text{ and } E_1 \subset E, \ E_1 \in \mathcal{B}\}.$$  

It is clear that a family $\mathcal{B}$ of symmetric entourages of the diagonal $\Delta_X$ is a base of some preuniformity $\mathcal{U}$ on $X$ iff $\mathcal{B}$ satisfies condition (2$\_U$) and

$(1\_U)$ If $E_1, E_2 \in \mathcal{B}$, then there is $E \in \mathcal{B}$ such that $E \subset E_1 \cap E_2$.

Let $(X, \rho)$ be a pseudometric space. For $\varepsilon > 0$, set

$$E(\rho, \varepsilon) = \{(x, y) \in X \times X : \rho(x, y) < \varepsilon\}.$$  

Then the family

$$\mathcal{B}(\rho) = \{E(\rho, \varepsilon) : \varepsilon > 0\}$$  

is a base of a preuniformity that will be denoted by $u(\rho)$.

**Proposition 1.2.** $u(\rho)$ is a uniformity iff $\rho$ is a metric.

Let $\rho$ be a (pseudo)metric on $X$ and let $d > 0$. Set

$$\rho_d(x, y) = \min\{d, \rho(x, y)\}.$$  

Evidently $\rho_d$ is a (pseudo)metric.

**Proposition 1.3.** Let $\rho$ be a pseudometric on $X$ and $d > 0$. Then $u(\rho) = u(\rho_d)$.

If $\mathcal{U}$ is a (pre)uniformity on $X$, then the pair $(X, \mathcal{U})$ is called a (pre)uniform space. Sometimes we shall denote a (pre)uniform space $(X, \mathcal{U})$ by $X$. By $\mathbb{R}$ we shall denote four different objects:

1. the set of all real numbers;
2. the metric space $(\mathbb{R}, \rho_E)$, where $\rho_E(x, y) = |x - y|$;
3. the topological space $(\mathbb{R}, \tau_{\rho_E})$;
Let \( (X, \mathcal{U}) \) be a uniform space. A pseudometric \( \rho \) on \( X \) is said to be \textit{uniformly continuous} if the mapping \( \rho : X \times X \to \mathbb{R} \) is uniformly continuous.

**Proposition 1.4.** A pseudometric \( \rho \) on a preuniform space \( (X, \mathcal{U}) \) is uniformly continuous iff the identity mapping \( (X, \mathcal{U}) \to (X, \rho) \) is uniformly continuous.

**Corollary 1.5.** A pseudometric \( \rho \) on a preuniform space \( (X, \mathcal{U}) \) is uniformly continuous iff \( \mathcal{E}(\rho, \varepsilon) \in \mathcal{U} \) for any \( \varepsilon > 0 \).

**Proposition 1.6.** Let \( f : X \to Y \) be a uniformly continuous mapping between preuniform spaces. If \( \rho \) is a uniformly continuous pseudometric on \( Y \), then \( \rho \circ (f \times f) \) is a uniformly continuous pseudometric on \( X \).

**Proposition 1.7.** Let \( (X, \mathcal{U}) \) be a preuniform space and \( E \in \mathcal{U} \). Then there is a bounded uniformly continuous pseudometric \( \rho \) on \( X \) such that \( \mathcal{E}(\rho, 1) \subset E \).

We shall say that a family \( R \) of uniformly continuous pseudometrics on a preuniform space \( (X, \mathcal{U}) \) \textit{generates the preuniformity} \( \mathcal{U} \) if for each \( E \in \mathcal{U} \) there exist \( \rho \in R \) and \( \varepsilon > 0 \) such that \( \mathcal{E}(\rho, \varepsilon) \subset E \).

**Proposition 1.8.** Let \( (X, \mathcal{U}) \) be a preuniform space. Then the family \( \mathcal{R}(\mathcal{U}) \) of all bounded uniformly continuous pseudometrics on \( X \) generates the preuniformity \( \mathcal{U} \).

**Proposition 1.9.** Let \( R \) be a family of pseudometrics on a set \( X \) satisfying the condition:

\[ \text{(UP1)} \quad \text{If } \rho_1, \rho_2 \in R, \text{ then there is } \rho \in R \text{ such that } \rho_1, \rho_2 \leq \rho. \text{ Then there is a unique preuniformity } \]

\[ \mathcal{U} \equiv u(R) \text{ on } X \text{ such that } R \text{ generates } \mathcal{U}. \]

Moreover, \( u(R) \) is a uniformity iff \( R \) satisfies the condition:

\[ \text{(UP2)} \quad \text{For any } x, y \in X, \ x \neq y, \text{ there is } \rho \in R \text{ such that } \rho(x, y) > 0. \]

Let \( (X, \mathcal{U}) \) be a preuniform space and \( E \in \mathcal{U} \). For an arbitrary \( x \in X \) set

\[ E(x) = \{ y \in X : (x, y) \in E \}. \]

A preuniform space \( (X, \mathcal{U}) \) is called \textit{precompact} if for any \( E \in \mathcal{U} \) there is a finite set \( \{x_1, \ldots, x_n\} \subset X \) such that

\[ X - \bigcup \{ E(x_i) : i = 1, \ldots, n \}. \]

An entourage \( E \) from a preuniformity \( \mathcal{U} \) is said to be \textit{precompact} if there is a finite set \( \{x_1, \ldots, x_n\} \subset X \) such that

\[ \bigcup \{ E(x_i) \times E(x_i) : i = 1, \ldots, n \} \in \mathcal{U}. \]
Proposition 1.10. A preuniform space \((X, \mathcal{U})\) is precompact iff each \(E \in \mathcal{U}\) is precompact.

Proposition 1.11. Let \((X, \mathcal{U})\) be a precompact space and let \(Y \subset X\). Then \((Y, \mathcal{U}|_Y)\) is a precompact space.

Let \(\mathcal{U}\) be a uniformity on \(X\) and \(p\mathcal{U} = \{ E \in \mathcal{U}: E\) is precompact\}.

Proposition 1.12. For an arbitrary (pre)uniformity \(\mathcal{U}\) the family \(p\mathcal{U}\) is the biggest precompact (pre)uniformity which is contained in \(\mathcal{U}\).

Proposition 1.13. A preuniform space \(X\) is precompact iff every uniformly continuous pseudometric on \(X\) is totally bounded.

Corollary 1.14. Let \((X, \rho)\) be a metric space. Then the uniformity \(u(\rho)\) is precompact iff \(\rho\) is totally bounded.

For a preuniform space \((X, \mathcal{U})\) by \(\tau(\mathcal{U})\) we denote a topology induced by \(\mathcal{U}\). In this topology a set \(U \subset X\) is open iff for any \(x \in U\) there is \(E \in \mathcal{U}\) such that \(E(x) \subset U\). A preuniform space \((X, \mathcal{U})\) is called compact if \((X, \tau(\mathcal{U}))\) is a compact space.

Proposition 1.15. Let \((X, \mathcal{U})\) be a preuniform space. Then the following conditions are equivalent:

(a) \(\mathcal{U}\) is a uniformity;
(b) \(\tau(\mathcal{U})\) is Hausdorff;
(c) \(\tau(\mathcal{U})\) is Tychonoff.

Proposition 1.16. A uniform space \(X\) is compact iff \(X\) is precompact and complete.

By \(\text{Unif}\) we denote the category of all uniform spaces and their uniformly continuous mappings. By \(c\text{Unif}, \ p\text{Unif}, \ c\text{plUnif}\) we denote full subcategories of \(\text{Unif}\) consisting respectively of all compact, precompact, complete uniform spaces. For a preuniform space \((X, \mathcal{U})\) by \(pX\) \((cpl X)\) we shall denote its precompactification \((pX, p\mathcal{U})\) \((cpl X, cpl \mathcal{U})\). Let \(C\) be a category of uniform spaces and let \(D\) be its full subcategory. A covariant functor \(r : C \rightarrow D\) is said to be a reflection if \(r \circ r = r\), and there is a natural transformation \(T : \text{Id} \rightarrow r\) of the identity functor \(\text{Id}\) such that for any \(X \in C\) and uniformly continuous mapping \(f : X \rightarrow Y \in D\) there is a unique uniformly continuous mapping \(f_0 : r(X) \rightarrow Y\) with \(f = f_0 \circ T_X\).

Proposition 1.17. The completion \(cpl : \text{Unif} \rightarrow cpl\text{Unif}\) is a reflection. A component \(T_X\) of a natural transformation \(T : \text{Id} \rightarrow cpl\) is the identity embedding \(X \rightarrow cpl X\).

Proposition 1.18. The precompactification \(p\text{Unif} \rightarrow p\text{Unif}\) is a reflection. A component \(T_X : X \rightarrow pX\) of a natural transformation \(T : \text{Id} \rightarrow p\) is the identity mapping.
A composition \( cpl \circ p \equiv S \) is called Samuel compactification. The compactification \( S(X, \mathcal{U}) \) we shall denote by \( S_\mathcal{U}X \) or \( SX \).

**Proposition 1.19.** \( S: \mathfrak{Unif} \to \mathfrak{ClUnif} \) is a reflection.

**Proposition 1.20** [9, II, Exercise 12]. Let \((X, \mathcal{U})\) be a uniform space. Then \( p(\mathcal{U} \times \mathcal{U}) = p\mathcal{U} \times p\mathcal{U} \) iff \( \mathcal{U} \) is precompact.

### 2. Pseudometrics on spaces of probability measures

Let \( X \) be a compact Hausdorff space. By \( C(X) \) we denote a Banach space of all real-valued continuous functions on \( X \). The dual space \( C(X)^* \) is equipped with the \(*\)-weak topology, i.e., the topology induced by the identity embedding \( C(X)^* \subset \mathbb{R}^{C(X)} \). By Riesz’ theorem the positive cone \( C(X)^*_+ \) is affinely isomorphic to the space \( M(X) \) of all Borel finite positive regular measures on \( X \). This space is also equipped with the \(*\)-weak topology. We shall identify measures \( \mu \in M(X) \) with linear functionals from \( C(X)^* \). So, sometimes, for \( \phi \in C(X) \) we shall write \( \mu(\phi) \) instead of \( \int \phi \, d\mu \). A measure \( \mu \in M(X) \) is said to be a probability measure if \( \mu(1_X) = 1 \). The set of all probability measures on \( X \) is denoted by \( P(X) \). The space \( P(X) \) is a convex compact subset of \( \mathbb{R}^{C(X)} \). By the definition of \(*\)-weak topology its open base consists of sets

\[
O(\mu, \varphi_1, \ldots, \varphi_k, \varepsilon) = \{ \mu' \in P(X): |\mu(\varphi_i) - \mu'(\varphi_i)| < \varepsilon, \; i = 1, \ldots, k \}, \tag{2.1}
\]

where \( \mu \in P(X), \varphi_i \in C(X), \varepsilon > 0 \).

If \( f: X \to Y \) is a continuous mapping, then the formula

\[
P(f)(\mu)(\varphi) = \mu(\varphi \circ f), \tag{2.2}
\]

where \( \mu \in P(X) \) and \( \varphi \in C(Y) \), defines a continuous mapping \( P(f): P(X) \to P(Y) \). So, \( P \) is a covariant functor acting in the category \( \mathfrak{Comp} \) of compact Hausdorff spaces and their continuous mappings. It is clear that the mapping \( P(f) \) can be defined in the following way:

\[
P(f)(\mu)(B) = \mu(f^{-1}B), \tag{2.3}
\]

where \( B \subset Y \) is an arbitrary Borel set.

Let \( X \) be a compact Hausdorff space and \( \mu \in P(X) \). Set

\[
\text{supp } \mu = \{ x \in X: \mu(Ox) > 0 \text{ for any arbitrary neighbourhood } Ox \}.
\]

This set \( \text{supp } \mu \) is called the support of \( \mu \). The next statement is evident.

**Proposition 2.1.** Let \( X \) be a compact Hausdorff space, \( F \subset X \) and let \( \mu \in P(X) \). Then \( F = \text{supp } \mu \) iff \( F \) is the smallest closed subset of \( X \) such that \( \mu(F) = 1 \).

Now let \( X \) be a Tychonoff space and let \( \beta X \) be its Stone–Čech compactification. Set

\[
P_\beta(X) = \{ \mu \in P(\beta X): \text{supp } \mu \subset X \}. \tag{2.4}
\]
Let $\gamma X$ be an arbitrary compactification of $X$ and let $\pi_\gamma : \beta X \to \gamma X$ be a natural projection.

**Proposition 2.2.** $P(\pi_\gamma)|P_\beta(X)$ is a homeomorphism.

In fact, $P(\pi_\gamma)|P_\beta(X)$ is evidently a one-to-one correspondence and $P_\beta(X) = P(\pi_\gamma)^{-1}P(\pi_\gamma)(P_\beta(X))$. Hence, topologically we can define $P_\beta(X)$ as:

$$P_\beta(X) = \{\mu \in P(\gamma X): \text{supp} \mu \subset X\},$$

where $\gamma X$ is an arbitrary compactification of $X$.

Let $f : X \to Y$ be a continuous mapping between Tychonoff spaces and let $\beta f : \beta X \to \beta Y$ be its Stone–Čech compactification. We set

$$G(f) = P(\beta f)|P_\beta(X).$$

Clearly, $G(f)(\beta(X)) \subset \beta(Y)$. Thus, $P_\beta$ is a covariant functor acting in the category $\text{Top}$ of Tychonoff spaces and their continuous mappings. Evidently, $P_\beta$ is an extension of the functor $P : \text{Comp} \to \text{Comp}$ to the category $\text{Top}$.

For $x \in X$, by $\delta(x)$ we denote the Dirac measure, which is defined by

$$\delta(x)(\varphi) = \varphi(x) \quad \text{or} \quad \delta(x)\{x\} = 1.$$

It is easy to see that the Dirac embedding

$$\delta : X \to P_\beta(X)$$

is a topological embedding. Usually we shall identify the spaces $X$ and $\delta(X) \subset P_\beta(X)$.

Let $X$ be a Tychonoff space and let $\rho$ be a pseudometric on $X$. We define a distance function $P_\beta(\rho)$ on $P_\beta(X)$ by:

$$P_\beta(\rho)(\mu_1, \mu_2) = \inf \{\lambda(\rho): \lambda \in A(\mu_1, \mu_2)\},$$

where

$$A(\mu_1, \mu_2) = \{\lambda \in P(X \times X): \text{pr}_i(\lambda) = \mu_i, \ i = 1, 2\},$$

$\text{pr}_i = P_\beta(p_i)$, and $p_i : X \times X \to X$ is the projection onto the $i$th factor.

**Proposition 2.3.** If $\rho$ is a bounded continuous pseudometric on a Tychonoff space $X$, then $P_\beta(\rho)$ is a continuous pseudometric on $P_\beta(X)$ such that $P_\beta(\rho)|X = \rho$ and $\text{diam} P_\beta(\rho) = \text{diam} \rho$.

Basically it was proved in [5] for a metric compact space $(X, \rho)$. For a general case look at [2,11].

We shall say that a family $R$ of pseudometrics on $X$ separates points and closed subsets if for each $x \in X$ and closed set $F \subset X$, $x \notin F$, there is a pseudometric $\rho \in R$ such that $\rho(x, F) > 0$, where

$$\rho(x, F) = \inf \{\rho(x, y): y \in F\}.$$
We shall say that a family \( R \) of continuous pseudometrics on \( X \) generates the topology of \( X \) if for each \( x \in X \) and each neighbourhood \( O_x \) there are \( \rho \in R \) and \( \varepsilon > 0 \) such that \( O^\rho(x, \varepsilon) \subset O_x \).

**Theorem 2.4.** Let \( R \) be a family of continuous bounded pseudometrics on \( X \) which is directed, i.e., satisfies (UP1), and separates points and closed subsets. Then the family

\[
P_\beta(R) = \{ P_\beta(\rho): \rho \in R \}
\]

generates the \(*\)-weak topology of \( P_\beta(X) \).

To prove this theorem we need some auxiliary results.

**Proposition 2.5.** Let \( X \) be a compact Hausdorff space, \( C \subset C(X) \) be a family of functions, which separates points of \( X \) and contains all finite products. Then

\[
B_C = \{ O(\mu, \varphi, \varepsilon): \varphi \in C, \varepsilon > 0 \}
\]

is a subbase of \( P(X) \).

**Proof.** It follows from the definition of \(*\)-weak topology that the set

\[
B_D = \{ O(\mu, \psi, \varepsilon): \psi \in D, \varepsilon > 0 \},
\]

where \( D \) is dense in \( C(X) \), is a subbase of \( P(X) \). So, it suffices to show that for some dense set \( D \subset C(X) \) and an arbitrary neighbourhood \( O(\mu, \psi, \varepsilon) \in B_D \) there is a smaller neighbourhood of \( \mu \) which is an intersection of a finite family of neighborhoods from \( B_C \). Let \( D \) be the smallest subring of \( C(X) \) containing \( C \) and all constants. The set \( D \) is dense in \( C(X) \) by the Weierstrass–Stone theorem. Since \( C \) contains all its finite products, each function \( \psi \in D \) has a form

\[
\psi = r_1 \varphi_1 + \cdots + r_k \varphi_k + r_{k+1},
\]

where \( \varphi_i \in C, r_i \in R \). Let

\[
r = \max \{ |r_i|: i = 1, \ldots, k \}, \quad \delta = \frac{\varepsilon}{kr}.
\]

It remains to show that

\[
\bigcap_{i=1}^k O(\mu, \varphi_i, \delta) \subset O(\mu, \psi, \varepsilon).
\]

Let \( \nu \in \bigcap_{i=1}^k O(\mu, \varphi_i, \delta) \equiv O(\mu, \varphi_1, \ldots, \varphi_k, \delta) \). Then

\[
|\mu(\psi) - \nu(\psi)| = \left| \sum_{i=1}^k r_i (\mu(\varphi_i) - \nu(\varphi_i)) + \mu(r_{k+1}) - \nu(r_{k+1}) \right|
\]

\[
= \left| \sum_{i=1}^k r_i (\mu(\varphi_i) - \nu(\varphi_i)) \right| \quad \text{(since } \mu(s) = \nu(s) = s \text{ for any constant } s)\]
Proposition 2.5 is proved. □

Now let $X$ be a Tychonoff space. We shall say that a family $\Phi$ of continuous functions $\varphi : X \to [0, 1]$ correctly separates points and closed subsets of $X$ if for any closed set $F \subset X$ and point $x \in X \setminus F$ there is a function $\varphi \in \Phi$ such that $\varphi(F) = 0$ and $\varphi(x) = 1$.

**Proposition 2.6.** Let $X$ be a Tychonoff space, $C_\beta(X)$ be a family of all bounded real-valued continuous functions of $X$, $\Phi \subset C_\beta(X)$ correctly separates points and closed subsets of $X$ and contains all its finite products. Then the family

$$\{O(\mu, \varphi_1, \ldots, \varphi_k, \varepsilon) : \mu \in P_\beta(X), \; \varphi_i \in \Phi, \; \varepsilon > 0\}$$

is a base of $P_\beta(X)$.

**Proof.** Evidently, the diagonal product $f : X \to I^\Phi$ of functions $\varphi \in \Phi$ is an embedding. Let us denote by $\gamma X$ the closure of $f(X)$ in $I^\Phi$. From definition of $f$ we have $\Phi \subset C_\gamma(X)$, where $C_\gamma(X) = C(\gamma X)|X$. Consequently, every function $\varphi \in \Phi$ can be extended to a function $\overline{\varphi} \in C(\gamma X)$. Let $\overline{f} : \gamma X \to I^\Phi$ be the diagonal product of functions $\varphi$, $\varphi \in \Phi$. Clearly, $\overline{f}$ is the identity embedding. Hence, the family $\overline{\Phi} = \{\overline{\varphi} : \varphi \in \Phi\}$ separates points of $\gamma X$. Moreover, it contains all its finite products. Thus, according to Proposition 2.5 the sets $O(\mu, \overline{\varphi}_1, \ldots, \overline{\varphi}_k, \varepsilon)$, $\overline{\varphi}_i \in \overline{\Phi}$, form a base of $P(\gamma X)$. Then in view of (2.5) their traces $O(\mu, \varphi_1, \ldots, \varphi_k, \varepsilon)$, $\varphi_i \in \Phi$, on $P_\beta(X)$ form a base on $P_\beta(X)$. Proposition 2.6 is proved. □

For a family $R$ of continuous pseudometrics on a Tychonoff space $X$ we shall denote by $\Phi(R)$ the set of all functions $\varphi \in C_\beta(X)$ which are uniformly continuous with respect to some pseudometric $\rho \in R$.

**Proposition 2.7.** Let $R$ be a directed family of continuous pseudometrics on $X$ separating points and closed subsets of $X$. Then the family

$$\{O(\mu, \varphi, 1) : \mu \in P_\beta(X), \; \varphi \in \Phi(R)\}$$

(2.7)

form a subbase of $P_\beta(X)$.

**Proof.** First of all let us check that $\Phi(R)$ is a ring over $\mathbb{R}$. Let $\varphi_1, \varphi_2 \in \Phi(R)$ and let $\varphi_i$ be uniformly continuous with respect to $\rho_i \in R$. There is $\rho \in R$ such that $\rho \geq \max\{\rho_1, \rho_2\}$. Then $\varphi_1, \varphi_2$ are uniformly continuous with respect to $\rho$. Hence, $\varphi_1 + \varphi_2$ and $\varphi_1 \cdot \varphi_2$ are uniformly continuous. Consequently, $\Phi(R)$, containing all constants, is a ring over $\mathbb{R}$. 
Further, $\Phi(R)$ correctly separates points and closed subsets of $X$. In fact, let $x_0 \in X \setminus F$. There is a pseudometric $\rho \in R$ such that $\rho(x_0,F) - a > 0$. Let

$$\varphi(x) = \min \left\{ 1, \frac{d(x,F)}{a} \right\}.$$ 

It is clear, that $\varphi \in \Phi(R)$ satisfies the condition of a correct separation of $x_0$ and $F$. Hence, by Proposition 2.6 the family

$$\{ O(\mu, \varphi, \epsilon) : \mu \in P_\beta(X), \, \varphi \in \Phi(R), \, \epsilon > 0 \}$$

forms a subbase of $P_\beta(X)$. But if $\varphi \in \Phi(R)$ and $r \in \mathbb{R}$, then $r \cdot \varphi \in \Phi(R)$. It yields that the families (2.8) and (2.7) coincide. Proposition 2.7 is proved. \square

Remark 2.8. It is easy to see that if a family

$$\{ O(\mu, \varphi, \epsilon) : \mu \in P_\beta(X), \, \varphi \in \Phi, \, \epsilon \in E \},$$

is a subbase of $P_\beta(X)$, then for an arbitrary $\mu_0 \in P_\beta(X)$ the family

$$\{ O(\mu_0, \varphi, \epsilon) : \varphi \in \Phi, \, \epsilon \in E \},$$

is a subbase of neighborhoods of $\mu_0$ in $P_\beta(X)$.

Proof of Theorem 2.4. According to Propositions 2.3, 2.7 and Remark 2.8 it suffices to prove that every subbasic neighbourhood $O(\mu_0, \varphi, 1)$, $\varphi \in \Phi(R)$, of a measure $\mu_0$ contains an $\epsilon$-neighbourhood of this measure with respect to a pseudometric $\rho \in R$. Set $M = ||\varphi|| + 1$. Since $\varphi$ is uniformly continuous, there is $\delta, 0 < \delta < 1/(4M)$, such that $\rho(x_1,x_2) < \delta$ implies $|\varphi(x_1) - \varphi(x_2)| < 1/2$. We are going to show that

$$O^P_\beta(\rho)(\mu_0, \epsilon) \subset O(\mu_0, \varphi, 1)$$

for $\epsilon = \delta^2$. Let $\mu \in P_\beta(X)$ and $P_\beta(\rho)(\mu_0, \mu) < \epsilon$. There is $\lambda \in A(\mu_0, \mu)$ such that $\lambda(\rho) < \epsilon$. Set

$$A = \{(x_1,x_2) \in X \times X : \rho(x_1,x_2) \geq \delta \}.$$ 

Then $\epsilon > \lambda(\rho) \geq \int_A \rho(x_1,x_2) \, d\lambda \geq \delta \lambda(A)$. Hence, $\lambda(A) < \epsilon/\delta = \delta$. Consequently,

$$|\mu_0(\varphi) - \mu(\varphi)| = \left| \int_{X \times X} \varphi(x_1) \, d\lambda - \int_{X \times X} \varphi(x_2) \, d\lambda \right|$$

$$\leq \int_{X \times X} |\varphi(x_1) - \varphi(x_2)| \, d\lambda$$

$$\leq \int_A |\varphi(x_1) - \varphi(x_2)| \, d\lambda + \int_{X \times X \setminus A} |\varphi(x_1) - \varphi(x_2)| \, d\lambda$$

$$\leq 2M\lambda(A) + \frac{1}{2}\lambda(X \times X \setminus A) < 2M\delta + \frac{1}{2} < 1.$$ 

Theorem 2.4 is proved. \square

Proposition 2.3 and Theorem 2.4 yield

Theorem 2.9. Let $(X, \rho)$ be a bounded metric space. Then $P_\beta(\rho)$ is a metric generating the $*$-weak topology of $P_\beta(X)$. 


This theorem was proved by Al-Kassas [1] for uniformly zero-dimensional \( X \) and by Sadovnichy [11] for the general case.

Let \( C \) be some category, whose objects are Tychonoff spaces with an additional structure (metric, group, uniform and so on) and let \( \mathcal{F} : C \to \text{Tych} \) be a “forgetful” functor. We say that a functor \( \mathcal{G} : \text{Tych} \to \text{Tych} \) is lifted unto the category \( C \) if there is a functor \( \tilde{\mathcal{G}} : C \to C \) such that \( \mathcal{F} \circ \tilde{\mathcal{G}} = \mathcal{G} \circ \mathcal{F} \). Theorem 2.9 implies

**Theorem 2.10.** The functor \( P_\beta \) is lifted unto the category \( \text{Metr}_b \) of all bounded metric spaces and their continuous mappings.

The just described lifting of \( P_\beta \) unto the category \( \text{Metr}_b \) we shall denote by \( P_{\beta}^{\text{Metr}_b} \).

Let us check some simple properties of this lifting.

**Proposition 2.11.** The functor \( P_{\beta}^{\text{Metr}_b} \) preserves isometric embeddings.

Proof is trivial.

**Lemma 2.12.** Let \( f : X \to Y \) be a continuous mapping between Tychonoff spaces, and \( \rho_1 \) and \( \rho_2 \) be continuous bounded pseudometrics on \( X \) and \( Y \) respectively. Let \( \mu, \nu \in P_\beta(X) \) and

\[
P_\beta(\rho_1)(\mu, \nu) = \int_{X \times X} \rho_1(x_1, x_2) \, d\lambda,
\]

where \( \lambda \in \Lambda(\mu, \nu) \). Then

\[
P_\beta(\rho_2)(P_\beta(f)(\mu), P_\beta(f)(\nu)) \leq \int_{X \times X} \rho_2(f(x_1), f(x_2)) \, d\lambda.
\]

This lemma for metric compact spaces was proved in [5]. For the general case the proof is the same.

**Corollary 2.13.** The functor \( P_{\beta}^{\text{Metr}_b} \) preserves nonexpansive mappings.

The next statement for metric compact spaces was also proved in [5]. We repeat the proof in view of the high importance of this statement.

**Lemma 2.14.** Let \( f : X \to Y \) be a continuous mapping between Tychonoff spaces, and let \( \rho_1 \) and \( \rho_2 \) be continuous pseudometrics of diameter \( \leq a \) on \( X \) and \( Y \) correspondingly. If \( f : (X, \rho_1) \to (Y, \rho_2) \) is an \( (\varepsilon, \delta) \)-uniformly continuous mapping of pseudometric spaces, then \( P_\beta(f) \) is \( (2\varepsilon, \varepsilon \delta / a) \)-uniformly continuous.

**Proof.** Let \( P_\beta(\rho_1)(\mu, \nu) < \varepsilon \delta / a \). Since \( \Lambda(\mu, \nu) \subset P(\text{supp} \mu \times \text{supp} \nu) \) is compact, there is \( \lambda \in \Lambda(\mu, \nu) \) such that

\[
P_\beta(\rho_1)(\mu, \nu) = \int_{X \times X} \rho_1(x_1, x_2) \, d\lambda.
\]
Then
\[ P_\beta(\rho_2)(P_\beta(f)(\mu), P_\beta(f)(\nu)) \leq \int_{X \times X} \rho_2(f(x_1), f(x_2)) \, d\lambda \quad \text{(by Lemma 2.12)} \]
\[ = \int_{\rho_1(x_1, x_2) < \delta} \rho_2(f(x_1), f(x_2)) \, d\lambda + \int_{\rho_1(x_1, x_2) \geq \delta} \rho_2(f(x_1), f(x_2)) \, d\lambda \]
\[ < \varepsilon + \frac{a}{\delta} \int_{\rho_1(x_1, x_2) \geq \delta} \rho_1(x_1, x_2) \, d\lambda \quad \text{(since } f \text{ is } (\varepsilon, \delta)-\text{uniformly continuous)} \]
\[ \leq \varepsilon + \frac{a}{\delta} P_\beta(\rho_1)(\mu, \nu) < \varepsilon + \frac{a}{\delta} \cdot \frac{\varepsilon \delta}{a} = 2\varepsilon. \]

Lemma 2.14 is proved. \( \square \)

**Corollary 2.15.** The functor \( P_\beta^{M_b} \) preserves uniformly continuous mappings.

Let us denote by \( \text{Metr}_{b} \) the subcategory of \( \text{Metr}_b \) consisting of all bounded metric spaces and all their uniformly continuous mappings. Theorem 2.10 and Corollary 2.15 imply

**Theorem 2.16.** The functor \( P_\beta \) is lifted onto the category \( \text{Metr}_{b} \).

We shall denote this lifted functor by \( P_\beta^{M_{b}} \).

### 3. Probability measures on uniform spaces

Let \((X, \mathcal{U})\) be a uniform space. Let \( R(\mathcal{U}) \) be a family of all bounded uniformly continuous pseudometrics on \((X, \mathcal{U})\). Then, evidently, the family \( P_\beta(R(\mathcal{U})) \) satisfies condition \((\text{UP1})\). Hence, the preuniformity \( u(P_\beta(R(\mathcal{U}))) \) (look at Proposition 1.9) on \( P_\beta(X) \) induces on \( X \) the preuniformity \( \mathcal{U} \) in view of Propositions 1.8 and 2.3. We shall denote this preuniformity by \( P_\beta(\mathcal{U}) \).

**Theorem 3.1 [13].** Let \((X, \mathcal{U})\) be a uniform space. Then \((P_\beta(X), P_\beta(\mathcal{U}))\) is a uniform space with \(*\)-weak topology.

**Proof.** In view of Proposition 1.9 it suffices to verify that the preuniformity \( P_\beta(\mathcal{U}) \) generates the \(*\)-weak topology. We shall deduce it from Theorem 2.4. For this we have to check that the family \( R(\mathcal{U}) \) separates points and closed subsets of \( X \). Let \( x \in X \setminus F \), where \( F \) is closed in \( X \). By definition of a uniform topology, there is \( E \in \mathcal{U} \) such that \( E(x) \subset X \setminus F \). According to Proposition 1.8 there are \( \rho \in R(\mathcal{U}) \) and \( \varepsilon > 0 \) such that \( E(\rho, \varepsilon) \subset E \). Then \( \rho(x, F) \geq \varepsilon > 0 \). Theorem 3.1 is proved. \( \square \)

**Proposition 3.2.** If a family \( R \) of bounded uniformly continuous pseudometrics on a uniform space \((X, \mathcal{U})\) generates the uniformity \( \mathcal{U} \), then the family \( P_\beta(R) \) generates the uniformity \( P_\beta(\mathcal{U}) \).
Proof. Let $E$ be an arbitrary entourage from the uniformity $P_\beta(U)$. By definition of this uniformity there are a pseudometric $\rho \in R(U)$ and $\varepsilon > 0$ such that $P_\beta(\rho)^{-1}[0, \varepsilon) \equiv E(P_\beta(\rho), \varepsilon)) \subset E$. Since $R$ generates $U$, there are $\rho_1 \in R$ and $\delta > 0$ such that $\rho_1^{-1}[0, \delta)) \subset \rho^{-1}[0, \varepsilon/2))$. Hence, the identity mapping $(X, \rho_1) \to (X, \rho)$ is $(\varepsilon/2, \delta)$-uniformly continuous. Let $a = \max\{\text{diam} \rho_1, \text{diam} \rho\}$. Consequently, by Lemma 2.14, the identity mapping

$$(P_\beta(X), P_\beta(\rho_1)) \to (P_\beta(X), P_\beta(\rho))$$

is $(\varepsilon, (\varepsilon \delta)/(2a))$-uniformly continuous. Therefore,

$P_\beta(\rho_1)^{-1}\left[0, \frac{\varepsilon \delta}{2a}\right) \subset P_\beta(\rho)^{-1}[0, \varepsilon) \subset E.$

Proposition 3.2 is proved. □

Corollary 3.3. If $(X, \rho)$ is a bounded metric space, then the uniformities $P_\beta(u(\rho))$ and $u(P_\beta(\rho))$ on $P_\beta(X)$ coincide.

The next statement is a corollary of both Proposition 1.6 and Lemma 2.14.

Proposition 3.4 [13]. If $f : (X, U) \to (Y, V)$ is a uniformly continuous mapping between uniform spaces, then the mapping

$P_\beta(f) : (P_\beta(X), P_\beta(U)) \to (P_\beta(Y), P_\beta(V))$

is also uniformly continuous.

Theorem 3.1 and Proposition 3.4 imply

Theorem 3.5 [13]. The functor $P_\beta : \text{Tych} \to \text{Tych}$ is lifted onto the category $\text{Unif}$.

We denote this lifted functor by $P^\beta$. Let $\mathcal{MUnif} \subset \text{Unif}$ be the category of all metrizable uniform spaces and their uniformly continuous mappings. We have

$P^\beta(\mathcal{MUnif}) \subset \mathcal{MUnif}$

according to Corollary 3.3 and Lemma 2.14. We shall denote the restriction $P^\beta|\mathcal{MUnif}$ by $P^\beta\mu$. Let $\mathcal{F}_u : \text{Metr}_bu \to \mathcal{MUnif}$ be the uniformization functor.

Proposition 3.6. The functor $P^\mu : \mathcal{MUnif} \to \mathcal{MUnif}$ is lifted to the functor

$P^\mu_{\text{Metr}_bu} : \text{Metr}_bu \to \text{Metr}_bu$.

Proof. It is sufficient to check that

$$\mathcal{F}_u \circ P^\mu_{\text{Metr}_bu} = P^\mu \circ \mathcal{F}_u.$$  \hspace{1cm} (3.1)

For spaces $(X, \rho) \in \text{Metr}_bu$, Eq. (3.1) follows from Corollary 3.3. For mappings $f \in \text{Metr}_bu$, Eq. (3.1) follows from Lemma 2.14 and Proposition 3.2. □
Proposition 3.7 [3]. If \( f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V}) \) is a uniform embedding, then
\[
P^\beta(f) : (P^\beta(X), P^\beta(\mathcal{U})) \rightarrow (P^\beta(Y), P^\beta(\mathcal{V}))
\]
is also a uniform embedding.

**Proof.** We may assume that \( f : X \rightarrow Y \) is the identity embedding. By (2.5) and Proposition 3.4, \( P^\beta(f) \) is a topological embedding and a uniformly continuous mapping. Let \( E \subset P^\beta(\mathcal{U}) \) be an arbitrary entourage. By definition of \( P^\beta(\mathcal{U}) \) there is a pseudometric \( \rho \in R(\mathcal{U}) \) such that \( \rho^{-1}[0, \varepsilon) \subset E \) for some \( \varepsilon > 0 \). This pseudometric \( \rho : X \times X \rightarrow \mathbb{R} \) can be extended to a bounded uniformly continuous pseudometric \( \rho_1 \) on \( (Y, \mathcal{V}) \) (see [8, 8.5.6]). Let \( E_1 = \rho_1^{-1}[0, \varepsilon) \). Then \( E_1 \in P^\beta(\mathcal{V}) \) by definition and evidently \( E_1 \cap (X \times X) = \rho^{-1}[0, \varepsilon) \subset E \). Proposition 3.7 is proved. \( \Box \)

Proposition 3.8 [3]. A uniform space \((X, \mathcal{U})\) is precompact iff \((P^\beta(X), P^\beta(\mathcal{U}))\) is precompact.

**Proof.** If \((P^\beta(X), P^\beta(\mathcal{U}))\) is precompact, then \((X, \mathcal{U})\) is precompact in view of Proposition 1.11 and a uniform embedding
\[
\delta : (X, \mathcal{U}) \rightarrow (P^\beta(X), P^\beta(\mathcal{U})).
\]
Conversely, if \((X, \mathcal{U})\) is precompact, then \((X, \mathcal{U}) \rightarrow S_\mathcal{U}X \) is a uniform embedding into a compact space. Then, by Proposition 3.7, \((P^\beta(X), P^\beta(\mathcal{U}))\) is a subspace of the compact uniform space \( P(S_\mathcal{U}X) \). Applying Proposition 1.11 once more, we get a precompactness of \((P^\beta(X), P^\beta(\mathcal{U}))\). The proposition is proved. \( \Box \)

Proposition 3.8 and Corollaries 3.3 and 1.14 imply

Proposition 3.9. A metric space \((X, \rho)\) is totally bounded iff \((P^\beta(X), P^\beta(\rho))\) is totally bounded.

Proposition 3.10. Let \((X, \mathcal{U})\) be a uniform space and let
\[
i \equiv i_X : X \times X \rightarrow P^\beta(X)
\]
be a mapping which is defined as follows:
\[
i(x_1, x_2) = \frac{9}{10} \delta(x_1) + \frac{1}{10} \delta(x_2).
\]
Then \( i : (X \times X, \mathcal{U} \times \mathcal{U}) \rightarrow (P^\beta(X), P^\beta(\mathcal{U})) \) is a uniform embedding.

**Proof.** It suffices to show that for every bounded uniformly continuous pseudometric \( \rho \) on \( X \) the mappings \( i \) and \( i^{-1} \) are uniformly continuous with respect to the pseudometric \( \rho \times \rho \) and \( P^\beta(\rho) \), where
\[
\rho \times \rho((x_1, x_2), (y_1, y_2)) = \rho(x_1, y_1) + \rho(x_2, y_2).
\]
We shall prove more: for any \( \xi, \eta \in X \times X \)
\[
\frac{1}{20} \rho \times \rho(\xi, \eta) \leq P^\beta(\rho)(i(\xi), i(\eta)) \leq \rho \times \rho(\xi, \eta).
\]
Let $\xi = (x_1, x_2)$, $\eta = (y_1, y_2)$, $\rho(x_1, y_1) = \rho_1$, $\rho(x_2, y_2) = \rho_2$. We are going to prove that

$$\frac{1}{10}(\rho_1 + \rho_2) \leq d \leq \rho_1 + \rho_2,$$

where $d = P_\beta(\rho)(i(\xi), i(\eta))$. Let $\lambda \in \Lambda(i(\xi), i(\eta))$. Then

$$\lambda = m_{11}\delta(x_1, y_1) + m_{22}\delta(x_2, y_2) + m_{12}\delta(x_1, y_2) + m_{21}\delta(x_2, y_1),$$

where $m_{ij} \geq 0$ and

$$m_{11} + m_{12} = m_{11} + m_{21} = \frac{9}{10},$$
$$m_{22} + m_{12} = m_{22} + m_{21} = \frac{1}{10}.$$

Let $m_{11} = a$, then $m_{12} = m_{21} = \frac{9}{10} - a$ and $m_{22} = a - \frac{8}{10}$. We have

$$d \leq \lambda(\rho) = a\rho_1 + \left(a - \frac{8}{10}\right)\rho_2 + \left(\frac{9}{10} - a\right)(\rho_{12} + \rho_{21})$$

for an arbitrary $a$ (let us note that $\frac{8}{10} \leq a \leq \frac{9}{10}$). In particular, for $a = \frac{9}{10}$,

$$d \leq \frac{9}{10}\rho_1 + \frac{1}{10}\rho_2 \leq \rho_1 + \rho_2.$$

Since $\Lambda(i(\xi), i(\eta))$ is compact, there is $a$ such that

$$d = a\rho_1 + \left(a - \frac{8}{10}\right)\rho_2 + \left(\frac{9}{10} - a\right)(\rho_{12} + \rho_{21}).$$

There are two possibilities: $a \geq 0.85$ and $a \leq 0.85$. Let $a \geq 0.85$. Then

$$d \geq a\rho_1 + (a - 0.8)\rho_2 \geq 0.85\rho_1 + 0.05\rho_2 \geq 0.05(\rho_1 + \rho_2).$$

Now let $a \leq 0.85$. We have

$$\rho_2 = \rho(x_2, y_2) \leq \rho(x_2, y_1) + \rho(y_1, x_1) + \rho(x_1, y_2).$$

Hence,

$$\rho_{12} + \rho_{21} \geq \rho_2 - \rho_1.$$

Then

$$d \geq a\rho_1 + (a - \frac{8}{10})\rho_2 + (0.9 - a)(\rho_2 - \rho_1) = (2a - 0.9)\rho_1 + 0.1\rho_2$$
$$\geq 0.7\rho_1 + 0.1\rho_2 \geq 0.1(\rho_1 + \rho_2).$$

Proposition 3.10 is proved. ☐

Let $\mathcal{C} = \{\mathcal{O}, \mathcal{M}\}$ be some category, where $\mathcal{O}$ is a family of its objects and $\mathcal{M}$ is a family of its morphisms. Let $\mathcal{F}, \mathcal{G}: \mathcal{C} \to \mathcal{C}$ be covariant functors. Let us recall that a family $T = \{T_X: \mathcal{F}(X) \to \mathcal{G}(X): X \in \mathcal{O}\}$ of morphisms from $\mathcal{M}$ is said to be a natural transformation of the functor $\mathcal{F}$ in to the functor $\mathcal{G}$ if for any morphism $f: X \to Y$ from $\mathcal{M}$ the following diagram

$$\begin{array}{ccc}
\mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y) \\
\mathcal{T}_X \downarrow & & \downarrow \mathcal{T}_Y \\
\mathcal{G}(X) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(Y)
\end{array}$$
is commutative. Morphisms $T_X$ are called components of a natural transformation $T$. A functor $F$ is called a subfunctor of a functor $G$ if there is a natural transformation $T : F \to G$ such that each component $T_X$ is a monomorphism.

By $\Pi^2$ we denote the functor of square: $\Pi^2(X) = X \times X$; if $f : X \to Y$ is a mapping, then $\Pi^2(f) = f \times f : X \times X \to Y \times Y$. The functor $\Pi^2$ acts in such categories as $\mathcal{T}_\text{Ych}$, $\text{Comp}$, $\text{Unif}$, $\text{cUnif}$ and so on.

**Theorem 3.11.** The embedding $i_X : X \times X \to P_\beta(X)$ from Proposition 3.10 can be extended to a natural transformation $i : \Pi^2 \to P_\beta^2$.

In fact, one has only to check that for every uniformly continuous mapping $f : X \to Y$ the following diagram

$$
\begin{array}{ccc}
X \times X & \xrightarrow{f \times f} & Y \times Y \\
\downarrow i_X & & \downarrow i_Y \\
P_\beta(X) & \xrightarrow{P_\beta(f)} & P_\beta(Y)
\end{array}
$$

is commutative. But this is evident.

**Remark 3.12.** It is clear that the embedding $i_X : X \times X \to P_\beta(X)$ can be considered as a component of a natural transformation of functors in the category $\mathcal{T}_\text{Ych}$.

**Proposition 3.13.** Let $(X, \mathcal{U})$ be a uniform space. Then $P_\beta(p\mathcal{U}) = p(P_\beta(\mathcal{U}))$ iff $\mathcal{U} = p\mathcal{U}$.

**Proof.** Sufficiency follows from Proposition 3.8. Now let

$$P_\beta(p\mathcal{U}) = p(P_\beta(\mathcal{U})). \quad (3.2)$$

From Proposition 3.10 we have

$$\mathcal{U} \times \mathcal{U} \subset P_\beta(\mathcal{U}), \quad (3.3)$$

$$p\mathcal{U} \times p\mathcal{U} \subset P_\beta(p\mathcal{U}). \quad (3.4)$$

But (3.3) implies

$$p(\mathcal{U} \times \mathcal{U}) \subset p(P_\beta(\mathcal{U})). \quad (3.5)$$

Hence, (3.2) and (3.5) give us

$$p(\mathcal{U} \times \mathcal{U}) \subset P_\beta(p\mathcal{U}). \quad (3.6)$$

By (3.4) and (3.6), both $i(X \times X, p\mathcal{U} \times p\mathcal{U})$ and $i(X \times X, p(\mathcal{U} \times \mathcal{U}))$ are subspaces of the uniform space $(P_\beta(X), p(P_\beta(\mathcal{U})))$. Consequently, $p\mathcal{U} \times p\mathcal{U} = p(\mathcal{U} \times \mathcal{U})$. Then $\mathcal{U} = p\mathcal{U}$ by Proposition 1.20. The proposition is proved. $\square$
4. Main results

Proposition 4.1. The identity transformation is the only natural transformation of the functor \( P : \text{Comp} \to \text{Comp} \) into itself.

Proof. Let \( T : P \to P \) be a natural transformation and let 
\[ n = \{0, \ldots, n - 1\} \]
be an \( n \)-point set.

Claim 1. Let 
\[ \mu = \frac{1}{n} \sum_{k=0}^{n-1} \delta(k). \]
Then \( T_n(\mu) = \mu \).

Proof. In fact, assume
\[ T_n(\mu) = \nu = \sum_{k=0}^{n-1} a_k \delta(k) \neq \mu. \]
Let \( a = \min\{a_k : k \in \mathbb{N}\} \) and \( b = \max\{a_k : k \in \mathbb{N}\} \). Let \( a = a_{k_0}, b = a_{k_1} \). Then \( a < b \) and \( k_0 \neq k_1 \). Define a mapping \( \varphi : n \to n \) by:
\[ \varphi(k_0) = k_1, \quad \varphi(k_1) = k_0, \quad \varphi(k) = k \quad \text{for} \ k \notin \{k_0, k_1\}. \]

By the definition of natural transformation we have \( T_n \circ P(\varphi) = P(\varphi) \circ T_n \). But \( P(\varphi)(\mu) = \mu \). Hence, \( (T_n \circ P(\varphi))(\mu) = T_n(\mu) = \nu \). On the other hand, \( (P(\varphi) \circ T_n)(\mu) = P(\varphi)(\nu) \neq \nu, \) since \( P(\varphi)(\nu)(k_1) = a < b = \nu(k_1) \). We arrive at a contradiction and Claim 1 is proved. \( \square \)

Claim 2. Let \( m_0, \ldots, m_{n-1} \) be positive integers and \( N = \sum_{k=0}^{n-1} m_k \). Let
\[ \mu = \sum_{k=0}^{n-1} \frac{m_k}{N} \delta(k). \]
Then \( T_n(\mu) = \mu \).

Proof. Indeed, define a mapping \( \varphi : N \to n \) by
\[ \varphi^{-1}(k) = [m_0 + \cdots + m_{k-1} + 1, \ m_0 + \cdots + m_{k-1} + m_k]. \]
Let
\[ \nu = \frac{1}{N} \sum_{l=0}^{N-1} \delta(l). \]
Clearly, \( P(\varphi)(\nu) = \mu \). On the other hand, according to Claim 1 we have \( T_N(\nu) = \nu \).

Hence,

\[
T_n(\mu) = (T_n \circ P(\varphi))(\nu) = (P(\varphi) \circ T_N)(\nu) = P(\varphi)(\nu) = \mu.
\]

Claim 2 is proved. \( \square \)

From Claim 2 we get \( T_n = \text{id}_{P(n)} \) for an arbitrary \( n > 0 \). Now let \( X \) be an arbitrary Hausdorff compact space, \( \mu \in P(X) \) and \( |\text{supp}\mu| = n < \infty \). There is an embedding \( \varphi : n \to X \) such that \( \varphi(n) = \text{supp}\mu \). From the equality \( P(\varphi) \circ T_n = T_X \circ P(\varphi) \) we get \( P(\varphi) = T_X \circ P(\varphi) \), since \( T_n = \text{id}_{P(n)} \). But \( P(\varphi) : P(n) \to P(X) \) is an embedding with \( P(\varphi)(P(n)) = P(\text{supp}\mu) \). Hence, there is a unique \( \nu \in P(n) \) such that \( P(\varphi)(\nu) = \mu \).

Then \( T_X(\mu) = T_X(P(\varphi)(\nu)) = P(\varphi)(\nu) = \mu \). Consequently, \( T_X(\mu) = \mu \) for an arbitrary measure \( \mu \in P(X) \) with finite support. But these measures are everywhere dense in \( T(X) \). So, \( T_X = \text{id}_{P(X)} \) for arbitrary \( X \in \text{Comp} \). Proposition 4.1 is proved. \( \square \)

**Proposition 4.2.** There is a unique natural transformation \( T : S \circ P_\beta^a \to P \circ S \).

**Proof.** *Uniqueness.* The category \( c\text{Unif} \subset \text{Unif} \) is invariant with respect to both functors \( S \circ P_\beta^a \) and \( P \circ S \). Hence, \( T|c\text{Unif} \) is a natural transformation of the functor \( P : c\text{Unif} \to c\text{Unif} \) into itself. By Proposition 4.1, \( T|c\text{Unif} = \text{Id} \). Let \( (X, U) \) be a uniform space. Then \( P_\beta(X) \) is everywhere dense in both \( S_{P_\beta(U)}(P_\beta(X)) \) and \( P(S_U(X)) \). Since \( T|c\text{Unif} = \text{Id} \), we have \( T_U|P(K) = \text{Id} \) for an arbitrary compact subset \( K \subset X \). Consequently, \( T_U|P_\beta(X) = \text{Id} \). So \( T_U \) is unique being uniquely defined on a dense subset.

*Existence.* The identity mapping \( (X, U) \to (X, pU) \) is uniformly continuous. Hence, the identity embedding \( i_U : (X, U) \to S_U(X) \) is uniformly continuous. Then the mapping

\[
P_\beta(i_U) : (P_\beta(X), P_\beta(U)) \to P(S_U(X))
\]

is also uniformly continuous. Applying to this mapping the functor \( p \) of the precompactification we get in view of Proposition 1.18 a uniform continuity of the mapping

\[
P_\beta(i_U) : (P_\beta(X), p(P_\beta(U))) \to P(S_U(X)).
\]

Now we extend this mapping on the completions and get by Proposition 1.17 the mapping

\[
T_U : S_{P_\beta(U)}(P_\beta(X)) \to P(S_U(X)).
\]

It is easy to verify that \( T_U \) is a component of a natural transformation \( T : S \circ P_\beta^a \to P \circ S \).

The proposition is proved. \( \square \)

**Theorem 4.3.** \( T_U \) is a homeomorphism iff \( U \) is a precompact uniformity.

**Proof.** Let \( (X, U) \) be a precompact space. Then \( (X, U) \to (X, pU) \) is a uniform isomorphism. Hence, \( i_U : (X, U) \to S_U(X) \) is a uniform embedding. By Proposition 3.7, mapping (4.1) is a uniform embedding too. Therefore, mapping (4.2), being equal to mapping (4.1), is a uniform embedding as well. So, \( T_U \) is the identity mapping of \( P(S_U(X)) \).
Now let $T_U$ is a homeomorphism. Then mapping (4.2) is an embedding. Hence, the uniformity $p(P_3(U))$ on $P_3(X)$ is equal to the uniformity $P_3(pU)$ on $P_3(X) \subset P(S_U(X))$. By Proposition 3.13, $U = pU$. The theorem is proved. \hfill \Box

Theorem 4.3 and Corollary 1.14 yield

**Corollary 4.4.** Let $(X, \rho)$ be a metric space. Then $T_{u(\rho)}$ is a homeomorphism iff $\rho$ is totally bounded.

By an embedding $U : \text{Tych} \to \text{Unif}$ we mean a certain functor of a uniformization, i.e., for an arbitrary Tychonoff space $X$, $U(X)$ is a uniform space with the original topology of $X$, and for an arbitrary continuous mapping $f : X \to Y$ the mapping $f : U(X) \to U(Y)$ is uniformly continuous. There are many uniformizations $U : \text{Tych} \to \text{Unif}$. For example:

1. $U(X)$ is the universal uniform space, i.e., the biggest uniform space on $X$;
2. $U(X)$ is the Stone–Čech uniform space, i.e., the biggest precompact uniform space on $X$ (for this uniform space we have $\varpi(U(X)) = \beta X$).

But the problem of a uniformization can become unsolvable if one adds some restrictions. The next assertion is an example of this.

**Proposition 4.5.** There is no embedding $U : \text{Tych} \to \text{Unif}$ such that

$$\Pi^2 \circ U = U \circ \Pi^2.$$  \hfill (4.3)

**Proof.** Assume that there is such an embedding $U$. Let $N$ be a discrete space of non-negative integers.

**Claim 1.** $U(N)$ is a universal uniform space.

**Proof.** Indeed, according to (4.3) we have

$$U(N) \times U(N) = U(N \times N).$$  \hfill (4.4)

On the other hand, any uniformity of type $U(N) \times U(N)$ contains a disjoint covering consisting of two infinite sets $A$ and $B$, for example, $A = \{0\} \times N, B = N^+ \times N$. Let $f : N \times N \to N \times N$ be some bijections such that $f(A) = \Delta_N$. Then $f : U(N \times N) \to U(N \times N)$ is a uniform isomorphism. Hence,

$$u = \{f(A), f(B)\}$$

is a uniform covering of $U(N \times N)$. Consequently, $u$ is a uniform covering of $U(N) \times U(N)$, because of (4.4). There is a uniform covering $v$ of $U(N)$ such that $v \times v$ refines $u$. Since $u = \{\Delta_N, N \times N \setminus \Delta_N\}$, it follows that $v$ consists of one-point sets. Claim 1 is proved. \hfill \Box

Let $\mathbb{Q} \subset \mathbb{R}$ be the space of rationals.

**Claim 2.** $U(\mathbb{Q})$ is a universal space.
**Proof.** In fact, we have to check that an arbitrary open covering of $Q$ is uniform. For this it suffices to show that an arbitrary disjoint covering $u$ of $Q$ consisting of clopen sets is uniform. There is an epimorphism $f: Q \to N$ such that

$$u = \{f^{-1}(n) : n \in N\}.$$ 

Since $f$ is continuous, $f: U(Q) \to U(N)$ is uniformly continuous. Then $u$ is a uniform covering according to Claim 1. Claim 2 is proved. □

Now let $f: Q \to Q \times Q$ be a homeomorphism. Then $f: U(Q) \to U(Q \times Q)$ is a uniform isomorphism. But $U(Q \times Q) = U(Q) \times U(Q)$ in view of (4.3). So, Claim 2 implies that the uniform space $U(Q) \times U(Q)$ is universal. But that is not correct. We arrive at a contradiction. Proposition 4.5 is proved. □

Proposition 4.5, Theorem 3.11 and Remark 3.12 imply

**Theorem 4.6.** There is no embedding $U: Tych \to Unif$ such that

$$P^u_\beta \circ U = U \circ P_\beta.$$

Analysis of the proof of Theorem 4.6 shows us that the next statement is true.

**Theorem 4.7.** There is no embedding $U$ of the category $M$ of all metrizable spaces into the category $MUnif$ of metrizable uniform spaces such that

$$P^u_\beta \circ U = U \circ P_\beta.$$

**Remark 4.8.** Banakh [2,3] considered the functors $P_t$ and $P_\tau$ (of Radon and $\tau$-additive measures respectively) in the category $Tych$. For an arbitrary Tychonoff space $X$ we have

$$P_\beta(X) \subset P_t(X) \subset P_\tau(X) \subset P(\beta X).$$

He lifted the functors $P_t$ and $P_\tau$ to the functors $P^u_t$ and $P^u_\tau$ acting in the category $Unif$ in the same manner as it was done in Section 2 for the functor $P_\beta$. For an arbitrary uniform space $(X,U)$ the following inclusions

$$(P_\beta(X), P_\beta(U)) \subset (P_t(X), P_t(U)) \subset (P_\tau(X), P_\tau(U))$$

are uniform. Moreover, we have the functor inclusions

$$P^u_\beta \subset P^u_t \subset P^u_\tau.$$

So all main results for the functor $P^u_\beta$ (Theorems 4.3, 4.6 and 4.7, Proposition 4.4) hold for the functors $P^u_t$ and $P^u_\tau$ as well.

**References**