On some categorical properties of uniform spaces of probability measures

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Abstract

We deal with the functor $P_\beta: Unif \rightarrow Unif$ of uniform spaces of probability measures, defined by Sadovnichy (1994). We show that there is a unique natural transformation $T: S \circ P_\beta \rightarrow P \circ S$, where $S: Unif \rightarrow cUnif$ is the functor of Samuel compactification. In our first main result (Theorem 4.3) it is established that for a uniform space $(X, U)$ the component $T_U$ of this natural transformation $T$ is a homeomorphism iff $U$ is a precompact uniformity. The second main result (Theorem 4.6) shows that there is no embedding $I: Tykh \rightarrow Unif$ such that $P_\beta \circ I = I \circ P_\beta$.

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Introduction

For a Tychonoff space $X$ by $P_\beta(X)$ we denote the set of all probability measures on $X$ with compact supports, i.e.,

$$P_\beta(X) = \{\mu \in P(\beta X): \mu(K) = 1 \text{ for some compact set } K \subset X\}.$$ 

This set $P_\beta(X)$ is equipped with the $*$-weak topology. $P_\beta$ is a covariant functor acting in the category Tykh of Tychonoff spaces. In [11–14] Sadovnichy lifted this functor onto the categories $Met_{rb}$ of bounded metric spaces and $Unif$ of uniform spaces, and investigated

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some properties of these liftings. We answer two questions arising in connection with these investigations.

Let $P^u_\beta$ be the lifting of $P_\beta$ onto the category $\text{Unif}$. We show (Proposition 4.2) that there is a unique natural transformation $T : S \circ P^u_\beta \to P \circ S$, where $S : \text{Unif} \to c\text{Unif}$ is the functor of Samuel compactification, and $P : \text{Comp} \to \text{Comp}$ is the probability measures functor. In our first main result (Theorem 4.3) it is established that for a uniform space $(X, \mathcal{U})$ the component $T_\mathcal{U}$ of this natural transformation $T$ is a homeomorphism iff $\mathcal{U}$ is a precompact uniformity. The second main result (Theorem 4.6) shows that there is no embedding (uniformization functor) $U : \text{Tyche} \to \text{Unif}$ such that $P^u_\beta \circ U = U \circ P_\beta$.

In Section 1 we recall all necessary notions and facts about pseudometrics and uniformities. More detailed information can be found in [4,9,10]. In Section 2 we give basic information about probability measures spaces and (pseudo)metrics on them. One can find additional information about spaces and functors of probability measures in [6,7]. The main result of Section 2 is Theorem 2.4. In this theorem sufficient conditions on a family of pseudometrics generating $*\text{-weak topology on } P^u_\beta(X)$ are given. Theorem 2.4 allows us to get a simple proof of Theorem 3.1 stating that for an arbitrary uniform space $(X, \mathcal{U})$ the uniformity $P^u_\beta(\mathcal{U})$ generates the $*$-weak topology. The main result of Section 3 (Theorem 3.11) establishes that the functor of square $P^2$ is a subfunctor of $P^u_\beta$. This theorem plays a crucial role in Section 4 which contains the main results of the article.

1. Pseudometrics and uniformities

A pair $(X, \rho)$, where $X$ is a set and $\rho$ is a pseudometric on $X$, is said to be a pseudometric space. Every pseudometric space $(X, \rho)$ is equipped with topology $\tau_\rho$. An open base of this topology is formed by open $\varepsilon$-balls

$$O(x, \varepsilon) = \{ y \in X : \rho(x, y) < \varepsilon \}, \quad x \in X, \varepsilon > 0.$$ 

The topology $\tau_\rho$ is Hausdorff iff $\rho$ is a metric.

Let $(X, \tau)$ be a topological space and $\rho$ be a pseudometric on $X$. This pseudometric is called continuous if the mapping

$$\rho : (X \times X, \tau \times \tau) \to \mathbb{R}$$

is continuous. It is clear that $\rho$ is continuous iff the identity mapping $(X, \tau) \to (X, \tau_\rho)$ is continuous.

Let $(X, \rho)$ be a pseudometric space. We denote by $(\rho)$ a binary relation on $X$ which is defined in the following way:

$$x(\rho)y \iff \rho(x, y) = 0.$$ 

Evidently, $(\rho)$ is an equivalence relation. The quotient set $X/(\rho)$ we denote by $X_\rho$, the quotient mapping $X \to X_\rho$ we denote by $\pi_\rho$. Let $\xi, \eta \in X_\rho, x, x' \in \pi^{-1}_\rho(\xi), y, y' \in \pi^{-1}_\rho(\eta)$. Then it is easy to see that $\rho(x, y) = \rho(x', y')$. So we can define a
mapping $\tilde{\rho}: X_\rho \times X_\rho \to \mathbb{R}$ by $\tilde{\rho}(\xi, \eta) = \rho(x, y)$ for any $x \in \pi_\rho^{-1}(\xi)$ and $y \in \pi_\rho^{-1}(\eta)$. Clearly, $\tilde{\rho}$ is a metric on $X_\rho$. The next statement is well known and trivial in proof.

**Proposition 1.1.** Let $X$ be a topological space and let $\rho$ be a continuous pseudometric on $X$. Then the metric $\tilde{\rho}$ is continuous on $X_\rho$ with respect to the quotient topology.

By a **uniformity** on a set $X$ we mean a family $\mathcal{U}$ of symmetric entourages of the diagonal $\Delta_X \subset X \times X$ such that:

1. If $E_1, E_2 \in \mathcal{U}$, then $E_1 \cap E_2 \in \mathcal{U}$.
2. If $E \in \mathcal{U}$, then there is $E_1 \in \mathcal{U}$ such that $E_1 \circ E_1 \subset E$.
3. If $E \in \mathcal{U}$, $E \subset E_1$, and $E_1$ is symmetric, then $E_1 \in \mathcal{U}$.
4. $\bigcap \{E: E \in \mathcal{U}\} = \Delta_X$.

If $\mathcal{U}$ satisfies Conditions (1)–(4), then we say that $\mathcal{U}$ is a **preuniformity**. A family $\mathcal{B} \subset \mathcal{U}$ is said to be a base of a preuniformity $\mathcal{U}$, if for any $E \in \mathcal{U}$ there is $E_1 \in \mathcal{B}$ such that $E \subset E_1$. If $\mathcal{B}$ is a base of preuniformity $\mathcal{U}$, then

$$\mathcal{U} = \{E \subset X \times X: E = E^{-1} \text{ and } E_1 \subset E, E_1 \in \mathcal{B}\}.$$ 

It is clear that a family $\mathcal{B}$ of symmetric entourages of the diagonal $\Delta_X$ is a base of some preuniformity $\mathcal{U}$ on $X$ iff $\mathcal{B}$ satisfies condition (2) and

(1') If $E_1, E_2 \in \mathcal{B}$, then there is $E \in \mathcal{B}$ such that $E \subset E_1 \cap E_2$.

Let $(X, \rho)$ be a pseudometric space. For $\varepsilon > 0$, set

$$E(\rho, \varepsilon) = \{(x, y) \in X \times X: \rho(x, y) < \varepsilon\}.$$ 

Then the family

$$\mathcal{B}(\rho) = \{E(\rho, \varepsilon): \varepsilon > 0\}$$

is a base of a preuniformity that will be denoted by $u(\rho)$.

**Proposition 1.2.** $u(\rho)$ is a uniformity iff $\rho$ is a metric.

Let $\rho$ be a (pseudo)metric on $X$ and let $d > 0$. Set

$$\rho_d(x, y) = \min\{d, \rho(x, y)\}.$$ 

Evidently $\rho_d$ is a (pseudo)metric.

**Proposition 1.3.** Let $\rho$ be a pseudometric on $X$ and $d > 0$. Then $u(\rho) = u(\rho_d)$.

If $\mathcal{U}$ is a (pre)uniformity on $X$, then the pair $(X, \mathcal{U})$ is called a (pre)uniform space. Sometimes we shall denote a (pre)uniform space $(X, \mathcal{U})$ by $X$. By $\mathbb{R}$ we shall denote four different objects:

1. the set of all real numbers;
2. the metric space $(\mathbb{R}, \rho_E)$, where $\rho_E(x, y) = |x - y|$;
3. the topological space $(\mathbb{R}, \tau_{\rho_E})$;
Proposition 1.4. A pseudometric $\rho$ on a preuniform space $(X, \mathcal{U})$ is uniformly continuous iff the identity mapping $(X, \mathcal{U}) \to (X, u(\rho))$ is uniformly continuous.

Corollary 1.5. A pseudometric $\rho$ on a preuniform space $(X, \mathcal{U})$ is uniformly continuous iff $E(\rho, \varepsilon) \in \mathcal{U}$ for any $\varepsilon > 0$.

Proposition 1.6. Let $f : X \to Y$ be a uniformly continuous mapping between preuniform spaces. If $\rho$ is a uniformly continuous pseudometric on $Y$, then $\rho \circ (f \times f)$ is a uniformly continuous pseudometric on $X$.

Proposition 1.7. Let $(X, \mathcal{U})$ be a preuniform space and $E \in \mathcal{U}$. Then there is a bounded uniformly continuous pseudometric $\rho$ on $X$ such that $E(\rho, 1) \subset E$.

We shall say that a family $R$ of uniformly continuous pseudometrics on a preuniform space $(X, \mathcal{U})$ generates the preuniformity $\mathcal{U}$ if for each $E \in \mathcal{U}$ there exist $\rho \in R$ and $\varepsilon > 0$ such that $E(\rho, \varepsilon) \subset E$.

Proposition 1.8. Let $(X, \mathcal{U})$ be a preuniform space. Then the family $R(U)$ of all bounded uniformly continuous pseudometrics on $X$ generates the preuniformity $\mathcal{U}$.

Proposition 1.9. Let $R$ be a family of pseudometrics on a set $X$ satisfying the condition:

$(\text{UP1})$ If $\rho_1, \rho_2 \in R$, then there is $\rho \in R$ such that $\rho_1, \rho_2 \leq \rho$. Then there is a unique preuniformity $\mathcal{U} \equiv u(R)$ on $X$ such that $R$ generates $\mathcal{U}$.

Moreover, $u(R)$ is a uniformity iff $R$ satisfies the condition:

$(\text{UP2})$ For any $x, y \in X$, $x \neq y$, there is $\rho \in R$ such that $\rho(x, y) > 0$.

Let $(X, \mathcal{U})$ be a preuniform space and $E \in \mathcal{U}$. For an arbitrary $x \in X$ set

$$E(x) = \{y \in X : (x, y) \in E\}.$$ 

A preuniform space $(X, \mathcal{U})$ is called precompact if for any $E \in \mathcal{U}$ there is a finite set $\{x_1, \ldots, x_n\} \subset X$ such that

$$X = \bigcup \{E(x_i) : i = 1, \ldots, n\}.$$ 

An entourage $E$ from a preuniformity $\mathcal{U}$ is said to be precompact if there is a finite set $\{x_1, \ldots, x_n\} \subset X$ such that

$$\bigcup \{E(x_i) \times E(x_i) : i = 1, \ldots, n\} \in \mathcal{U}.$$
Proposition 1.10. A preuniform space \((X, U)\) is precompact iff each \(E \in U\) is precompact.

Proposition 1.11. Let \((X, U)\) be a precompact space and let \(Y \subset X\). Then \((Y, U|Y)\) is a precompact space.

Let \(U\) be a uniformity on \(X\) and \(pU = \{E \in U: E\text{ is precompact}\}\).

Proposition 1.12. For an arbitrary (pre)uniformity \(U\) the family \(pU\) is the biggest precompact (pre)uniformity which is contained in \(U\).

Proposition 1.13. A preuniform space \(X\) is precompact iff every uniformly continuous pseudometric on \(X\) is totally bounded.

Corollary 1.14. Let \((X, \rho)\) be a metric space. Then the uniformity \(u(\rho)\) is precompact iff \(\rho\) is totally bounded.

For a preuniform space \((X, U)\) by \(\tau(U)\) we denote a topology induced by \(U\). In this topology a set \(U \subset X\) is open iff for any \(x \in U\) there is \(E \in U\) such that \(E(x) \subset U\). A preuniform space \((X, U)\) is called compact if \((X, \tau(U))\) is a compact space.

Proposition 1.15. Let \((X, U)\) be a preuniform space. Then the following conditions are equivalent:

(a) \(U\) is a uniformity;
(b) \(\tau(U)\) is Hausdorff;
(c) \(\tau(U)\) is Tychonoff.

Proposition 1.16. A uniform space \(X\) is compact iff \(X\) is precompact and complete.

By \(\text{Unif}\) we denote the category of all uniform spaces and their uniformly continuous mappings. By \(\text{cUnif}, \text{pUnif}, \text{cplUnif}\) we denote full subcategories of \(\text{Unif}\) consisting respectively of all compact, precompact, complete uniform spaces. For a preuniform space \((X, U)\) by \(\tau(U)\) we shall denote its precompactification \((pX, pU)\) (completion \((\text{cpl} X, \text{cpl} U)\)). Let \(C\) be a category of uniform spaces and let \(D\) be its full subcategory. A covariant functor \(r: C \to D\) is said to be a reflection if \(r \circ r = r\), and there is a natural transformation \(T: \text{Id} \to r\) of the identity functor \(\text{Id}\) such that for any \(X \in C\) and uniformly continuous mapping \(f: X \to Y \in D\) there is a unique uniformly continuous mapping \(f_0: r(X) \to Y\) with \(f = f_0 \circ T_X\).

Proposition 1.17. The completion \(\text{cpl}: \text{Unif} \to \text{cplUnif}\) is a reflection. A component \(T_X\) of a natural transformation \(T: \text{Id} \to \text{cpl}\) is the identity embedding \(X \to \text{cpl} X\).

Proposition 1.18. The precompactification \(p: \text{Unif} \to \text{pUnif}\) is a reflection. A component \(T_X: X \to pX\) of a natural transformation \(T: \text{Id} \to p\) is the identity mapping.
A composition $cpl \circ p \equiv S$ is called Samuel compactification. The compactification $S(X, U)$ we shall denote by $S_\mathcal{U}X$ or $SX$.

**Proposition 1.19.** $S: \mathcal{U}nif \to cl\mathcal{U}nif$ is a reflection.

**Proposition 1.20** [9, II, Exercise 12]. Let $(X, \mathcal{U})$ be a uniform space. Then $p(\mathcal{U} \times \mathcal{U}) = p\mathcal{U} \times p\mathcal{U}$ iff $\mathcal{U}$ is precompact.

2. Pseudometrics on spaces of probability measures

Let $X$ be a compact Hausdorff space. By $C(X)$ we denote a Banach space of all real-valued continuous functions on $X$. The dual space $C(X)^*$ is equipped with the $*$-weak topology, i.e., the topology induced by the identity embedding $C(X)^* \subset \mathbb{R}^{C(X)}$. By Riesz' theorem the positive cone $C(X)^+_+$ is affinely isomorphic to the space $M(X)$ of all Borel finite positive regular measures on $X$. This space is also equipped with the $*$-weak topology. We shall identify measures $\mu \in M(X)$ with linear functionals from $C(X)^*$. So, sometimes, for $\varphi \in C(X)$ we shall write $\mu(\varphi)$ instead of $\int \varphi \, d\mu$. A measure $\mu \in M(X)$ is said to be a probability measure if $\mu(1_X) = 1$. The set of all probability measures on $X$ is denoted by $P(X)$. The space $P(X)$ is a convex compact subset of $\mathbb{R}^{C(X)}$. By the definition of $*$-weak topology its open base consists of sets

$$O(\mu, \varphi_1, \ldots, \varphi_k, \varepsilon) = \{ \mu' \in P(X) : |\mu(\varphi_i) - \mu'(\varphi_i)| < \varepsilon, \ i = 1, \ldots, k \}, \quad (2.1)$$

where $\mu \in P(X), \varphi_i \in C(X), \varepsilon > 0$.

If $f : X \to Y$ is a continuous mapping, then the formula

$$P(f)(\mu)(\varphi) = \mu(\varphi \circ f), \quad (2.2)$$

where $\mu \in P(X)$ and $\varphi \in C(Y)$, defines a continuous mapping $P(f) : P(X) \to P(Y)$. So, $P$ is a covariant functor acting in the category $\text{Comp}$ of compact Hausdorff spaces and their continuous mappings. It is clear that the mapping $P(f)$ can be defined in the following way:

$$P(f)(\mu)(B) = \mu(f^{-1}B), \quad (2.3)$$

where $B \subset Y$ is an arbitrary Borel set.

Let $X$ be a compact Hausdorff space and $\mu \in P(X)$. Set

$$\text{supp} \mu = \{ x \in X : \mu(Ox) > 0 \text{ for any arbitrary neighbourhood } Ox \}.$$ 

This set $\text{supp} \mu$ is called the support of $\mu$. The next statement is evident.

**Proposition 2.1.** Let $X$ be a compact Hausdorff space, $F \subset X$ and let $\mu \in P(X)$. Then $F = \text{supp} \mu$ iff $F$ is the smallest closed subset of $X$ such that $\mu(F) = 1$.

Now let $X$ be a Tychonoff space and let $\beta X$ be its Stone–Cech compactification. Set

$$P_\beta(X) = \{ \mu \in P(\beta X) : \text{supp} \mu \subset X \}. \quad (2.4)$$
Let $\gamma X$ be an arbitrary compactification of $X$ and let $\pi_\gamma : \beta X \to \gamma X$ be a natural projection.

**Proposition 2.2.** $P(\pi_\gamma)|P_\beta(X)$ is a homeomorphism.

In fact, $P(\pi_\gamma)|P_\beta(X)$ is evidently a one-to-one correspondence and $P_\beta(X) = P(\pi_\gamma)^{-1}P(\pi_\gamma)(P_\beta(X))$. Hence, topologically we can define $P_\beta(X)$ as:

$$P_\beta(X) = \{\mu \in P(\gamma X): \text{supp} \mu \subset X\}, \quad (2.5)$$

where $\gamma X$ is an arbitrary compactification of $X$.

Let $f : X \to Y$ be a continuous mapping between Tychonoff spaces and let $\beta f : \beta X \to \beta Y$ be its Stone–Čech compactification. We set

$$P_\beta(f) = P(\beta f)|P_\beta(X).$$

Clearly, $P_\beta(f)(P_\beta(X)) \subset P_\beta(Y)$. Thus, $P_\beta$ is a covariant functor acting in the category $Tych$ of Tychonoff spaces and their continuous mappings. Evidently, $P_\beta$ is an extension of the functor $P : Comp \to Comp$ to the category $Tych$.

For $x \in X$, by $\delta(x)$ we denote the Dirac measure, which is defined by

$$\delta(x)(\varphi) = \varphi(x) \quad \text{or} \quad \delta(x)\{x\} = 1.$$

It is easy to see that the Dirac embedding

$$\delta : X \to P_\beta(X)$$

is a topological embedding. Usually we shall identify the spaces $X$ and $\delta(X) \subset P_\beta(X)$.

Let $X$ be a Tychonoff space and let $\rho$ be a pseudometric on $X$. We define a distance function $P_\beta(\rho)$ on $P_\beta(X)$ by:

$$P_\beta(\rho)(\mu_1, \mu_2) = \inf\{\lambda(\rho): \lambda \in \Lambda(\mu_1, \mu_2)\}, \quad (2.6)$$

where

$$\Lambda(\mu_1, \mu_2) = \{\lambda \in P(X \times X): \text{pr}_i(\lambda) = \mu_i, \ i = 1, 2\},$$

$\text{pr}_i = P_\beta(p_i)$, and $p_i : X \times X \to X$ is the projection onto the $i$th factor.

**Proposition 2.3.** If $\rho$ is a bounded continuous pseudometric on a Tychonoff space $X$, then $P_\beta(\rho)$ is a continuous pseudometric on $P_\beta(X)$ such that $P_\beta(\rho)|X = \rho$ and $\text{diam} P_\beta(\rho) = \text{diam} \rho$.

Basically it was proved in [5] for a metric compact space $(X, \rho)$. For a general case look at [2,11].

We shall say that a family $R$ of pseudometrics on $X$ separates points and closed subsets if for each $x \in X$ and closed set $F \subset X$, $x \notin F$, there is a pseudometric $\rho \in R$ such that $\rho(x, F) > 0$, where

$$\rho(x, F) = \inf\{\rho(x, y): y \in F\}.$$
We shall say that a family $R$ of continuous pseudometrics on $X$ generates the topology of $X$ if for each $x \in X$ and each neighbourhood $O_x$ there are $\rho \in R$ and $\varepsilon > 0$ such that $O^\rho(x, \varepsilon) \subset O_x$.

**Theorem 2.4.** Let $R$ be a family of continuous bounded pseudometrics on $X$ which is directed, i.e., satisfies (UP1), and separates points and closed subsets. Then the family

$$P_\beta(R) = \{P_\beta(\rho): \rho \in R\}$$

generates the $*$-weak topology of $P_\beta(X)$.

To prove this theorem we need some auxiliary results.

**Proposition 2.5.** Let $X$ be a compact Hausdorff space, $C \subset C(X)$ be a family of functions, which separates points of $X$ and contains all finite products. Then

$$B_C = \{O(\mu, \varphi, \varepsilon): \varphi \in C, \varepsilon > 0\}$$

is a subbase of $P(X)$.

**Proof.** It follows from the definition of $*$-weak topology that the set

$$B_D = \{O(\mu, \psi, \varepsilon): \psi \in D, \varepsilon > 0\},$$

where $D$ is dense in $C(X)$, is a subbase of $P(X)$. So, it suffices to show that for some dense set $D \subset C(X)$ and an arbitrary neighbourhood $O(\mu, \psi, \varepsilon) \in B_D$ there is a smaller neighbourhood of $\mu$ which is an intersection of a finite family of neighborhoods from $B_C$. Let $D$ be the smallest subring of $C(X)$ containing $C$ and all constants. The set $D$ is dense in $C(X)$ by the Weierstrass–Stone theorem. Since $C$ contains all its finite products, each function $\psi \in D$ has a form

$$\psi = r_1 \varphi_1 + \cdots + r_k \varphi_k + r_{k+1},$$

where $\varphi_i \in C$, $r_i \in R$. Let

$$r = \max\{|r_i|: i = 1, \ldots, k\}, \quad \delta = \frac{\varepsilon}{kr}.$$

It remains to show that

$$\bigcap_{i=1}^k O(\mu, \varphi_i, \delta) \subset O(\mu, \psi, \varepsilon).$$

Let $\nu \in \bigcap_{i=1}^k O(\mu, \varphi_i, \delta) \equiv O(\mu, \varphi_1, \ldots, \varphi_k, \delta)$. Then

$$|\mu(\psi) - \nu(\psi)|$$

$$= \sum_{i=1}^k r_i (\mu(\varphi_i) - \nu(\varphi_i)) + \mu(r_{k+1}) - \nu(r_{k+1})$$

$$= \sum_{i=1}^k r_i (\mu(\varphi_i) - \nu(\varphi_i))$$

(since $\mu(s) = \nu(s) = s$ for any constant $s$)
Proposition 2.5 is proved. □

Now let $X$ be a Tychonoff space. We shall say that a family $\Phi$ of continuous functions $\varphi : X \rightarrow [0, 1]$ correctly separates points and closed subsets of $X$ if for any closed set $F \subseteq X$ and point $x \in X \setminus F$ there is a function $\varphi \in \Phi$ such that $\varphi(F) = 0$ and $\varphi(x) = 1$.

**Proposition 2.6.** Let $X$ be a Tychonoff space, $C_\beta(X)$ be a family of all bounded real-valued continuous functions of $X$, $\Phi \subseteq C_\beta(X)$ correctly separates points and closed subsets of $X$ and contains all its finite products. Then the family

$$\{O(\mu, \varphi_1, \ldots, \varphi_k, \varepsilon) : \mu \in P_\beta(X), \ \varphi_i \in \Phi, \ \varepsilon > 0\}$$

is a base of $P_\beta(X)$.

**Proof.** Evidently, the diagonal product $f : X \rightarrow I^\Phi$ of functions $\varphi \in \Phi$ is an embedding. Let us denote by $\gamma X$ the closure of $f(X)$ in $I^\Phi$. From definition of $f$ we have $\Phi \subseteq C_\gamma(X)$, where $C_\gamma(X) = C(\gamma X)|X$. Consequently, every function $\varphi \in \Phi$ can be extended to a function $\overline{\varphi} \in C_\gamma(X)$. Let $\overline{\gamma} : \gamma X \rightarrow I^\Phi$ be the diagonal product of functions $\varphi$, $\varphi \in \Phi$. Clearly, $\overline{\gamma}$ is the identity embedding. Hence, the family $\overline{\Phi} = \{\overline{\varphi} : \varphi \in \Phi\}$ separates points of $\gamma X$. Moreover, it contains all its finite products. Thus, according to Proposition 2.5 the sets $O(\mu, \overline{\varphi}_1, \ldots, \overline{\varphi}_k, \varepsilon)$, $\overline{\varphi}_i \in \overline{\Phi}$, form a base of $P(\gamma X)$. Then in view of (2.5) their traces $O(\mu, \varphi_1, \ldots, \varphi_k, \varepsilon)$, $\varphi_i \in \Phi$, on $P_\beta(X)$ form a base on $P_\beta(X)$. Proposition 2.6 is proved. □

For a family $R$ of continuous pseudometrics on a Tychonoff space $X$ we shall denote by $\Phi(R)$ the set of all functions $\varphi \in C_\beta(X)$ which are uniformly continuous with respect to some pseudometric $\rho \in R$.

**Proposition 2.7.** Let $R$ be a directed family of continuous pseudometrics on $X$ separating points and closed subsets of $X$. Then the family

$$\{O(\mu, \varphi, 1) : \mu \in P_\beta(X), \ \varphi \in \Phi(R)\}$$

form a subbase of $P_\beta(X)$.

**Proof.** First of all let us check that $\Phi(R)$ is a ring over $\mathbb{R}$. Let $\varphi_1, \varphi_2 \in \Phi(R)$ and let $\varphi_i$ be uniformly continuous with respect to $\rho_i \in R$. There is $\rho \in R$ such that $\rho \geq \max\{\rho_1, \rho_2\}$. Then $\varphi_1, \varphi_2$ are uniformly continuous with respect to $\rho$. Hence, $\varphi_1 + \varphi_2$ and $\varphi_1 \cdot \varphi_2$ are uniformly continuous. Consequently, $\Phi(R)$, containing all constants, is a ring over $\mathbb{R}$. 

\[
\begin{align*}
\left| \sum_{i=1}^{k} r_i \right| |\mu(\varphi_i) - \nu(\varphi_i)| & \\
\leq \sum_{i=1}^{k} r_i \left| \frac{\varepsilon}{k^r} \right| & \text{(in view of $\nu \in O(\mu, \varphi_i, \delta)$)} \\
\leq \varepsilon.
\end{align*}
\]
Further, $\mathcal{I}(R)$ correctly separates points and closed subsets of $X$. In fact, let $x_0 \in X \setminus F$. There is a pseudometric $\rho \in R$ such that $\rho(x_0, F) = \alpha > 0$. Let
\[
\varphi(x) = \min\left\{1, \frac{d(x, F)}{\alpha}\right\}.
\]
It is clear, that $\varphi \in \mathcal{I}(R)$ satisfies the condition of a correct separation of $x_0$ and $F$. Hence, by Proposition 2.6 the family
\[
\{O(\mu, \varphi, \varepsilon) : \mu \in \mathcal{P}_\beta(X), \varphi \in \mathcal{I}(R), \varepsilon > 0\}
\]
forms a subbase of $\mathcal{P}_\beta(X)$. But if $\varphi \in \mathcal{I}(R)$ and $r \in \mathbb{R}$, then $r \cdot \varphi \in \mathcal{I}(R)$. It yields that the families (2.8) and (2.7) coincide. Proposition 2.7 is proved. □

Remark 2.8. It is easy to see that if a family
\[
\{O(\mu, \varphi, \varepsilon) : \mu \in \mathcal{P}_\beta(X), \varphi \in \Phi, \varepsilon \in E\},
\]
is a subbase of $\mathcal{P}_\beta(X)$, then for an arbitrary $\mu_0 \in \mathcal{P}_\beta(X)$ the family
\[
\{O(\mu_0, \varphi, \varepsilon) : \varphi \in \Phi, \varepsilon \in E\},
\]
is a subbase of neighborhoods of $\mu_0$ in $\mathcal{P}_\beta(X)$.

Proof of Theorem 2.4. According to Propositions 2.3, 2.7 and Remark 2.8 it suffices to prove that every subbasic neighbourhood $O(\mu_0, \varphi, 1), \varphi \in \mathcal{I}(R)$, of a measure $\mu_0$ contains an $\varepsilon$-neighbourhood of this measure with respect to a pseudometric $\rho \in R$. Set $M = ||\varphi|| + 1$. Since $\varphi$ is uniformly continuous, there is $\delta, 0 < \delta < 1/(4M)$, such that $\rho(x_1, x_2) < \delta$ implies $|\varphi(x_1) - \varphi(x_2)| < 1/2$. We are going to show that
\[
O^{P_\beta}(\mu_0, \varphi, \varepsilon) \subset O(\mu_0, \varphi, 1)
\]
for $\varepsilon = \delta^2$. Let $\mu \in \mathcal{P}_\beta(X)$ and $P_\beta(\rho)(\mu_0, \mu) < \varepsilon$. There is $\lambda \in \mathcal{L}(\mu_0, \mu)$ such that $\lambda(\rho) < \varepsilon$. Set
\[
A = \{(x_1, x_2) \in X \times X : \rho(x_1, x_2) \geq \delta\}.
\]
Then $\varepsilon > \lambda(\rho) \geq \int_A \mu_0(x_1, x_2) \, d\lambda \geq \delta \lambda(A)$. Hence, $\lambda(A) < \varepsilon / \delta = \delta$. Consequently,
\[
|\mu_0(\varphi) - \mu(\varphi)| = \left| \int_{X \times X} \varphi(x_1) \, d\lambda - \int_{X \times X} \varphi(x_2) \, d\lambda \right|
\leq \int_{X \times X} |\varphi(x_1) - \varphi(x_2)| \, d\lambda
\leq \int_A |\varphi(x_1) - \varphi(x_2)| \, d\lambda + \int_{X \times X \setminus A} |\varphi(x_1) - \varphi(x_2)| \, d\lambda
\leq 2M \lambda(A) + \frac{1}{2} \lambda(X \times X \setminus A) < 2M \delta + \frac{1}{2} < 1.
\]
Theorem 2.4 is proved. □

Proposition 2.3 and Theorem 2.4 yield

Theorem 2.9. Let $(X, \rho)$ be a bounded metric space. Then $P_\beta(\rho)$ is a metric generating the $*$-weak topology of $P_\beta(X)$. 
This theorem was proved by Al-Kassas [1] for uniformly zero-dimensional $X$ and by Sadovnichy [11] for the general case.

Let $C$ be some category, whose objects are Tychonoff spaces with an additional structure (metric, group, uniform and so on) and let $F: C \to \text{Tych}$ be a “forgetful” functor. We say that a functor $G: \text{Tych} \to \text{Tych}$ is lifted onto the category $C$ if there is a functor $\tilde{G}: C \to C$ such that $F \circ \tilde{G} = G \circ F$. Theorem 2.9 implies

**Theorem 2.10.** The functor $P_\beta$ is lifted onto the category $\text{Metr}_b$ of all bounded metric spaces and their continuous mappings.

The just described lifting of $P_\beta$ onto the category $\text{Metr}_b$ we shall denote by $P_\beta^{\text{Metr}_b}$. Let us check some simple properties of this lifting.

**Proposition 2.11.** The functor $P_\beta^{\text{Metr}_b}$ preserves isometric embeddings.

Proof is trivial.

**Lemma 2.12.** Let $f: X \to Y$ be a continuous mapping between Tychonoff spaces, and $\rho_1$ and $\rho_2$ be continuous bounded pseudometrics on $X$ and $Y$ respectively. Let $\mu, \nu \in P_\beta(X)$ and

$$P_\beta(\rho_1)(\mu, \nu) = \int_{X \times X} \rho_1(x_1, x_2) \, d\lambda,$$

where $\lambda \in \Lambda(\mu, \nu)$. Then

$$P_\beta(\rho_2)\left(P_\beta(f)(\mu), P_\beta(f)(\nu)\right) \leq \int_{X \times X} \rho_2(f(x_1), f(x_2)) \, d\lambda.$$

This lemma for metric compact spaces was proved in [5]. For the general case the proof is the same.

**Corollary 2.13.** The functor $P_\beta^{\text{Metr}_b}$ preserves nonexpansive mappings.

The next statement for metric compact spaces was also proved in [5]. We repeat the proof in view of the high importance of this statement.

**Lemma 2.14.** Let $f: X \to Y$ be a continuous mapping between Tychonoff spaces, and let $\rho_1$ and $\rho_2$ be continuous pseudometrics of diameter $\leq a$ on $X$ and $Y$ correspondingly. If $f: (X, \rho_1) \to (Y, \rho_2)$ is an $(\varepsilon, \delta)$-uniformly continuous mapping of pseudometric spaces, then $P_\beta(f)$ is $(2\varepsilon, \varepsilon \delta / a)$-uniformly continuous.

**Proof.** Let $P_\beta(\rho_1)(\mu, \nu) < \varepsilon \delta / a$. Since $\Lambda(\mu, \nu) \subset P(\text{supp} \mu \times \text{supp} \nu)$ is compact, there is $\lambda \in \Lambda(\mu, \nu)$ such that

$$P_\beta(\rho_1)(\mu, \nu) = \int_{X \times X} \rho_1(x_1, x_2) \, d\lambda.$$
Then
\[ P_\beta(\rho_2)(P_\beta(f)(\mu), P_\beta(f)(\nu)) \]
\[ \leq \int_{X \times X} \rho_2(f(x_1), f(x_2)) \, d\lambda \quad \text{by Lemma 2.12} \]
\[ = \int_{\rho_1(x_1, x_2) < \delta} \rho_2(f(x_1), f(x_2)) \, d\lambda + \int_{\rho_1(x_1, x_2) \geq \delta} \rho_2(f(x_1), f(x_2)) \, d\lambda \]
\[ \leq \varepsilon + \frac{a}{\delta} \int_{\rho_1(x_1, x_2) > \delta} \rho_1(x_1, x_2) \, d\lambda \quad \text{(since } f \text{ is } (\varepsilon, \delta)\text{-uniformly continuous)} \]
\[ \leq \varepsilon + \frac{a}{\delta} P_\beta(\rho_1)(\mu, \nu) < \varepsilon + \frac{a}{\delta} \cdot \frac{\varepsilon \delta}{a} = 2\varepsilon. \]

Lemma 2.14 is proved. \( \square \)

**Corollary 2.15.** The functor \( P_\beta^{\text{Mtr}} \) preserves uniformly continuous mappings.

Let us denote by \( \text{Metr}_{\text{bu}} \) the subcategory of \( \text{Metr}_b \) consisting of all bounded metric spaces and all their uniformly continuous mappings. Theorem 2.10 and Corollary 2.15 imply

**Theorem 2.16.** The functor \( P_\beta \) is lifted onto the category \( \text{Metr}_{\text{bu}} \).

We shall denote this lifted functor by \( P_\beta^{\text{Mtr}_{\text{bu}}} \).

3. **Probability measures on uniform spaces**

Let \((X, U)\) be a uniform space. Let \( R(U) \) be a family of all bounded uniformly continuous pseudometrics on \((X, U)\). Then, evidently, the family \( P_\beta(R(U)) \) satisfies condition (UP1). Hence, the preuniformity \( u(P_\beta(R(U))) \) (look at Proposition 1.9) on \( P_\beta(X) \) induces on \( X \) the preuniformity \( U \) in view of Propositions 1.8 and 2.3. We shall denote this preuniformity by \( P_\beta(U) \).

**Theorem 3.1** [13]. Let \((X, U)\) be a uniform space. Then \((P_\beta(X), P_\beta(U))\) is a uniform space with \( \ast \)-weak topology.

**Proof.** In view of Proposition 1.9 it suffices to verify that the preuniformity \( P_\beta(U) \) generates the \( \ast \)-weak topology. We shall deduce it from Theorem 2.4. For this we have to check that the family \( R(U) \) separates points and closed subsets of \( X \). Let \( x \in X \setminus F \), where \( F \) is closed in \( X \). By definition of a uniform topology, there is \( E \in U \) such that \( E(x) \subset X \setminus F \). According to Proposition 1.8 there are \( \rho \in R(U) \) and \( \varepsilon > 0 \) such that \( E(\rho, \varepsilon) \subset E \). Then \( \rho(x, F) \geq \varepsilon > 0 \). Theorem 3.1 is proved. \( \square \)

**Proposition 3.2.** If a family \( R \) of bounded uniformly continuous pseudometrics on a uniform space \((X, U)\) generates the uniformity \( U \), then the family \( P_\beta(R) \) generates the uniformity \( P_\beta(U) \).
Proof. Let $E$ be an arbitrary entourage from the uniformity $P_\beta(U)$. By definition of this uniformity there are a pseudometric $\rho \in R(U)$ and $\varepsilon > 0$ such that $P_\beta(\rho)^{-1}[0, \varepsilon) \subseteq E(P_\beta(\rho), \varepsilon)) \subset E$. Since $R$ generates $U$, there are $\rho_1 \in R$ and $\delta > 0$ such that $\rho_1^{-1}[0, \delta) \subseteq \rho^{-1}[0, \varepsilon/2)$. Hence, the identity mapping $(X, \rho_1) \to (X, \rho)$ is $(\varepsilon/2, \delta)$-uniformly continuous. Let $a = \max\{\text{diam } \rho_1, \text{diam } \rho\}$. Consequently, by Lemma 2.14, the identity mapping

$$\left( P_\beta(X), P_\beta(\rho_1) \right) \to \left( P_\beta(X), P_\beta(\rho) \right)$$

is $(\varepsilon, (\varepsilon \delta)/(2a))$-uniformly continuous. Therefore,

$$P_\beta(\rho_1)^{-1}\left[0, \frac{\varepsilon \delta}{2a}\right) \subseteq P_\beta(\rho)^{-1}[0, \varepsilon) \subset E.$$

Proposition 3.2 is proved. □

Corollary 3.3. If $(X, \rho)$ is a bounded metric space, then the uniformities $P_\beta(u(\rho))$ and $u(P_\beta(\rho))$ on $P_\beta(X)$ coincide.

The next statement is a corollary of both Proposition 1.6 and Lemma 2.14.

Proposition 3.4 [13]. If $f : (X, U) \to (Y, V)$ is a uniformly continuous mapping between uniform spaces, then the mapping

$$P_\beta(f) : (P_\beta(X), P_\beta(U)) \to (P_\beta(Y), P_\beta(V))$$

is also uniformly continuous.

Theorem 3.1 and Proposition 3.4 imply

Theorem 3.5 [13]. The functor $P_\beta : \text{Tych} \to \text{Tych}$ is lifted onto the category $\text{Unif}$.

We denote this lifted functor by $P_\beta$. Let $\mathcal{M}\text{Unif} \subset \text{Unif}$ be the category of all metrizable uniform spaces and their uniformly continuous mappings. We have

$$P_\beta(\mathcal{M}\text{Unif}) \subset \mathcal{M}\text{Unif}$$

according to Corollary 3.3 and Lemma 2.14. We shall denote the restriction $P_\beta|\mathcal{M}\text{Unif}$ by $P_\beta^{\text{mu}}$. Let $\mathcal{F}_u : \text{Metr}_{\text{bu}} \to \text{Metr}_{\text{bu}}$ be the uniformization functor.

Proposition 3.6. The functor $P_\beta^{\text{mu}} : \mathcal{M}\text{Unif} \to \mathcal{M}\text{Unif}$ is lifted to the functor

$$P_\beta^{\mathcal{M}\text{bu}} : \text{Metr}_{\text{bu}} \to \text{Metr}_{\text{bu}}.$$

Proof. It is sufficient to check that

$$\mathcal{F}_u \circ P_\beta^{\mathcal{M}\text{bu}} = P_\beta^{\text{mu}} \circ \mathcal{F}_u.$$

(3.1)

For spaces $(X, \rho) \in \text{Metr}_{\text{bu}}$, Eq. (3.1) follows from Corollary 3.3. For mappings $f \in \text{Metr}_{\text{bu}}$, Eq. (3.1) follows from Lemma 2.14 and Proposition 3.2. □
Proposition 3.7 [3]. If \( f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V}) \) is a uniform embedding, then
\[
P^\beta(f): (P^\beta(X), P^\beta(\mathcal{U})) \rightarrow (P^\beta(Y), P^\beta(\mathcal{V}))
\]
is also a uniform embedding.

Proof. We may assume that \( f: X \rightarrow Y \) is the identity embedding. By (2.5) and Proposition 3.4, \( P^\beta(f) \) is a topological embedding and a uniformly continuous mapping. Let \( E \subseteq P^\beta(\mathcal{U}) \) be an arbitrary entourage. By definition of \( P^\beta(\mathcal{U}) \) there is a pseudometric \( \rho \in \mathcal{R}(\mathcal{U}) \) such that \( \rho^{-1}[0, \varepsilon) \subseteq E \) for some \( \varepsilon > 0 \). This pseudometric \( \rho: X \times X \rightarrow \mathbb{R} \) can be extended to a bounded uniformly continuous pseudometric \( \rho_1 \) on \((Y, \mathcal{V})\) (see [8, 8.5.6]). Let \( E_1 = \rho_1^{-1}[0, \varepsilon) \). Then \( E_1 \in P^\beta(\mathcal{V}) \) by definition and evidently \( E_1 \cap (X \times X) = \rho^{-1}[0, \varepsilon) \subseteq E \). Proposition 3.7 is proved. \( \square \)

Proposition 3.8 [3]. A uniform space \((X, \mathcal{U})\) is precompact iff \( (P^\beta(X), P^\beta(\mathcal{U})) \) is precompact.

Proof. If \( (P^\beta(X), P^\beta(\mathcal{U})) \) is precompact, then \((X, \mathcal{U})\) is precompact in view of Proposition 1.11 and a uniform embedding
\[
\delta: (X, \mathcal{U}) \rightarrow (P^\beta(X), P^\beta(\mathcal{U})).
\]
Conversely, if \((X, \mathcal{U})\) is precompact, then \((X, \mathcal{U}) \rightarrow S_\mathcal{U}X\) is a uniform embedding into a compact space. Then, by Proposition 3.7, \((P^\beta(X), P^\beta(\mathcal{U}))\) is a subspace of the compact uniform space \( P(S_\mathcal{U}X) \). Applying Proposition 1.11 once more, we get a precompactness of \((P^\beta(X), P^\beta(\mathcal{U}))\). The proposition is proved. \( \square \)

Proposition 3.8 and Corollaries 3.3 and 1.14 imply

Proposition 3.9. A metric space \((X, \rho)\) is totally bounded iff \((P^\beta(X), P^\beta(\rho))\) is totally bounded.

Proposition 3.10. Let \((X, \mathcal{U})\) be a uniform space and let
\[
i \equiv i_X: X \times X \rightarrow P^\beta(X)
\]
be a mapping which is defined as follows:
\[
i(x_1, x_2) = \frac{9}{10} \delta(x_1) + \frac{1}{10} \delta(x_2).
\]
Then \( i: (X \times X, \mathcal{U} \times \mathcal{U}) \rightarrow (P^\beta(X), P^\beta(\mathcal{U})) \) is a uniform embedding.

Proof. It suffices to show that for every bounded uniformly continuous pseudometric \( \rho \) on \( X \) the mappings \( i \) and \( i^{-1} \) are uniformly continuous with respect to the pseudometric \( \rho \times \rho \) and \( P^\beta(\rho) \), where
\[
\rho \times \rho((x_1, x_2), (y_1, y_2)) = \rho(x_1, y_1) + \rho(x_2, y_2).
\]
We shall prove more: for any \( \xi, \eta \in X \times X \)
\[
\frac{1}{30} \rho \times \rho(\xi, \eta) \leq P^\beta(\rho)(i(\xi), i(\eta)) \leq \rho \times \rho(\xi, \eta).
\]
Let $\xi = (x_1, x_2)$, $\eta = (y_1, y_2)$, $\rho(x_1, y_1) = \rho_1$, $\rho(x_2, y_2) = \rho_2$. We are going to prove that
\[
\frac{1}{10}(\rho_1 + \rho_2) \leq d \leq \rho_1 + \rho_2,
\]
where $d = P_\beta(i(\xi), i(\eta))$. Let $\lambda \in \Lambda(i(\xi), i(\eta))$. Then
\[
\lambda = m_{11}\delta(x_1, y_1) + m_{22}\delta(x_2, y_2) + m_{12}\delta(x_1, y_2) + m_{21}\delta(x_2, y_1),
\]
where $m_{ij} \geq 0$ and
\[
m_{11} + m_{12} = m_{11} + m_{21} = \frac{9}{10},
m_{22} + m_{12} = m_{22} + m_{21} = \frac{1}{10}.
\]
Let $m_{11} = a$, then $m_{12} = m_{21} = \frac{9}{10} - a$ and $m_{22} = a - \frac{8}{10}$. We have
\[
d \leq \lambda(\rho) = a\rho_1 + (a - \frac{8}{10})\rho_2 + \left(\frac{9}{10} - a\right)(\rho_{12} + \rho_{21})
\]
for an arbitrary $a$ (let us note that $\frac{8}{10} \leq a \leq \frac{9}{10}$). In particular, for $a = \frac{9}{10}$,
\[
d \leq \frac{9}{10}\rho_1 + \frac{1}{10}\rho_2 \leq \rho_1 + \rho_2.
\]
Since $\Lambda(i(\xi), i(\eta))$ is compact, there is a such that
\[
d = a\rho_1 + (a - \frac{8}{10})\rho_2 + \left(\frac{9}{10} - a\right)(\rho_{12} + \rho_{21}).
\]
There are two possibilities: $a \geq 0.85$ and $a \leq 0.85$. Let $a \geq 0.85$. Then
\[
d \geq a\rho_1 + (a - 0.8)\rho_2 \geq 0.85\rho_1 + 0.05\rho_2 \geq 0.05(\rho_1 + \rho_2).
\]
Now let $a \leq 0.85$. We have
\[
\rho_2 = \rho(x_2, y_2) \leq \rho(x_2, y_1) + \rho(y_1, x_1) + \rho(x_1, y_2).
\]
Hence,
\[
\rho_{12} + \rho_{21} \geq \rho_2 - \rho_1.
\]
Then
\[
d \geq a\rho_1 + (a - \frac{8}{10})\rho_2 + (0.9 - a)(\rho_2 - \rho_1) = (2a - 0.9)\rho_1 + 0.1\rho_2
\geq 0.7\rho_1 + 0.1\rho_2 \geq 0.1(\rho_1 + \rho_2).
\]
Proposition 3.10 is proved. \(\square\)

Let $C = \{O, M\}$ be some category, where $O$ is a family of its objects and $M$ is a family of its morphisms. Let $F, G : C \to C$ be covariant functors. Let us recall that a family $T = \{T_X : F(X) \to G(X) : X \in O\}$ of morphisms from $M$ is said to be a natural transformation of the functor $F$ in to the functor $G$ if for any morphism $f : X \to Y$ from $M$ the following diagram

\[
\begin{array}{ccc}
F(X) & \overset{F(f)}{\longrightarrow} & F(Y) \\
\downarrow\scriptstyle T_X & & \downarrow\scriptstyle T_Y \\
G(X) & \overset{G(f)}{\longrightarrow} & G(Y)
\end{array}
\]
is commutative. Morphisms $T_X$ are called *components* of a natural transformation $T$. A functor $F$ is called a *subfunctor* of a functor $G$ if there is a natural transformation $T : F \to G$ such that each component $T_X$ is a monomorphism.

By $\Pi^2$ we denote the functor of square: $\Pi^2(X) = X \times X$; if $f : X \to Y$ is a mapping, then $\Pi^2(f) = f \times f : X \times X \to Y \times Y$. The functor $\Pi^2$ acts in such categories as $\mathcal{T}_{ych}$, $\mathcal{Comp}$, $\mathcal{Unif}$, $\mathcal{cUnif}$ and so on.

**Theorem 3.11.** The embedding $i_X : X \times X \to P_\beta(X)$ from Proposition 3.10 can be extended to a natural transformation $i : \Pi^2 \to P_\beta^\#$.

In fact, one has only to check that for every uniformly continuous mapping $f : X \to Y$ the following diagram

\[
\begin{array}{ccc}
X \times X & \xrightarrow{f \times f} & Y \times Y \\
\downarrow i_X & & \downarrow i_Y \\
P_\beta(X) & \xrightarrow{P_\beta(f)} & P_\beta(Y)
\end{array}
\]

is commutative. But this is evident.

**Remark 3.12.** It is clear that the embedding $i_X : X \times X \to P_\beta(X)$ can be considered as a component of a natural transformation of functors in the category $\mathcal{T}_{ych}$.

**Proposition 3.13.** Let $(X, \mathcal{U})$ be a uniform space. Then $P_\beta(p\mathcal{U}) = p(P_\beta(\mathcal{U}))$ iff $\mathcal{U} = p\mathcal{U}$.

**Proof.** Sufficiency follows from Proposition 3.8. Now let

$$P_\beta(p\mathcal{U}) = p(P_\beta(\mathcal{U})).$$  \hfill (3.2)

From Proposition 3.10 we have

$$\mathcal{U} \times \mathcal{U} \subset P_\beta(\mathcal{U}),$$  \hfill (3.3)

$$p\mathcal{U} \times p\mathcal{U} \subset P_\beta(p\mathcal{U}).$$  \hfill (3.4)

But (3.3) implies

$$p(\mathcal{U} \times \mathcal{U}) \subset p(P_\beta(\mathcal{U})).$$  \hfill (3.5)

Hence, (3.2) and (3.5) give us

$$p(\mathcal{U} \times \mathcal{U}) \subset P_\beta(p\mathcal{U}).$$  \hfill (3.6)

By (3.4) and (3.6), both $i(X \times X, p\mathcal{U} \times p\mathcal{U})$ and $i(X \times X, p(\mathcal{U} \times \mathcal{U}))$ are subspaces of the uniform space $(P_\beta(X), p(P_\beta(\mathcal{U})))$. Consequently, $p\mathcal{U} \times p\mathcal{U} = p(\mathcal{U} \times \mathcal{U})$. Then $\mathcal{U} = p\mathcal{U}$ by Proposition 1.20. The proposition is proved. \hfill $\square$
4. Main results

Proposition 4.1. The identity transformation is the only natural transformation of the functor $P: \text{Comp} \rightarrow \text{Comp}$ into itself.

Proof. Let $T: P \rightarrow P$ be a natural transformation and let

$$n = \{0, \ldots, n - 1\}$$

be an $n$-point set.

Claim 1. Let

$$\mu = \frac{1}{n} \sum_{k=0}^{n-1} \delta(k).$$

Then $T_n(\mu) = \mu$.

Proof. In fact, assume

$$T_n(\mu) = \nu = \sum_{k=0}^{n-1} a_k \delta(k) \neq \mu.$$

Let $a = \min\{a_k: k \leq n\}$ and $b = \max\{a_k: k \leq n\}$. Let $a = a_{k_0}$, $b = a_{k_1}$. Then $a < b$ and $k_0 \neq k_1$. Define a mapping $\varphi: n \rightarrow n$ by:

$$\varphi(k_0) = k_1, \quad \varphi(k_1) = k_0, \quad \varphi(k) = k \text{ for } k \notin \{k_0, k_1\}.$$

By the definition of natural transformation we have $T_n \circ P(\varphi) = P(\varphi) \circ T_n$. But $P(\varphi)(\mu) = \mu$. Hence, $(T_n \circ P(\varphi))(\mu) = T_n(\mu) = \nu$. On the other hand, $(P(\varphi) \circ T_n)(\mu) = P(\varphi)(\nu) \neq \nu$, since $P(\varphi)(\nu)(k_1) = a < b = \nu(k_1)$. We arrive at a contradiction and Claim 1 is proved. □

Claim 2. Let $m_0, \ldots, m_{n-1}$ be positive integers and $N = \sum_{k=0}^{n-1} m_k$. Let

$$\mu = \sum_{k=0}^{n-1} \frac{m_k}{N} \delta(k).$$

Then $T_n(\mu) = \mu$.

Proof. Indeed, define a mapping $\varphi: N \rightarrow n$ by

$$\varphi^{-1}(k) = [m_0 + \cdots + m_{k-1} + 1, m_0 + \cdots + m_{k-1} + m_k].$$

Let

$$\nu = \frac{1}{N} \sum_{l=0}^{N-1} \delta(l).$$
Clearly, \( P(\varphi)(\nu) = \mu \). On the other hand, according to Claim 1 we have \( T_N(\nu) = \nu \). Hence,

\[
T_n(\mu) = (T_n \circ P(\varphi))(\nu) = (P(\varphi) \circ T_N)(\nu) = P(\varphi)(\nu) = \mu.
\]

Claim 2 is proved. \( \square \)

From Claim 2 we get \( T_n = \text{id}_{P(n)} \) for an arbitrary \( n > 0 \). Now let \( X \) be an arbitrary Hausdorff compact space, \( \mu \in P(X) \) and \( |\text{supp}\mu| = n < \infty \). There is an embedding \( \varphi: n \to X \) such that \( \varphi(n) = \text{supp}\mu \). From the equality \( P(\varphi) \circ T_n = T_X \circ P(\varphi) \) we get \( P(\varphi) = T_X \circ P(\varphi) \), since \( T_n = \text{id}_{P(n)} \). But \( P(\varphi): P(n) \to P(X) \) is an embedding with \( P(\varphi)(P(n)) = P(\text{supp}\mu) \). Hence, there is a unique \( \nu \in P(n) \) such that \( P(\varphi)(\nu) = \mu \).

Then \( T_X(\mu) = T_X(P(\varphi)(\nu)) = P(\varphi)(\nu) = \mu \). Consequently, \( T_X(\mu) = \mu \) for an arbitrary measure \( \mu \in P(X) \) with finite support. But these measures are everywhere dense in \( T(X) \). So, \( T_X = \text{id}_{P(X)} \) for arbitrary \( X \in \text{Comp} \). Proposition 4.1 is proved. \( \square \)

**Proposition 4.2.** There is a unique natural transformation \( T: S \circ P_\beta^a \to P \circ S \).

**Proof.** *Uniqueness.* The category \( c\text{Unif} \subset \text{Unif} \) is invariant with respect to both functors \( S \circ P_\beta^a \) and \( P \circ S \). Hence, \( T|c\text{Unif} \) is a natural transformation of the functor \( P: c\text{Unif} \to c\text{Unif} \) into itself. By Proposition 4.1, \( T|c\text{Unif} = \text{Id} \). Let \( (X, \mathcal{U}) \) be a uniform space. Then \( P_\beta(X) \) is everywhere dense in both \( S_{P_\beta(\mathcal{U})}(P_\beta(X)) \) and \( P(S_{\mathcal{U}}(X)) \). Since \( T|c\text{Unif} = \text{Id} \), we have \( T_{\mathcal{U}}|P(K) = \text{id} \) for an arbitrary compact subset \( K \subset X \). Consequently, \( T_{\mathcal{U}}|P_\beta(X) = \text{id} \). So \( T_{\mathcal{U}} \) is unique being uniquely defined on a dense subset.

*Existence.* The identity mapping \( (X, \mathcal{U}) \to (X, p\mathcal{U}) \) is uniformly continuous. Hence, the identity embedding \( i_{\mathcal{U}}: (X, \mathcal{U}) \to S_{\mathcal{U}}(X) \) is uniformly continuous. Then the mapping

\[
P_{\beta}(i_{\mathcal{U}}): (P_\beta(X), P_\beta(\mathcal{U})) \to P(S_{\mathcal{U}}(X))
\]

is also uniformly continuous. Applying to this mapping the functor \( p \) of the precompactification we get in view of Proposition 1.18 a uniform continuity of the mapping

\[
P_{\beta}(i_{\mathcal{U}}): (P_\beta(X), p(P_\beta(\mathcal{U}))) \to P(S_{\mathcal{U}}(X)).
\]

Now we extend this mapping on the completions and get by Proposition 1.17 the mapping

\[
T_{\mathcal{U}}: S_{P_\beta(\mathcal{U})}(P_\beta(X)) \to P(S_{\mathcal{U}}(X)).
\]

It is easy to verify that \( T_{\mathcal{U}} \) is a component of a natural transformation \( T: S \circ P_\beta^a \to P \circ S \). The proposition is proved. \( \square \)

**Theorem 4.3.** \( T_{\mathcal{U}} \) is a homeomorphism iff \( \mathcal{U} \) is a precompact uniformity.

**Proof.** Let \( (X, \mathcal{U}) \) be a precompact space. Then \( (X, \mathcal{U}) \to (X, p\mathcal{U}) \) is a uniform isomorphism. Hence, \( i_{\mathcal{U}}: (X, \mathcal{U}) \to S_{\mathcal{U}}(X) \) is a uniform embedding. By Proposition 3.7, mapping (4.1) is a uniform embedding too. Therefore, mapping (4.2), being equal to mapping (4.1), is a uniform embedding as well. So, \( T_{\mathcal{U}} \) is the identity mapping of \( P(S_{\mathcal{U}}(X)) \).
Now let $T_{\mathcal{U}}$ is a homeomorphism. Then mapping (4.2) is an embedding. Hence, the uniformity $p(P_{3}(\mathcal{U}))$ on $P_{3}(X)$ is equal to the uniformity $P_{3}(p\mathcal{U})$ on $P_{3}(X) \subset P(S_{\mathcal{U}}(X))$. By Proposition 3.13, $\mathcal{U} = p\mathcal{U}$. The theorem is proved. \hfill \square

Theorem 4.3 and Corollary 1.14 yield

**Corollary 4.4.** Let $(X, \rho)$ be a metric space. Then $T_{\psi(\rho)}$ is a homeomorphism iff $\rho$ is totally bounded.

By an embedding $U : \text{Tych} \rightarrow \text{Unif}$ we mean a certain functor of a uniformization, i.e., for an arbitrary Tychonoff space $X$, $U(X)$ is a uniform space with the original topology of $X$, and for an arbitrary continuous mapping $f : X \rightarrow Y$ the mapping $f : U(X) \rightarrow U(Y)$ is uniformly continuous. There are many uniformizations $U : \text{Tych} \rightarrow \text{Unif}$. For example:

1. $U(X)$ is the universal uniform space, i.e., the biggest uniform space on $X$;
2. $U(X)$ is the Stone–Čech uniform space, i.e., the biggest precompact uniform space on $X$ (for this uniform space we have $@_{\mathcal{U}}(U(X)) = \beta X$).

But the problem of a uniformization can become unsolvable if one adds some restrictions. The next assertion is an example of this.

**Proposition 4.5.** There is no embedding $U : \text{Tych} \rightarrow \text{Unif}$ such that

$$II^{2} \circ U = U \circ II^{2}. \quad (4.3)$$

**Proof.** Assume that there is such an embedding $U$. Let $N$ be a discrete space of non-negative integers.

**Claim 1.** $U(N)$ is a universal uniform space.

**Proof.** Indeed, according to (4.3) we have

$$U(N) \times U(N) = U(N \times N). \quad (4.4)$$

On the other hand, any uniformity of type $U(N) \times U(N)$ contains a disjoint covering consisting of two infinite sets $A$ and $B$, for example, $A = \{0\} \times N$, $B = N^{+} \times N$. Let $f : N \times N \rightarrow N \times N$ be some bijections such that $f(A) = \Delta_N$. Then $f : U(N \times N) \rightarrow U(N \times N)$ is a uniform isomorphism. Hence,

$$u = \{f(A), f(B)\}$$

is a uniform covering of $U(N \times N)$. Consequently, $u$ is a uniform covering of $U(N) \times U(N)$, because of (4.4). There is a uniform covering $v$ of $U(N)$ such that $v \times v$ refines $u$. Since $u = \{\Delta_N, N \times N \setminus \Delta_N\}$, it follows that $v$ consists of one-point sets. Claim 1 is proved. \hfill \square

Let $\mathbb{Q} \subset \mathbb{R}$ be the space of rationals.

**Claim 2.** $U(\mathbb{Q})$ is a universal space.
Proof. In fact, we have to check that an arbitrary open covering of \( Q \) is uniform. For this it suffices to show that an arbitrary disjoint covering \( \mathcal{U} \) of \( Q \) consisting of clopen sets is uniform. There is an epimorphism \( f : Q \to N \) such that
\[
\mathcal{U} = \{ f^{-1}(n) : n \in N \}.
\]
Since \( f \) is continuous, \( f : U(Q) \to U(N) \) is uniformly continuous. Then \( \mathcal{U} \) is a uniform covering according to Claim 1. Claim 2 is proved. \( \square \)

Now let \( f : Q \to Q \times Q \) be a homeomorphism. Then \( f : U(Q) \to U(Q \times Q) \) is a uniform isomorphism. But \( U(Q \times Q) = U(Q) \times U(Q) \) in view of (4.3). So, Claim 2 implies that the uniform space \( U(Q) \times U(Q) \) is universal. But that is not correct. We arrive at a contradiction. Proposition 4.5 is proved. \( \square \)

Proposition 4.5, Theorem 3.11 and Remark 3.12 imply

**Theorem 4.6.** There is no embedding \( U : Tych \to Unif \) such that
\[
P^u_\beta \circ U = U \circ P_\beta.
\]

Analysis of the proof of Theorem 4.6 shows us that the next statement is true.

**Theorem 4.7.** There is no embedding \( U \) of the category \( M \) of all metrizable spaces in to the category \( MUnif \) of metrizable uniform spaces such that
\[
P^u_\beta \circ U = U \circ P_\beta.
\]

**Remark 4.8.** Banakh [2,3] considered the functors \( P_t \) and \( P_\tau \) (of Radon and \( \tau \)-additive measures respectively) in the category \( Tych \). For an arbitrary Tychonoff space \( X \) we have
\[
P_\beta(X) \subset P_t(X) \subset P_\tau(X) \subset P(\beta X).
\]
He lifted the functors \( P_t \) and \( P_\tau \) to the functors \( P_t^u \) and \( P_\tau^u \) acting in the category \( Unif \) in the same manner as it was done in Section 2 for the functor \( P_\beta \). For an arbitrary uniform space \( (X, U) \) the following inclusions
\[
(P_\beta(X), P_\beta(U)) \subset (P_t(X), P_t(U)) \subset (P_\tau(X), P_\tau(U))
\]
are uniform. Moreover, we have the functor inclusions
\[
P_\beta^u \subset P_t^u \subset P_\tau^u.
\]
So all main results for the functor \( P_\beta^u \) (Theorems 4.3, 4.6 and 4.7, Proposition 4.4) hold for the functors \( P_t^u \) and \( P_\tau^u \) as well.

References