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Subquadrangles of order s of generalized quadrangles of order (s, s^2), Part I

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Abstract

In this paper, subquadrangles of order s of generalized quadrangles (GQs) of order (s, s^2) are investigated. In the case where $\mathcal{S}$ is a dual flock GQ of order (s, s^2), s even, the classification of subquadrangles of order s is completed. In the case where $\mathscr{E}$ is an egg good at an element and the corresponding translation generalized quadrangle $T(\mathscr{E})$ of order (s, s^2), s even, has base point (∞) it is proved that if $T(\mathscr{E})$ has a subquadrangle of order s not containing (∞), then $T(\mathscr{E})$ is the classical GQ $Q(5, s)$. Also a new strong characterization of the classical GQ $Q(5, s)$, s even or odd, is obtained. In Part II, the case s odd is considered.
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1. Introduction and definitions

In this paper, we investigate subquadrangles of order s of generalized quadrangles (GQs) of order (s, s^2). In the case where $\mathcal{S}$ is a dual flock GQ of order (s, s^2), s even, we complete the classification of subquadrangles of order s. In the case where $\mathscr{E}$ is an egg good at an element and the translation generalized quadrangle (TGQ) $T(\mathscr{E})$ of order (s, s^2), s even, has base point (∞) we prove that if $T(\mathscr{E})$ has a subquadrangle of order s not containing (∞), then $T(\mathscr{E})$ is the classical GQ $Q(5, s)$. Also we obtain a new strong characterization of the classical GQ $Q(5, s)$. As a preparatory result to these theorems we prove that for any GQ of order (s^2, s) with a regular point X such

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that the associated dual net is H_s^3, any subquadrangle of $\mathcal{S}$ of order s not containing X is isomorphic to $T_2(\mathcal{O})$ for some oval $\mathcal{O}$. In Part II the case s odd is considered.

Note that for convenience we will sometimes switch between subquadrangles of order s of GQ of order (s, s^2) and the dual situation.

We begin with some definitions and fundamental results.

A (finite) *generalized quadrangle* (GQ) (see [14] for a comprehensive introduction) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (non-empty) sets of objects called *points* and *lines*, respectively, and for which $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line,
(ii) Each line is incident with $1 + s$ points ($s \geqslant 1$) and two distinct lines are incident with at most one point,
(iii) If X is a point and ℓ is a line not incident with X, then there is a unique pair $(Y, m) \in \mathcal{P} \times \mathcal{B}$ for which $X \mathrm{I} m \mathrm{I} Y \mathrm{I} \ell$.

The integers s and t are the *parameters* of the GQ and $\mathcal{S}$ is said to have *order* (s, t). If $s = t$, then $\mathcal{S}$ is said to have order s. If $\mathcal{S}$ has order (s, t), then it follows that $|\mathcal{P}| = (s + 1)(st + 1)$ and $|\mathcal{B}| = (t + 1)(st + 1)$ [14, 1.2.1]. A *subquadrangle* $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ of $\mathcal{S}$ is a GQ such that $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{B}' \subseteq \mathcal{B}$ and I' is the restriction of I to $(\mathcal{P}' \times \mathcal{B}') \cup (\mathcal{B}' \times \mathcal{P}')$. If $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a GQ of order (s, t) then the incidence structure $\mathcal{S}^* = (\mathcal{B}, \mathcal{P}, \mathrm{I})$ is a GQ of order (t, s) called the *dual* of $\mathcal{S}$. As there is a point-line duality for GQ of order (s, t), we assume without further notice that the dual of a given theorem or definition has also been given.

The classical GQs of order s (s a prime power) are $Q(4, s)$ that arises as the points and lines of the non-singular (parabolic) quadric in $\mathrm{PG}(4, s)$, and $W(s)$ which is defined as the singular points and lines of a symplectic polarity in $\mathrm{PG}(3, s)$. The classical GQ of order (s, s^2) is $Q(5, s)$ that arises as the points and lines of the non-singular elliptic quadric in $\mathrm{PG}(5, s)$ and the classical GQ of order (s^2, s) is $H(3, s^2)$ that arises as the points and lines of the non-singular Hermitian variety in $\mathrm{PG}(3, s^2)$. We also have that $W(s) \cong Q(4, s)^*$, $W(s) \cong Q(4, s)$ if and only if s is even, while $Q(5, s) \cong H(3, s^2)^*$ (see [14, Chapter 3]).

Given two (not necessarily distinct) points X, X' of $\mathcal{S}$, we write $X \sim X'$ and say that X and X' are *collinear*, provided that there is some line ℓ for which $X \mathrm{I} \ell \mathrm{I} X'$. Dually, for $\ell, \ell' \in \mathcal{B}$, we write $\ell \sim \ell'$ if ℓ and ℓ' are *concurrent*. For $X \in \mathcal{P}$ we define $X^\perp = \{X' \in \mathcal{P} : X \sim X'\}$, while for $A \subset \mathcal{P}$ we define $A^\perp = \cap\{X^\perp : X \in A\}$ and $A^{\perp\perp} = \{X \in \mathcal{P} : X \in Y^\perp \text{ for all } Y \in A^\perp\}$.

If $X \sim X'$, $X \neq X'$, or if $X \nsim X'$ and $|\{X, X'\}^{\perp\perp}| = t + 1$, where $X, X' \in \mathcal{P}$, we say that the pair $\{X, X'\}$ is *regular*. The point X is *regular* provided $\{X, X'\}$ is regular for all $X' \in \mathcal{P}$, $X' \neq X$. Regularity for lines is defined dually. A *triad* of $\mathcal{S}$ is a set consisting of three pairwise non-collinear points. Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, s^2), $s \neq 1$. If $\{X, X', X''\}$ is a triad of $\mathcal{S}$, then we say that $\{X, X', X''\}$ is 3-*regular* provided $|\{X, X', X''\}^{\perp\perp}| = s+1$. The point X is 3-*regular* provided each triad containing X is 3-regular.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, s^2), $s \neq 1$. Let X_1, Y_1 be distinct collinear points. We say that the pair $\{X_1, Y_1\}$ has *Property* (G), or that $\mathcal{S}$ has *Property* (G) *at* $\{X_1, Y_1\}$, if every triad $\{X_1, X_2, X_3\}$ of points, with $Y_1 \in \{X_1, X_2, X_3\}^{\perp}$, is 3-regular. The GQ $\mathcal{S}$ has *Property* (G) *at the line* ℓ, or the line ℓ has *Property* (G), if each pair of points of $\{X, Y\}$, $X \neq Y$ and $X \,\mathrm{I}\, \ell \,\mathrm{I}\, Y$, has Property (G). If (X, ℓ) is a flag, then we say that $\mathcal{S}$ has *Property* (G) at (X, ℓ), or that (X, ℓ) has *Property* (G), if every pair $\{X, Y\}$, $X \neq Y$ and $Y \mathrm{I} \ell$, has Property (G).

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, t), $s \neq 1$, $t \neq 1$. A collineation θ of $\mathcal{S}$ is an *elation* about the point P if $\theta = id$ or if θ fixes all lines incident with P and fixes no point of $\mathcal{P} \backslash P^{\perp}$. If there is a group G of elations about P acting regularly on $\mathcal{P} \backslash P^{\perp}$, then we say that $\mathcal{S}$ is an *elation generalized quadrangle* (EGQ) with *elation group G* and *base point P*. Briefly, we say that $(\mathcal{S}^{(P)}, G)$ or $\mathcal{S}^{(P)}$ is an EGQ. If the group G is abelian, then we say that the EGQ $(\mathcal{S}^{(P)}, G)$ is a *translation generalized quadrangle* (TGQ) and G is the *translation group*. For any TGQ $\mathcal{S}^{(P)}$ the point P is coregular, that is, each line incident with P is regular. Further for any TGQ $\mathcal{S}^{(P)}$ each line ℓ incident with P is an *axis of symmetry*, that is, there is a (maximal) group of q collineations of $\mathcal{S}$ fixing $\ell^{\perp}$ elementwise (see [14, Chapter 8]).

In $\mathrm{PG}(2n+m-1, q)$ consider a set $O(n, m, q)$ of $q^m + 1$ $(n-1)$-dimensional subspaces $\mathrm{PG}^{(0)}(n-1, q), \mathrm{PG}^{(1)}(n-1, q), \ldots, \mathrm{PG}^{(q^m)}(n-1, q)$, every three of which generate a $\mathrm{PG}(3n-1, q)$ and such that each element $\mathrm{PG}^{(i)}(n-1, q)$ of $O(n, m, q)$ is contained in a $\mathrm{PG}^{(i)}(n+m-1, q)$ having no point in common with any $\mathrm{PG}^{(j)}(n-1, q)$ for $j \neq i$. It is easy to check that $\mathrm{PG}^{(i)}(n+m-1, q)$ is uniquely determined, $i = 0, 1, \ldots, q^m$. The space $\mathrm{PG}^{(i)}(n+m-1, q)$ is called the *tangent space* of $O(n, m, q)$ at $\mathrm{PG}^{(i)}(n-1, q)$. For $n = m = 1$ such a set $O(1, 1, q)$ is an oval in $\mathrm{PG}(2, q)$ and more generally for $n = m$ such a set $O(n, n, q)$ is called a *pseudo-oval* of $\mathrm{PG}(3n-1, q)$. For $m = 2n = 2$ such a set $O(1, 2, q)$ is an ovoid of $\mathrm{PG}(3, q)$ and more generally for $m = 2n$ such a set $O(n, 2n, q)$ is called an *egg*.

Now embed $\mathrm{PG}(2n+m-1, q)$ in a $\mathrm{PG}(2n+m, q)$, and construct a point-line geometry $T(n, m, q)$ as follows.

Points are of three types:

(i) the points of $\mathrm{PG}(2n+m, q) \backslash \mathrm{PG}(2n+m-1, q)$, called the *affine points*,
(ii) the $(n+m)$-dimensional subspaces of $\mathrm{PG}(2n+m, q)$ which intersect $\mathrm{PG}(2n+m-1, q)$ in one of the $\mathrm{PG}^{(i)}(n+m-1, q)$,
(iii) the symbol (∞).

Lines are of two types:

(a) the n-dimensional subspaces of $\mathrm{PG}(2n+m, q)$ which intersect $\mathrm{PG}(2n+m-1, q)$ in a $\mathrm{PG}^{(i)}(n-1, q)$,
(b) the elements of $O(n, m, q)$.

Incidence in $T(n,m,q)$ is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of $\mathrm{PG}(2n+m,q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of $O(n,m,q)$ contained in it. Point (∞) is incident with no line of type (a) and with all lines of type (b).

Theorem 1.1 (Payne and Thas [14, 8.7.1]). *The incidence geometry $T(n,m,q)$ is a TGQ of order (q^n,q^m) with base point (∞). Conversely, every TGQ is isomorphic to a $T(n,m,q)$. It follows that the theory of TGQ is equivalent to the theory of the sets $O(n,m,q)$.*

In the case where $n=m=1$ and $O(1,1,q)$ is the oval $\mathcal{O}$ the GQ $T(1,1,q)$ is the Tits GQ $T_2(\mathcal{O})$. When $m=2n=2$ and $O(1,2,q)$ is the ovoid Ω, the GQ $T(1,2,q)$ is the Tits GQ $T_3(\Omega)$. Note that $T_2(\mathcal{O})\cong Q(4,q)$, if and only if $\mathcal{O}$ is a conic and non-classical otherwise, while $T_3(\Omega)\cong Q(5,q)$ if and only if Ω is an elliptic quadric (see [14, Chapter 3]).

An egg $O(n,2n,q)=\{\mathrm{PG}(n-1,q),\mathrm{PG}^{(1)}(n-1,q),\mathrm{PG}^{(2)}(n-1,q),\ldots,\mathrm{PG}^{(q^{2n})}(n-1,q)\}$ of $\mathrm{PG}(4n-1,q)$ is *good* at the element $\mathrm{PG}(n-1,q)$ if any triple $\{\mathrm{PG}(n-1,q),\mathrm{PG}^{(i)}(n-1,q),\mathrm{PG}^{(j)}(n-1,q)\}$ with $i,j\in\{1,2,\ldots,q^{2n}\}$, $i\neq j$, spans a $\mathrm{PG}(3n-1,q)$ containing exactly q^n+1 elements of $O(n,2n,q)$. In such a case the corresponding TGQ $T(n,2n,q)$ contains $q^{3n}+q^{2n}$ subquadrangles of order q^{2n} containing both the base point of $T(n,2n,q)$ and the line corresponding to the good element of $O(n,2n,q)$ (see [20]).

Let $\mathcal{K}$ be a quadratic cone with vertex V in $\mathrm{PG}(3,s)$. A partition $\mathcal{F}$ of $\mathcal{K}\setminus\{V\}$ into s disjoint conics is called a *flock* of $\mathcal{K}$. Then, by Thas [19] with $\mathcal{F}$ there corresponds a GQ $\mathcal{S}(\mathcal{F})$ of order (s^2,s) (see [21] for details and notation of the construction). The GQ $\mathcal{S}(\mathcal{F})$ is an EGQ with base point (∞) and elation group G. In the case where s is even to each flock $\mathcal{F}$ of a quadratic cone is associated a set $H(\mathcal{F})=\{\mathcal{O}_1,\mathcal{O}_2,\ldots,\mathcal{O}_{s+1}\}$ of $s+1$ ovals of $\mathrm{PG}(2,s)$ called a *herd*. For each $i=1,2,\ldots,s+1$ the GQ $\mathcal{S}(\mathcal{F})$ has s^2 subquadrangles isomorphic to $T_2(\mathcal{O}_i)$ containing the point (∞) and equivalent under the action of G, yielding s^3+s^2 in total.

Returning to regularity in GQs, we make the following definition (the connection of which to regularity in GQs will become clear via the theorem following the definition):

A (finite) *net* of *order* $k(\geqslant 2)$ and *degree* $r(\geqslant 2)$ is an incidence structure $\mathcal{N}=(\mathcal{P},\mathcal{B},\mathrm{I})$ satisfying

(i) each point is incident with r lines and two distinct points are incident with at most one line,
(ii) each line is incident with k points and two distinct lines are incident with at most one point,
(iii) if X is a point and ℓ is a line not incident with X, then there is a unique line m incident with X and not concurrent with ℓ.

For a net of order k and degree r we have $|\mathcal{P}|=k^2$ and $|\mathcal{B}|=kr$. The following result links nets with GQs with a regular point.

Theorem 1.2 (Payne and Thas [14, 1.3.1]). *Let X be a regular point of the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of order (s,t), $s \geqslant 2$. Then the incidence structure with pointset $X^\perp \backslash \{X\}$, with lineset consisting of the sets $\{Y, Z\}^{\perp\perp}$, where $Y, Z \in X^\perp \backslash \{X\}$, $Y \not\sim Z$, and with the natural incidence, is the dual of a net of order s and degree $t+1$. If in particular $s = t > 1$, there arises a dual affine plane of order s. Also, in the case $s = t > 1$ the incidence structure π_X with pointset $X^\perp$, with lineset the set of spans $\{Y, Z\}^{\perp\perp}$, where $Y, Z \in X^\perp$, $Y \neq Z$, and with the natural incidence, is a projective plane of order s.*

Affine planes are examples of nets and we will now describe another important class of nets. Denote by H_s^n, $n > 2$, the incidence structure with points the points of $\mathrm{PG}(n,s)$ not in a given subspace $\mathrm{PG}(n-2,s) \subset \mathrm{PG}(n,s)$, lines the lines of $\mathrm{PG}(n,s)$ which have no point in common with $\mathrm{PG}(n-2,s)$, and incidence the natural one. Then H_s^n is the dual of a net and we denote the net by $(H_s^n)^*$. Thas and De Clerck [23] proved that the H_s^n are the only dual nets that are not dual affine planes and satisfy the axiom of Veblen.

We will be particularly interested in the case where $n = 3$. For this dual net H_s^3, constructed from a given line ℓ, say, in $\mathrm{PG}(3,s)$, there are $s+1$ points on a line and s^2 lines on a point. For π a plane of $\mathrm{PG}(3,s)$ if π contains the line ℓ, then the lines of the *net* contained in π (that is, the points of $\pi \backslash \ell$) form a *parallel class* of lines of $(H_s^3)^*$. On the other hand, if π does not contain the line ℓ, then it meets ℓ in some point P, say, and the dual affine plane $\pi \backslash \{P\}$ corresponds to an affine plane subnet of $(H_s^3)^*$.

2. Technical preliminaries

2.1. Subquadrangles of order s of a GQ of order (s, s^2)

In this section, we review some results on subquadrangles that will be needed for later sections.

Lemma 2.1. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s^2, s) and let $\mathcal{S}'$, $\mathcal{S}''$ be two distinct subquadrangles of $\mathcal{S}$ of order s. Then there are three possibilities for $\mathcal{S}' \cap \mathcal{S}''$:*

(i) *a line ℓ, the $s+1$ points of $\mathcal{S}'$ and $\mathcal{S}''$ incident with ℓ and the lines incident with one of these points,*
(ii) *a spread of $\mathcal{S}'$ and $\mathcal{S}''$,*
(iii) *a dual grid.*

Proof. Follows from [14, 2.2, 2.3]. $\square$

Also, as an application of this lemma, we have the following result.

Lemma 2.2 (O'Keefe and Penttila [11]). *Let X and Y be two non-collinear points of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of order (s^2, s), $s > 1$. If X and Y are contained in two subquadrangles $\mathcal{S}'$ and $\mathcal{S}''$ of $\mathcal{S}$ of order s, then $\{X, Y\}$ is regular in each of $\mathcal{S}, \mathcal{S}'$ and $\mathcal{S}''$ and lie in at most $s+1$ subquadrangles of $\mathcal{S}$ of order s. Further $\mathcal{S}' \cap \mathcal{S}''$ is the dual grid with pointset $\{X, Y\}^{\perp\perp} \cup \{X, Y\}^{\perp}$.*

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, s^2), s even, and assume that the triad $\{X, Y, Z\}$ is 3-regular. Now, we define the following incidence structure $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$. The set $\mathcal{P}'$ consists of all points incident with the lines connecting a point of $\{X, Y, Z\}^{\perp}$ to a point of $\{X, Y, Z\}^{\perp\perp}$. The set $\mathcal{B}'$ consists of all lines of $\mathcal{B}$ incident with at least two points of $\mathcal{P}'$. Finally, I' is the restriction of I to $(\mathcal{P}' \times \mathcal{B}') \cup (\mathcal{B}' \times \mathcal{P}')$. Then we have the following theorem.

Theorem 2.3 (Thas [18]). *The incidence structure $\mathcal{S}'$ is a subquadrangle of order s of the GQ $\mathcal{S}$.*

The next lemma will be very useful for later sections.

Lemma 2.4. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, s^2), $s \neq 1$. If $\mathcal{S}$ is any line of $\mathcal{S}$, then $\mathcal{S}$ contains at most $s^3 + s^2$ subquadrangles of order s containing the line ℓ.*

Proof. Any two lines m_1, m_2, with $m_1 \sim \ell \sim m_2$, $m_1 \nsim m_2$, belong to at most $s+1$ subquadrangles of order s; see 2.2 of Payne and Thas [14]. It easily follows that ℓ is contained in at most $s^3 + s^2$ subquadrangles of order s. $\square$

2.2. GQs with a regular point and covers of nets

In this section, we discuss briefly the connection between GQs with a regular point and covers of the net associated with the regular point. In particular, we shall consider subquadrangles of order s of GQs of order (s, s^2) in this context.

Let $\mathcal{N} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a net of order k and degree r. A *t-fold cover* of $\mathcal{N}$ is a geometry $\overline{\mathcal{N}} = (\overline{\mathcal{P}}, \overline{\mathcal{B}}, \overline{\mathrm{I}})$ such that there exists an onto map $p : \overline{\mathcal{P}} \to \mathcal{P}$, satisfying: (i) for any $P \in \mathcal{P}$, $p^{-1}(P)$ consists of t pairwise non-collinear points; (ii) for any pair $\{P, Q\}$ of collinear points of $\mathcal{N}$, $p^{-1}(\{P, Q\})$ consists of t disjoint pairs of collinear points; (iii) for any pair $\{P, Q\}$ of non-collinear points of $\mathcal{N}$, $p^{-1}(\{P, Q\})$ is a set of pairwise non-collinear points of $\overline{\mathcal{N}}$; and (iv) for any line ℓ of $\mathcal{N}$, if $\mathcal{P}_\ell = \{P \in \mathcal{P} : \mathcal{P}\mathrm{I}\ell\}$, then $p^{-1}(P_\ell)$ consists of the points of t disjoint lines of $\overline{\mathcal{N}}$. The map p is called the *covering map*. A point or line of $\overline{\mathcal{N}}$ is said to be a *cover* of its image under p.

Now let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, t) with a regular point X, and let $\mathcal{N}_X$ be the associated net of order s and degree $t+1$. The geometry $\overline{\mathcal{N}_X}$ with pointset $\mathcal{P} \backslash X^{\perp}$, lineset $\mathcal{B} \backslash \{\ell \in \mathcal{B} : X\mathrm{I}\ell\}$ and incidence inherited from $\mathcal{S}$ is a t-fold cover of

$\mathcal{N}_X$. The covering map p maps the point $P \in \mathcal{P} \backslash X^\perp$ to the point $\{X, P\}^\perp$ of $\mathcal{N}_X$. The cover has the additional property that for a non-incident point, line pair (P, ℓ) of $\overline{\mathcal{N}_X}$, that either $p(P) \in p(Y)$ with $Y \mathrm{I} \ell$ and $Y \in \{P, X\}^{\perp\perp}$, or P is collinear with a unique point of $\overline{\mathcal{N}_X}$ incident with ℓ. In general, given a net of order s and degree $t+1$ with such a t-fold cover it is possible to construct a GQ of order (s, t), which in the above case is a reconstruction of $\mathcal{S}$. (A more detailed study of the above will appear in an upcoming paper of the first author.)

Suppose that $\mathcal{N}_X$ has a proper subnet $\mathcal{N}'_X$ of order $s'(\geqslant 2)$ and degree $t+1$. Then the subset $\mathcal{P}'$ of $\mathcal{P}$ consisting of X, the lines of $\mathcal{N}'_X$ and $P \in \mathcal{P} \backslash X^\perp$ such that $\{P, X\}^\perp$ is a point of $\mathcal{N}'_X$ is the pointset of a subquadrangle $\mathcal{S}'$ of order (s', t) of $\mathcal{S}$. The lines of $\mathcal{S}'$ are the lines of $\mathcal{S}$ incident with two (and hence $s'+1$) points of $\mathcal{P}'$. The point X is a regular point of $\mathcal{S}'$ with associated net $\mathcal{N}'_X$. It follows that $s = s'$, $t = s^2$ and so the subnet $\mathcal{N}'_X$ is an affine plane of order s. For more details and proofs see [27].

2.3. The dual net H_s^3 and subquadrangles

Let $\mathcal{S}$ be a GQ of order (s^2, s) with a regular point X such that the associated dual net $\mathcal{N}^*_X$ is isomorphic to H_s^3. In this section, we consider subquadrangles of order s of $\mathcal{S}$. Examples of classes of such GQs are given in the following two theorems.

Theorem 2.5 (Thas and Van Maldeghem [26]). *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s^2, s), s even, satisfying Property (G) at the point (∞). Then (∞) is regular in $\mathcal{S}$ and the dual net $\mathcal{N}^*_{(\infty)}$ defined by (∞) is isomorphic to H_s^3.*

Theorem 2.6 (Thas and Van Maldeghem [26]). *Let $\mathcal{S}^{(P)} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a TGQ of order (s, s^2), $s \neq 1$, with base point P. Then the dual net $\mathcal{N}^*_\delta$ defined by the regular line δ, with $P \mathrm{I} \delta$, is isomorphic to H_s^3 if and only if the egg $O(n, 2n, q)$ which corresponds to $\mathcal{S}^{(P)}$ is good at its element which corresponds to δ.*

Now we have the main theorem of this section.

Theorem 2.7. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s^2, s) with a regular point X such that the associated dual net $\mathcal{N}^*_X$ is isomorphic to H_s^3. Then X is contained in the maximal number $s^3 + s^2$ subquadrangles of order s. If $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ is any subquadrangle of order s not containing X, then $\mathcal{S}'$ is isomorphic to the dual of $T_2(\mathcal{O})$, for some oval $\mathcal{O}$ of* $\mathrm{PG}(2, s)$.

Proof. Suppose that the net $\mathcal{N}_X \cong (H_s^3)^*$ is constructed from the line ℓ of $\mathrm{PG}(3, s)$; that is by taking as points the lines of $\mathrm{PG}(3, s)$ not meeting ℓ, as lines the points of $\mathrm{PG}(3, s) \backslash \ell$, and the incidence from $\mathrm{PG}(3, s)$. Then each plane π of $\mathrm{PG}(3, s)$ *not* containing ℓ gives rise to an affine plane subnet of $(H_s^3)^*$, which is the dual of π with the point $\pi \cap \ell$ and the lines on $\pi \cap \ell$ removed. From Section 2.2, we see that π gives rise to a subquadrangle of order s. The subquadrangle has points X, the points of π not on ℓ and the points of $\mathscr{P}\backslash X^{\perp}$ that are covers of a line of π not meeting ℓ. The lines of the subquadrangle are the lines of $\mathscr{S}$ incident with X and the lines of $\mathscr{S}$ not incident with X that are covers of a point of $\pi\backslash(\pi\cap\ell)$. This gives s^3+s^2 distinct subquadrangles of order s containing X, the maximal number by Lemma 2.4.

Now suppose that $\mathscr{S}'=(\mathscr{P}',\mathscr{B}',\mathrm{I}')$ is a subquadrangle of order s of $\mathscr{S}$, not containing X. Recall from Section 2.2 that the geometry $\mathscr{S}\backslash X^{\perp}$ is an s-fold cover of the net $(H_s^3)^*$ with covering map p taking the point $P\in\mathscr{S}\backslash X^{\perp}$ to the point $\{X,P\}^{\perp}$ of $\mathscr{N}_X\cong(H_s^3)^*$. The subquadrangle $\mathscr{S}'$ contains a unique line, m say, incident with X. The points of $\mathscr{S}$ incident with m, but distinct from X, form a parallel class of H_s^3 the elements of which are contained in a plane of $\mathrm{PG}(3,s)$ containing ℓ which we will denote by $p(m)$. The subquadrangle $\mathscr{S}'$ contains $s+1$ points of m, which we denote by $\mathscr{O}$. In H_s^3 the set $\mathscr{O}$ is a set of $s+1$ points on the plane $p(m)$, none of which is incident with ℓ. Consider a line n of $\mathscr{S}'$ not concurrent with m. Thus $p(n)$ is a point of $\mathrm{PG}(3,s)$ not on the plane $p(m)$. Further, since no two lines of $\mathscr{S}'$ may be incident with a common point of $X^{\perp}$ not on m, it follows that the covering map p gives a one-to-one correspondence between the s^3 lines of $\mathscr{S}'$ not concurrent with m and the s^3 points of $\mathrm{PG}(3,s)\backslash p(m)$. Each point of $\mathscr{S}'$ not incident with m is collinear with a unique point of $\mathscr{O}$ and so under the map p is a line of $\mathrm{PG}(3,s)$ meeting $p(m)$ in a point of $\mathscr{O}$. Since no two lines of $\mathscr{S}'$ are concurrent in a point of $X^{\perp}$ not on m, it must also be the case that no two points of $\mathscr{S}'$, not incident with m, correspond under p to the same line of $\mathrm{PG}(3,s)$. Thus p gives a one-to-one correspondence between the set $\mathscr{P}'\backslash m$ and the lines of $\mathrm{PG}(3,s)$ not in $p(m)$ meeting $p(m)$ in a point of $\mathscr{O}$. It is now a straightforward exercise to verify that $\mathscr{O}$ is an oval and that $\mathscr{S}'$ is isomorphic to the dual of $T_2(\mathscr{O})$. $\square$

We have an immediate corollary.

Theorem 2.8. *Let $\mathscr{S}=(\mathscr{P},\mathscr{B},\mathrm{I})$ be a GQ of order (s^2,s), s odd, with a regular point X such that the associated dual net $\mathscr{N}_X^*$ is isomorphic to H_s^3. If $\mathscr{S}'=(\mathscr{P}',\mathscr{B}',\mathrm{I}')$ is any subquadrangle of order s not containing X, then $\mathscr{S}'$ is the classical GQ $Q(4,s)$.*

Note that for the GQs of order (s^2,s) with a regular point and associated dual net H_s^3 in Theorem 2.5 and Theorem 2.6, the existence of the s^3+s^2 subquadrangles containing X follows from [20, Section 2.3] when s is even and [20, Section 4.3] when s is odd.

2.4. A characterisation of $T_2(\mathscr{O})$, $\mathscr{O}$ a translation oval

If an oval $\mathscr{O}$ of $\mathrm{PG}(2,s)$, s even, has a tangent line ℓ such that there exists a group of s elations of $\mathrm{PG}(2,s)$ each element of which has axis ℓ and fixes $\mathscr{O}$, then $\mathscr{O}$ is called a *translation oval*. The line ℓ is called an *axis* of $\mathscr{O}$. It was proved in [12] that every translation oval of $\mathrm{PG}(2,s)$ is projectively equivalent to an oval of the form $\{(1,t,t^{\alpha}): t \in \mathrm{GF}(s)\} \cup \{(0,0,1)\}$, for some generator α of $\mathrm{Aut}(\mathrm{GF}(s))$.

In this section, we prove a result characterizing the GQs $T_2(\mathcal{O})$, with $\mathcal{O}$ a translation oval, amongst the $T_2(\mathcal{O})$ where $\mathcal{O}$ may be any oval of $\mathrm{PG}(2,s)$.

Theorem 2.9. *Let $\mathcal{O}$ be an oval of* $\mathrm{PG}(2,s)$, *s even. The GQ $T_2(\mathcal{O})$ has a regular pair of non-collinear affine points $\{P,Q\}$ with $(\infty) \notin \{P,Q\}^{\perp\perp}$ if and only if $\mathcal{O}$ is a translation oval.*

Proof. Construct the GQ $T_2(\mathcal{O})$ from $\mathcal{O} \subset \mathrm{PG}(2,s) \subset \mathrm{PG}(3,s)$ in the usual way. First, suppose that $\mathcal{O}$ is a translation oval of $\pi_\infty = \mathrm{PG}(2,s)$ with axis the line $\ell \subset \pi_\infty$. Then by [14, 12.4.5] every plane of $\mathrm{PG}(3,s)$ meeting π_∞ in ℓ is a regular point of $T_2(\mathcal{O})$. For such a plane π if P and Q are two distinct points of $\pi \backslash \ell$, not collinear in $T_2(\mathcal{O})$, then they are a regular pair of affine points of $T_2(\mathcal{O})$. If P and Q have the property that the line $\langle P,Q \rangle$ of $\mathrm{PG}(3,s)$ does not contain the nucleus of $\mathcal{O}$, then $(\infty) \notin \{P,Q\}^{\perp\perp}$.

Next, suppose that $T_2(\mathcal{O})$ has a regular pair of affine points $\{P,Q\}$ with $(\infty) \notin \{P,Q\}^{\perp\perp}$. It follows that the line $\langle P,Q \rangle$ of $\mathrm{PG}(3,s)$ meets π_∞ in a point R not contained in $\mathcal{O} \cup \{N\}$, where N is the nucleus of $\mathcal{O}$. Let ℓ be the tangent to $\mathcal{O}$ spanned by R and N, meeting $\mathcal{O}$ at X. The set $\{P,Q\}^{\perp}$ consists of s affine points of $T_2(\mathcal{O})$ and the plane $\pi_1 = \langle P,Q,N \rangle$. Since $\{P,Q\}$ is a regular pair it follows that $\{P,Q\}^{\perp\perp} = \{U,V\}^{\perp}$ for any two points U,V of $\{P,Q\}^{\perp}$ and so consists of s affine points and some plane π_2 distinct from π_1 and meeting π_∞ in ℓ. Now $\mathcal{O}_1 = (\{P,Q\}^{\perp\perp} \backslash \{\pi_2\}) \cup \{X\}$ is an oval of π_1 equivalent to $\mathcal{O}$ (since it is essentially the projection of $\mathcal{O}$ onto π_1 by an affine point of $\{P,Q\}^{\perp}$), and similarly $\mathcal{O}_2 = (\{P,Q\}^{\perp} \backslash \{\pi_1\}) \cup \{X\}$ is an oval of π_2 equivalent to $\mathcal{O}$. Now, let Y and Z be any two distinct points of $\mathcal{O}_1 \backslash \{X\}$ and let ϕ be the unique elation of $\mathrm{PG}(3,s)$ that has axis π_∞ and maps Y to Z (and so necessarily has a centre on ℓ). Since the points Y and Z both project the oval $\mathcal{O}$ onto $\mathcal{O}_2$ in π_2 it must be that ϕ fixes the oval $\mathcal{O}_2$. By the same argument applied to points on $\mathcal{O}_2$ we also have that ϕ must fix $\mathcal{O}_1$. Restricting ϕ to the plane π_1 gives an elation with axis ℓ that fixes $\mathcal{O}_1$. Since Y and Z were chosen arbitrarily on $\mathcal{O}_1 \backslash \{X\}$ it follows that we have $s-1$ non-trivial elations of π_1 with axis ℓ that fix $\mathcal{O}_1$. Since the set of elations of π_1 with axis ℓ that fix $\mathcal{O}_1$ form a group which has maximal size s, we have such a group and so by definition $\mathcal{O}_1$ is a translation oval. Since $\mathcal{O}$ is equivalent to $\mathcal{O}_1$, it also follows that $\mathcal{O}$ is a translation oval. $\square$

Remark. An alternative proof of Theorem 2.9 is as follows. By using the action of the group of $T_2(\mathcal{O})$ fixing π_1 on the set $\{P,Q\}^{\perp\perp}$ it is easily shown that each pair of points in $\pi_1^{\perp}$ is regular. Hence π_1 itself is regular and by [14, 12.4.6] $\mathcal{O}$ is a translation oval.

We have the following immediate corollary of Theorem 2.9.

Corollary 2.10. *Let $\mathcal{O}$ be an oval of* $\mathrm{PG}(2,s)$, *s even. The GQ* $T_2(\mathcal{O})$ *has a regular pair of non-collinear points* $\{P,\pi\}$ *with P an affine point and* $\pi\sim(\infty)$ *if and only if* $\mathcal{O}$ *is a translation oval and* $\ell=\pi\cap\mathrm{PG}(2,s)$ *is an axis of* $\mathcal{O}$.

Remark. It was brought to our attention by S.E. Payne that Theorem 2.9 is also contained in his paper [15].

3. A new characterization of $Q(5,s)$

In this section, we will prove a strong characterization theorem for $Q(5,s)$, that will be used in Section 4.

Theorem 3.1. *Let* $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ *be a GQ of order* (s,s^2), $s\neq 1$, *and assume one of the following conditions is satisfied*:

(i) *s is odd and* $\mathcal{S}$ *satisfies Property* (G) *at the flag* (X,ℓ), *or*
(ii) *s is even,* $\mathcal{S}$ *is the dual of a flock GQ,* ℓ *is the corresponding base line of* $\mathcal{S}$, *and* (X,ℓ) *is a flag of* $\mathcal{S}$.

If at least one triad $\{Z_1,Z_2,X\}$, *where* $\{Z_1,Z_2,X\}^\perp$ *does not contain a point incident with* ℓ, *is* 3-*regular, then* $\mathcal{S}$ *is classical.*

Proof. Let $Y\mathrm{I}\ell$, $X\neq Y$. Now we introduce the following incidence structure $\mathcal{S}_{XY}=(\mathcal{P}_{XY},\mathcal{B}_{XY},\mathrm{I}_{XY})$:

(i) $\mathcal{P}_{XY}=X^\perp\backslash\{X,Y\}^{\perp\perp}$.
(ii) Elements of $\mathcal{B}_{XY}$ are of two types: (a) the sets $\{Y,Z,V\}^{\perp\perp}\backslash\{Y\}$, with $\{Y,Z,V\}$ a triad for which $X\in\{Y,Z,V\}^\perp$, and (b) the sets $\{X,W\}^\perp\backslash\{X\}$, with $X\sim W\nsim Y$.
(iii) I_{XY} is containment.

Then $\mathcal{S}_{XY}$ is the design of points and lines of an affine space $\mathrm{AG}(3,s)$; see Thas [20]. The s^2 lines of type (b) of $\mathcal{S}_{XY}$ are parallel, so they define a point ∞ of the projective completion $\mathrm{PG}(3,s)$ of $\mathrm{AG}(3,s)$. Let $V\in\mathcal{P}$ with $X\nsim V\nsim Y$ and let $V_{XY}=\{X,Z\}^\perp$. Further, let R be the point of V_{XY} incident with ℓ. Then by Thas [22] the set $(V_{XY}\backslash\{R\})\cup\{\infty\}=\mathcal{O}_V$ is an ovoid of $\mathrm{PG}(3,s)$.

For s odd any ovoid of $\mathrm{PG}(3,s)$ is an elliptic quadric; see e.g. Hirschfeld [8]. For s even, $\mathcal{S}$ is the dual of a flock GQ, so the ovoid is again an elliptic quadric by Thas [22].

Let $\{Z_1,Z_2,X\}$ be a 3-regular triad for which $\{Z_1,Z_2,X\}^\perp$ does not contain a point incident with ℓ. As $\{Z_1,Z_2,X\}$ is 3-regular, we have $|\{Z_1,Z_2,X\}^{\perp\perp}|=s+1$. Let $\{Z_1,Z_2,X\}^{\perp\perp}=\{Z_1,Z_2,\dots,Z_s,X\}$ and let V_i be the unique point incident with ℓ and collinear with Z_i, with $i=1,2,\dots,s$. As $\{Z_1,Z_2,X\}^\perp$ does not contain a point incident with ℓ, the mapping $X\mapsto X$, $Z_i\mapsto V_i$, with $i=1,2,\dots,s$, is a bijection of

$\{Z_1, Z_2, X\}^{\perp\perp}$ onto the pointset of ℓ. Let Y be the point V_1. The elliptic quadrics $\mathcal{O}_{Z_2}, \mathcal{O}_{Z_3}, \ldots, \mathcal{O}_{Z_s}$ of $\mathrm{PG}(3,s)$, defined in the foregoing paragraph, have $\{Z_1, Z_2, X\}^{\perp}$ in common. By Thas [20] the set $\{Z_1, X\}^{\perp}\backslash\{Y\}$ is an affine plane of $\mathrm{AG}(3,s)$. As $\{Z_1, Z_2, X\}^{\perp} \subset \{Z_1, X\}^{\perp}\backslash\{Y\}$ it is a conic $\mathscr{C}$. The tangent plane π_∞ of $\mathcal{O}_{Z_i}$ at ∞, which is the plane at infinity of $\mathrm{AG}(3,s)$, intersects $\mathscr{C}$ in points $V_1, V_2 \in \mathrm{PG}(3,s^2)\backslash\mathrm{PG}(3,s)$. So $\mathcal{O}_{Z_i} \cap \pi_\infty = \ell_\infty^{(i)} \cup m_\infty^{(i)} = \infty V_1 \cup \infty V_2$ is independent of i; let $\ell_\infty^{(i)} = \ell_\infty$ and $m_\infty^{(i)} = m_\infty$, $i = 2, 3, \ldots, s$.

From Thas [22] it now follows that any elliptic quadric $\mathcal{O}_V$, with $X \nsim V \nsim Y$, contains ℓ_∞ and m_∞ (over $\mathrm{GF}(s^2)$). Now, by the proof of Theorem 6.1 in Thas [22], the GQ $\mathscr{S}$ is isomorphic to the classical GQ $Q(5,s)$. $\square$

As a corollary, we obtain the following well-known characterization of the classical GQ $Q(5,s)$, s odd; see Thas [17].

Corollary 3.2. *Let $\mathscr{S}$ be a GQ of order (s,s^2), $s>1$, with s odd. Then $\mathscr{S}$ is isomorphic to the GQ $Q(5,s)$ if and only if it has a 3-regular point.*

Proof. Each point of the GQ $Q(5,s)$ is 3-regular. Now let $\mathscr{S}$ be a GQ of order (s,s^2), $s>1$, with s odd, having a 3-regular point X. Then the conditions in the statement of Theorem 3.1 are satisfied for any flag (X,ℓ) and any triad $\{Z_1, Z_2, X\}$, where $\{Z_1, Z_2, X\}^{\perp}$ does not contain a point incident with ℓ. Hence $\mathscr{S} \cong Q(5,s)$. $\square$

Remark. In the even case, the condition "$\mathscr{S}$ is the dual of a flock GQ" may be replaced by "the ovoids $\mathcal{O}_{Z_2}, \mathcal{O}_{Z_3}, \ldots, \mathcal{O}_{Z_s}$ are elliptic quadrics".

4. Subquadrangles of flock GQs, the even case

In this section, we complete the classification of subquadrangles of order s of flock quadrangles of order (s^2,s), s even.

Let $\mathscr{S} = (\mathscr{P}, \mathscr{B}, \mathrm{I})$ be the point-line dual of a flock GQ of order (s^2,s), s even. If ℓ is a base line of $\mathscr{S}$ (*the* base line if $\mathscr{S}$ is not classical), then let $X \mathrm{I} \ell \mathrm{I} Y$, $X \neq Y$, and let $\{X, X_1, X_2\}$ be a triad with $Y \in \{X, X_1, X_2\}^{\perp}$. Then by Payne [13] the triad $\{X, X_1, X_2\}$ is 3-regular, so, by Theorem 2.3, there arise $s^3 + s^2$ subquadrangles of order s containing the line ℓ; see also Thas [20]. By Lemma 2.4 $s^3 + s^2$ is the maximum number of subquadrangles of order s of a GQ $\mathscr{S}$ of order (s,s^2), $s \neq 1$, containing a given line of $\mathscr{S}$. This gives in an elementary way the following result of O'Keefe and Penttila [11].

Theorem 4.1. *If $\mathscr{S}$ is a flock GQ of order (s^2,s), s even, with base point (∞), then $\mathscr{S}$ contains exactly $s^3 + s^2$ subquadrangles of order s containing the point (∞).*

Now, we classify those subquadrangles not containing the point (∞). In fact, we will show that if a flock quadrangle has such a subquadrangle then it must be the classical generalized quadrangle $H(3,s^2)$. Essential to this proof will be the following result of O'Keefe and Penttila.

Theorem 4.2 (O'Keefe and Penttila [10]). *Let s be even and let* H *be a herd of ovals in* $\mathrm{PG}(2,s)$. *If* H *contains at least one translation oval then it is either classical or FTWKB.*

A herd is called *classical* if the associated GQ is isomorphic to $H(3,s^2)$, which is equivalent to saying that all ovals in the herd are conics [10]. A herd is called *FTWKB* if the associated flock arises from the geometrical construction of Fisher and Thas [7, Theorem 3.10]; in this case the corresponding translation planes were discovered by Walker [28] (using flocks) and independently by Betten [1]. The associated GQ was discovered by Kantor [9].

Let $\mathscr{F}$ be a flock and $\mathscr{S}(\mathscr{F}) = (\mathscr{P},\mathscr{B},\mathrm{I})$ the corresponding flock quadrangle of order (s^2,s), s even, with elation group G about (∞) and associated herd $H = \{\mathscr{O}_1,\dots,\mathscr{O}_{s+1}\}$. From the construction of $\mathscr{S}(\mathscr{F})$ (see [21], for instance) we have that the point (∞) is a centre of symmetry, that is, there is a (maximal) group of s automorphisms of $\mathscr{S}(\mathscr{F})$ that fix $(\infty)^\perp$ elementwise. From [13] $\mathscr{S}(\mathscr{F})$ satisfies Property (G) at the point (∞) and so by Theorem 2.5 the dual net associated with (∞) is isomorphic to H_s^3. Note that any symmetry about (∞) induces the identity on H_s^3.

In Section 2.3, we saw that if H_s^3 is constructed from the line ℓ in $\mathrm{PG}(3,s)$, then for each plane π of $\mathrm{PG}(3,s)$ not containing ℓ there is a unique subquadrangle of order s of $\mathscr{S}(\mathscr{F})$ that contains (∞) as a regular point and for which the associated dual net is π with the pencil of lines on the point $\pi\cap\ell$ removed. Any symmetry about (∞) fixes each of these subquadrangles. In this way we have s^3+s^2 subquadrangles of $\mathscr{S}(\mathscr{F})$ of order s containing the point (∞). O'Keefe and Penttila classified the subquadrangles of order s of $\mathscr{S}(\mathscr{F})$ containing (∞) and each one is isomorphic to $T_2(\mathscr{O}_i)$ for some $i\in\{1,\dots,s+1\}$. Further, for a fixed $i\in\{1,\dots,s+1\}$ suppose that $\mathscr{S}'$ and $\mathscr{S}''$ are two of the s^2 subquadrangles of $\mathscr{S}(\mathscr{F})$ equivalent to $T_2(\mathscr{O}_i)$ (and to each other under the action of G). It is straight-forward to check that $\mathscr{S}'\cap\mathscr{S}''$ is a line (incident with (∞)), the $s+1$ points of $\mathscr{S}'$ and $\mathscr{S}''$ incident with this line, and the lines of $\mathscr{S}'$ and $\mathscr{S}''$ incident with these points. Given this intersection it must be the case that the planes of $\mathrm{PG}(3,s)$ corresponding to $\mathscr{S}'$ and $\mathscr{S}''$ meet ℓ in the *same* point, P_i say. In this way there is a one-to-one correspondence between the elements of the herd and the points of ℓ. Also the s^2 subquadrangles equivalent to $T_2(\mathscr{O}_i)$ correspond to the s^2 planes of $\mathrm{PG}(3,s)$ containing P_i but not ℓ.

So now let $\mathscr{S}'$ be a subquadrangle of $\mathscr{S}(\mathscr{F})$ of order s, not containing the point (∞). By Theorem 2.7 $\mathscr{S}'$ is isomorphic to the dual of $T_2(\mathscr{O})$ for some oval $\mathscr{O}$. In particular, as in the proof of Theorem 2.7, we can consider $\mathscr{O}$ as an oval in the plane

$p(m)$ on the line ℓ in the construction of H_s^3. We now consider the intersection of $\mathcal{S}'$ with the subquadrangles isomorphic to $T_2(\mathcal{O}_i)$ for a fixed i. In particular, let π be a plane of $\mathrm{PG}(3,s)$ containing P_i, but not ℓ, and also two points of $\mathcal{O}$. Let $\mathcal{S}'(\pi)$ be the subquadrangle constructed from π. If we consider the action of the symmetries about (∞) on $\mathcal{S}'$, then we have s distinct subquadrangles that partition the points of $\mathcal{S}\backslash(\infty)^\perp$. Thus, we may assume without loss of generality that $\mathcal{S}'(\pi)$ and $\mathcal{S}'$ have at least one point of $\mathcal{S}(\mathcal{F})\backslash(\infty)^\perp$ in common. Since we also know that $\mathcal{O}$ and π have two points in common, it follows that $\mathcal{S}'(\pi)$ and $\mathcal{S}'$ have two points in common on the line m which is incident with (∞). Now $\mathcal{S}'(\pi)$ and $\mathcal{S}'$ must have a pair of non-collinear points in common and so from Lemma 2.2 it must be that $\mathcal{S}'(\pi)\cong T_2(\mathcal{O}_i)$ has a regular pair of points one of which is affine. Hence by Corollary 2.10 the oval $\mathcal{O}_i$ is a translation oval and the line m corresponds to a point of $\mathcal{O}_i$ on an axis of $\mathcal{O}_i$. By applying Theorem 4.2 H is either classical or FTWKB. Now by [9] if $\mathcal{F}$ is the FTWKB flock, then the group of $\mathcal{S}(\mathcal{F})$ fixing (∞) is transitive on the lines through (∞). By the above arguments it now follows that each point of a herd oval is on an axis of the oval. Hence each oval in the herd is a conic, a contradiction. Thus the herd and flock must be classical and $\mathcal{S}(\mathcal{F})\cong H(3,s^2)$.

We have now proved the following theorem.

Theorem 4.3. *Let $\mathcal{S}$ be a non-classical flock GQ of order (s^2,s), s even, with base point (∞). Then any subquadrangle of $\mathcal{S}$ of order s contains (∞).*

4.1. Alternative proof relying on Theorem 3.1 instead of Theorem 4.2

Let $\mathcal{S}^*$ be the point-line dual of a non-classical flock GQ of order (s^2,s), s even. The base line of $\mathcal{S}^*$ will be denoted by ℓ. Suppose, by way of contradiction, that $\mathcal{S}'$ is a subquadrangle of order s of $\mathcal{S}^*$ which does not contain ℓ. Let X be the point of $\mathcal{S}'$ incident with ℓ and let $Y\mathrm{I}\ell$, $X\neq Y$. Further, let $\mathcal{O}$ be the ovoid of $\mathcal{S}'$ consisting of the s^2+1 points of $\mathcal{S}'$ collinear with Y. Also, let $Z\in\mathcal{O}\backslash\{X\}$, let m be a line of $\mathcal{S}'$ incident with X, let n be a line of $\mathcal{S}'$ incident with Z, and let $m\nsim n$. Let $m\mathrm{I}U\sim Z$ and $n\mathrm{I}V\sim X$. Then by Theorem 2.3 there is a subquadrangle $\mathcal{S}''$ of order s of $\mathcal{S}^*$ containing Y,U,V,X,Z. By Lemma 2.1 $\mathcal{S}'\cap\mathcal{S}''$ is a grid containing the lines m,n. Hence the pair $\{m,n\}$ is regular. Any line of $\mathcal{S}'$ is incident with a point of $\mathcal{O}$, so, for m fixed, $\{m,n\}$ is regular for any line n of $\mathcal{S}'$ not concurrent with m. Consequently m is a regular line of $\mathcal{S}'$, so in $\mathcal{S}'$ any line incident with X is regular. Now by 1.5.2 of Payne and Thas [14], the point X is regular in $\mathcal{S}'$. If W is a point of $\mathcal{S}'$ not collinear with X, then $\{X,W\}^{\perp'}$ and $\{X,W\}^{\perp'\perp'}$ contain $s+1$ points of $\mathcal{S}'$. Let $\{X,W\}^{\perp'}=A$ and $\{X,W\}^{\perp'\perp'}=B$. If $W'\in B\backslash\{X,W\}$, then $\{X,W,W'\}$ is 3-regular in $\mathcal{S}$ and $\{X,W,W'\}^\perp=A$ does not contain a point incident with ℓ. By Theorem 3.1, the GQ $\mathcal{S}^*$ is classical, a contradiction.

5. Subquadrangles of a TGQ with egg good at an element, not containing (∞), the even case

In this section, we will suppose that $\mathcal{S}^{(\infty)} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a TGQ of order (s, s^2), $s \neq 1$, with $\mathcal{S} = T(\mathcal{E})$ and $\mathcal{E}$ good at the element δ. By Theorem 2.6 the dual net $\mathcal{N}_\delta^*$ is isomorphic to H_s^3. Suppose that H_s^3 is constructed from the line ℓ in $\mathrm{PG}(3,s)$ and that p is the covering map from $\mathcal{S} \backslash \delta^\perp$ to H_s^3.

Let G be the translation group of $\mathcal{S}$ about (∞). Then G has order s^4, fixes each line incident with (∞) and acts regularly on the elements of $\mathcal{P} \backslash (\infty)^\perp$. Since G fixes $\mathcal{S}$ and δ, it induces an automorphism group of the dual net $\mathcal{N}_\delta^* = H_s^3$ of order s^3 (by factoring out the symmetries about δ). This group is induced by a collineation group of $\mathrm{PG}(3,s)$ that acts faithfully on H_s^3 and which we will denote by G_δ (we leave the proof of this as an exercise for the reader). If we denote the plane of $\mathrm{PG}(3,s)$ corresponding to the point (∞) by $p(\infty)$, then each element of G_δ must be an axial collineation with axis $p(\infty)$. If $g \in G$ fixes a line of $\delta^\perp$ not incident with (∞), then it follows that g is a symmetry about δ. Hence G_δ acts regularly on the points of $\mathrm{PG}(3,s) \backslash p(\infty)$ and must be the group of elations of $\mathrm{PG}(3,s)$ with axis $p(\infty)$. We will identify G_δ with the corresponding group of $\mathrm{PG}(3,s)$ it induces.

Since $\mathcal{E}$ is good at δ it follows that $\mathcal{S}$ has $s^3 + s^2$ subquadrangles of order s containing both the base point (∞) and the line δ. By Lemma 2.4 $\mathcal{S}$ has no other subquadrangles containing δ. Now suppose that $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ is a subquadrangle of order s of $\mathcal{S}$ not containing the point (∞); then δ cannot be a line of $\mathcal{S}'$. Consequently, by applying Theorem 2.7 it follows that $\mathcal{S}' \cong T_2(\mathcal{O})$, for some oval $\mathcal{O}$ of $p(M) = \mathrm{PG}(2,s)$, where M is the point of δ contained in $\mathcal{S}'$.

Lemma 5.1. *Let $\mathcal{S}^{(\infty)} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a TGQ of order (s, s^2), $s \neq 1$, with $\mathcal{S} = T(\mathcal{E})$ and $\mathcal{E}$ good at the element δ. Suppose that $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ is a subquadrangle of order s of $\mathcal{S}$ not containing the point (∞). Then $\mathcal{S}'$ is isomorphic to the classical GQ $Q(4,s)$.*

Proof. Since by Theorem 2.7 $\mathcal{S}' \cong T_2(\mathcal{O})$, we show that $\mathcal{O}$ is a conic in $p(M)$, with M the unique point of $\mathcal{S}'$ on the line δ. Let V be a point of $p(\infty)$ not on ℓ and let ϕ be any non-trivial elation of $\mathrm{PG}(3,s)$ with axis $p(\infty)$ and centre V. Let $\bar{\phi}$ be an automorphism of $\mathcal{S}$ that induces ϕ in $\mathrm{PG}(3,s)$. Under p the subquadrangle $\mathcal{S}'' = \bar{\phi}(\mathcal{S}')$ is mapped to $T_2(\phi(\mathcal{O}))$ where $\phi(\mathcal{O})$ is an oval in the plane $\phi(p(M))$. Under the action of the symmetry group about δ, the subquadrangle $\mathcal{S}''$ gives rise to s subquadrangles $\mathcal{S}_1'' = \mathcal{S}'', \mathcal{S}_2'', \ldots, \mathcal{S}_s''$ which intersect pairwise in the lines of $\mathcal{S}$ corresponding to the points of $\phi(\mathcal{O})$ and the points on these lines. We now consider the intersection between $\mathcal{S}'$ and $\mathcal{S}_i''$. The only possible lines that lie in both $\mathcal{S}'$ and $\mathcal{S}_i''$, for some i, are mapped by p to a line of $\mathrm{PG}(3,s)$ incident with a point of $\mathcal{O}$ and a point of $\phi(\mathcal{O})$. The subquadrangle $\mathcal{S}'$ contains $s^2 + 2s + 1$ such lines which are partitioned by the $\mathcal{S}_i''$. Now if $\mathcal{S}'$ and $\mathcal{S}_i''$ share such a line, then by Lemma 2.1 they share $s+1$ or $2s+2$ such lines. It follows that there exists a $j \in \{1, \dots, s\}$ such that $\mathcal{S}'$ and $\mathcal{S}''_j$ share $2s+2$ lines, that is, they meet in a grid. Under the covering map p this grid must become a regulus of $\mathrm{PG}(3, s)$, intersecting $p(M)$ in $\mathcal{O}$. It follows that $\mathcal{O}$ must be a conic (see [8, Chapter 15] for instance). $\square$

We now prove that when s is even that given the hypotheses of Lemma 5.1 the GQ $\mathcal{S}$ must be the classical GQ $Q(5, s)$. Note that when s is odd the equivalent result is Theorem 4.3 of [6].

The proof of this theorem will make use of the following ideas. Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is any GQ of order (s, s^2) and $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ a subquadrangle of order s. Let $P \in \mathcal{P} \backslash \mathcal{P}'$, then the elements of $\mathcal{P}'$ collinear with P form an ovoid of $\mathcal{S}'$ (see [14, 2.2.1]). Such an ovoid is said to be *subtended* by the point P. Thas and Payne proved the following theorem regarding subtended ovoids.

Theorem 5.2 (Thas and Payne [24]). *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, s^2), s even, having a subquadrangle $\mathcal{S}'$ isomorphic to $Q(4, s)$. If in $\mathcal{S}'$ each ovoid $\mathcal{O}_X$ consisting of all the points collinear with a given point X of $\mathcal{S} \backslash \mathcal{S}'$ is an elliptic quadric, then $\mathcal{S}$ is isomorphic to $Q(5, s)$.*

The equivalent theorem was proved in the odd case in [3] and a proof valid for both s odd and even may be found in [2,5].

We are now equipped to prove the classification theorem.

Theorem 5.3. *Let $\mathcal{S}^{(\infty)} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a TGQ of order (s, s^2), s even, with $\mathcal{S} = T(\mathcal{E})$ and $\mathcal{E}$ good at the element δ. Suppose that $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ is a subquadrangle of order s of $\mathcal{S}$ not containing the point (∞). Then $\mathcal{S}$ is the classical GQ $Q(5, s)$.*

Proof. By Lemma 2.8 if M is the point incident with the line δ and contained in $\mathcal{S}'$, then $\mathcal{S}' \cong T_2(\mathcal{C})$ where $\mathcal{C}$ is a conic in the plane $p(M)$ of $\mathrm{PG}(3, s)$. Let P be a point of $\mathcal{P} \backslash \mathcal{P}'$, such that $P \not\sim M$ and let Ω_P be the ovoid of $\mathcal{S}'$ subtended by P. We now show that Ω_P is an elliptic quadric ovoid of $\mathcal{S}'$. Let ϕ be a non-trivial elation of $\mathrm{PG}(3, s)$ with axis $p(\infty)$ and centre on the line ℓ. Let $\bar{\phi}$ be an automorphism of $\mathcal{S}$ that induces ϕ in $\mathrm{PG}(3, s)$. By Lemma 2.1, the subquadrangles $\mathcal{S}'$ and $\bar{\phi}(\mathcal{S}')$ can share either 0, 1, 2 or $s+1$ lines on the point M. Consequently, $\mathcal{C}$ and $\phi(\mathcal{C})$ may share 0, 1, 2 or $s+1$ points. However, ϕ cannot fix $\mathcal{C}$ so the possible intersection numbers are 0, 1 and 2. We may choose ϕ such that $|\mathcal{C} \cap \phi(\mathcal{C})| = 2$, say $\mathcal{C} \cap \phi(\mathcal{C}) = \{X, Y\}$. By applying the symmetry group about δ we may assume that P is contained in $\bar{\phi}(\mathcal{S}')$. In this case it follows from Lemma 2.1 that the intersection of $\mathcal{S}'$ and $\bar{\phi}(\mathcal{S}')$ is a grid. Under the covering map p the lines of the grid are mapped to X, Y and the $2s$ lines on X or Y, not in $p(M)$, in a fixed plane π of $\mathrm{PG}(3, s)$. Since the point P of $\mathcal{S}$ is not contained in $\mathcal{S}'$, but is contained in $\bar{\phi}(\mathcal{S}')$ it follows that $p(P) \notin \pi$. Each of the $s+1$ lines of $\bar{\phi}(\mathcal{S}')$ incident with P is incident with a unique point of $\mathcal{S}' \cap \bar{\phi}(\mathcal{S}')$. Looking in $\mathrm{PG}(3, s)$ at the $T_2(\mathcal{C})$ and $T_2(\phi(\mathcal{C}))$ models for $\mathcal{S}'$ and $\bar{\phi}(\mathcal{S}')$, respectively, we see that this set of $s+1$ points must contain the $s-1$ points of the projection of $\phi(\mathscr{C})\backslash\{X,Y\}$ from $p(P)$ onto π, that is $s-1$ points of a conic (completed to a conic by adding X and Y). Under the isomorphism from $T_2(\mathscr{C})$ to $W(s)$ (see [4] for an explicit such isomorphism) these $s-1$ points are also coplanar and contained in a conic. Under this isomorphism the ovoid Ω_P of $T_2(\mathscr{C})$ is mapped to an ovoid Ω of $W(s)$, which by [16] is also an ovoid of $\mathrm{PG}(3,s)$. Consequently, the $s-1$ points of the conic must complete to a conic contained in Ω and by [4] the Ω is an elliptic quadric, and hence Ω_P is an elliptic quadric ovoid of $T_2(\mathscr{C})$.

Now suppose that $P\in\mathscr{P}\backslash\mathscr{P}'$ is such that $P\sim M$ with $P\neq M$. Again let Ω_P be the ovoid of $\mathscr{S}'$ subtended by P. Let n be any line of $\mathscr{S}$ incident with P but not M. Then n meets $\mathscr{S}'$ in a unique point, N say. Each of the $s-1$ points in the set $n\backslash\{P,N\}$ subtends an elliptic quadric ovoid in $\mathscr{S}'$. Since the s points of $n\backslash\{N\}$ subtend s ovoids that partition the points of $\mathscr{S}'$ not collinear with N, it follows that the ovoid subtended by P is determined by the $s-1$ ovoids subtended by the points of $n\backslash\{P,N\}$. Looking in the $Q(4,s)$ or $W(s)$ model of $\mathscr{S}'$ we see that Ω_P is forced to be an elliptic quadric ovoid.

Since each point of $\mathscr{P}\backslash\mathscr{P}'$ subtends an elliptic quadric ovoid in $\mathscr{S}'$, it follows from Theorem 5.2 that $\mathscr{S}$ is isomorphic to the classical GQ $Q(5,s)$. $\square$

Note that a similar proof may be possible in the case when s is odd. However, given that the characterizations of an elliptic quadric ovoid of $Q(4,s)$ are much weaker than in the s even case, any such proof will be more complicated and probably longer than that given in Theorem 4.3 of [6].

6. Subquadrangles of order s of a TGQ with egg good at an element, containing (∞)

Suppose that $\mathscr{S}^{(\infty)}=(\mathscr{P},\mathscr{B},\mathrm{I})$ is a TGQ of order (s,s^2), s even, with $\mathscr{S}=T(\mathscr{E})$ and $\mathscr{E}$ is good at the element δ. Since $\mathscr{E}$ is good at δ it follows that $\mathscr{S}$ has s^3+s^2 subquadrangles of order s containing both the base point (∞) and the line δ. By Lemma 2.4 $\mathscr{S}$ has no other subquadrangles containing δ. By Theorem 2.6 the dual net $\mathscr{N}^*_\delta$ is isomorphic to H^3_s. So if $\mathscr{S}'$ is a subquadrangle of $\mathscr{S}$ of order s containing (∞), then $\mathscr{S}'$ is either one of the s^3+s^2 subquadrangles also containing δ, or does not contain δ and by Theorem 2.7 is isomorphic to $T_2(\mathscr{O})$ for some oval $\mathscr{O}$. The GQ $T_3(\Omega)$, for Ω an ovoid of $\mathrm{PG}(3,s)$, provide examples of GQ with such subquadrangles, and so consequently the classification of such GQ seems perhaps more difficult than in the odd case where there is only the classical example (see [6, Theorem 4.1]). In this case, we will prove only a result on the egg associated with such a GQ.

Theorem 6.1. *Let* $\mathscr{E}=\{\mathrm{PG}(n-1,s),\mathrm{PG}^{(1)}(n-1,s),\ldots,\mathrm{PG}^{(s^{2n})}(n-1,s)\}$ *be an egg of* $\mathrm{PG}(4n-1,s)$ *good at the element* $\mathrm{PG}(n-1,s)$. *Suppose that* $\mathscr{O}$ *is a pseudo-oval contained in* $\mathscr{E}$, *but not containing* $\mathrm{PG}(n-1,s)$, *then* $\mathscr{O}$ *is regular.*

Proof. Since the egg $\mathscr{E}$ contains the pseudo-oval $\mathcal{O}$ it follows that the GQ $T(\mathscr{E})$ contains a subquadrangle isomorphic to $T(\mathcal{O})$ not containing $\mathrm{PG}(n-1,s)$ and having the point (∞) of $T(\mathscr{E})$ as the base point in the construction. By Theorem 2.7 the $T(\mathcal{O})$ is isomorphic to $T_2(\hat{\mathcal{O}})$ for some oval $\hat{\mathcal{O}}$ with (∞) equivalent to the translation point of $T(\mathcal{O})$ in the construction. Hence $\mathcal{O}$ is regular; see e.g. Theorem 5 of [25]. □

Note that this result is valid for both s odd and even, however in the odd case Theorem 4.1 of [6] shows that such a TGQ is classical.

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