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THE GENERATING FUNCTION OF THE NUMBER OF SUBPATTERNS OF A D0L SEQUENCE

Peter JOHANSEN
Datalogisk Institut, Københavns Universitet, Sigurdsgade 41, DK-2200 København N, Denmark

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Abstract. The generating function of the number of subpatterns of a D0L sequence is shown to be rational. The computation of the generating function is based on a recursion formula which expresses this function by the generating functions of subpatterns of smaller length and by the Magnus transform of the homomorphism.

1. Introduction

The goal of the analysis of algorithms is to be able to predict the behaviour of an algorithm based on its description. Any algorithm can be written in normal form as

$$\textit{while} \quad B(S) \quad \textit{do} \quad S := f(S)$$

where S is a suitable state space and where f is a function expressible by conditional statements and basic operators. A very basic problem is then to study the behaviour of the sequence of states which is formed by iteration of a single function on this state space. A D0L sequence is such a sequence where the space is the set of words Σ^* on a finite set Σ, and where the function to be iterated is a homomorphism of this state space with respect to concatenation. D0L sequences have been studied extensively. A survey of L systems can be found in Rozenberg and Salomaa [6]. In programming terms, this situation is a model of a simple macro expansion; the homomorphism is a set of macros without arguments; a single application to a string of letters is a one level macro expansion of this string, and the nth iterated application is the complete n level macro expansion.

This paper studies the subpatterns of a D0L sequence. First it is shown how the formal sum of subpatterns of a word can be expressed as the result of a simple algebraic transformation. Then the subpatterns of the homomorphic image of a word are found from the subpatterns of the word. Finally the generating function of a given subpattern is found from a recursion equation. The generating function is rational. As a corollary it is shown decidable if two D0L sequences passes the same number of all subpatterns of length at most N.

For any subpattern. the coefficients of the generating function depend only on the initial word and on the homomorphism. Formulas for the computation of the coefficients are given.

During revision of the present paper the author learned that Ochsenschläger [4] independently has found Theorem 4.1 and the Corollaries 3.13, 3.14 and 4.2.

2. Preliminaries

Let $\Sigma\{\sigma_1,\ldots,\sigma_k\}$ be a finite set. Let $A(\Sigma)$ be the ring of polynomials with integer coefficients in the noncommuting variables of Σ. Let the unit element be 1 and the null element 0. A *monomial* of degree n is a polynomial which consists of a single term $ca_1\ldots a_n$ where $a_i\in\Sigma$ and where c is an integer. Let $P(\Sigma)$ be the ring of formal power series.

Σ^* can be considered a subset of $A(\Sigma)$ by identifying any word w with the monomial $1\cdot w$. The empty word Λ is identified with 1.

Let X_N be the subring generated by all monomials of degree greater than or equal to N. X_N is an ideal. Denote the quotient ring $A(\Sigma)/X_{N+1}$ by $A_N(\Sigma)$. The dimension of $A_N(\Sigma)$ is

$$1+|\Sigma|+|\Sigma|^2+\cdots+|\Sigma|^N.$$

$A(\Sigma)$ is a vector space. A basis consists of all monomials with coefficient 1. It can be written $1\cup\Sigma\cup\Sigma^2\cup\cdots$. To a ring homomorphism f corresponds a linear mapping of the vector space.

If $f(X_{N+1})\subset X_{N+1}$ then a linear mapping is induced in the quotient space $A_N(\Sigma)$. A basis of the vector space $A_N(\Sigma)$ consists of $\{w+X_{N+1}: w$ is a monomial of degree $\leq N$ with coefficient $1\}$. $w+X_{N+1}$ is often denoted simply by w.

Example 2.1. Let

$$g=\begin{cases} a\to b\\ b\to a+b+ab\end{cases}$$

and consider the induced map in $A_2(\Sigma)$. A basis of $A_2(\Sigma)$ is $\{1, a, b, aa, ab, ba, bb\}$. Strictly speaking the basis is $\{1+X_3, a+X_3, b+X_3,\ldots, bb+X_3\}$.

The induced map is described by a matrix. The columns are the images of the base vectors. The column index is the base vectors and the entries are the coefficients of the base vectors occurring as row indices.

$$\begin{array}{c|ccccccc} & 1 & a & b & aa & ab & ba & bb \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ b & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ aa & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ ab & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ ba & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ bb & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}$$

$$g(ab) \equiv b(a+b+ab) \equiv ba + bb \mod X_3,$$

$$g(ba) \equiv (a+b+ab)b \equiv ab + bb \mod X_3,$$

$$g(bb) \equiv (a+b+ab)(a+b+ab) \equiv aa + ab + ba + bb \mod X_3.$$

A monomial with coefficient 1 and degree n is often called a tensor of degree n. Often product is then denoted by $\otimes$. Let V be a vector space with base B. $V \otimes V$ is defined as the vector space with base B^2.

$$\underbrace{V \otimes \cdots \otimes V}_{n \text{ times}} \text{ is}$$

the vector space with B^n.

Let $f: U_1 \to U_2$ and $g: V_1 \to V_2$ be linear maps. $f \otimes g: U_1 \otimes V_1 \to U_2 \otimes V_2$ is defined by $f \otimes g(u, v) = f(u) \otimes g(v)$. If $U_1 = U_2$, then $f \otimes f$ and

$$f^{\otimes n} = \underbrace{f \otimes \cdots \otimes f}_{n \text{ times}}$$

is defined.

Example 2.2. Let B be $\{a, b\}$. $V \otimes V$ has basis $\{a \otimes a, a \otimes b, b \otimes a, b \otimes b\}$. Let f be described by the matrix

$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$

$f \otimes f$ is described by the matrix

$$\begin{array}{cc} & \begin{array}{cccc} aa & ab & ba & bb \end{array} \\ \begin{array}{c} aa \\ ab \\ ba \\ bb \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Here juxtaposition is used instead of $\otimes$.

3. Subpatterns and the Magnus transform

The subpattern representation of a word is defined. An algebraic transformation from a word to its subpattern representation is shown. The mapping which takes a subpattern representation of its homomorphic image is found and it is shown how to compute it.

A *n-subpattern* of the word $w=a_1\ldots a_m$, $a_i\in\Sigma$ is constructed by omitting $m-n$ occurrences of letters. There exists 2^m subpatterns of w each corresponding to a particular choice of letters. The number of occurrences of v as subpattern in w is denoted $\binom{w}{v}$ by Eilenberg [1, p. 238].

Example 3.1. $w=abaa$. ba is a 2-subpattern which can be constructed in two ways; by omitting the first and last letters or by omitting the first and third letters.

b is a 1-subpattern.
a is a 1-subpattern.
a is a 1-subpattern which occurs 3 times.

The *subpattern representation* of a word w is an element of $A(\Sigma)$. The coefficient of each subpattern is the number of times that subpattern occurs in w. Thus the subpattern representation of w can be expressed as $\sum_{v\in\Sigma^*}\binom{w}{v}v$.

Example 3.2. The subpattern representation of baa equals $1+2a+b+aa+2ba+baa$. The subpattern λ occurs once: if all letters are omitted. It is denoted by the unit element of $A(\Sigma)$.

Let $M:A(\Sigma)\to A(\Sigma)$ be the ring homomorphism defined by $M(\sigma)=1+\sigma$ for $\sigma\in\Sigma$. M is known as the Magnus transform. It is introduced in Magnus *et al.* [3].

The next lemma shows that the Magnus transform of a word equals its subpattern representation.

Lemma 3.3 *Let $w\in\Sigma^*$. The coefficient of the monomial v in $M(w)$ is* $\binom{w}{v}$.

Proof. Let $w=a_1\ldots a_n$. Then $M(w)=(1+a_1)\ldots(1+a_n)$. This product results in a sum of 2^n terms. Each term is obtained by selecting either 1 or a_i from the factor $(1+a_i)$ and so corresponds to the subpattern which omits the letters corresponding to the factors where 1 has been selected.

Corollary 3.4. *The subpattern representation of the product uv is the product of the subpattern representations of u and v.*

Proof. $M(uv)=M(u)M(v)$.

Example 3.5.

$$M(baa)=(1+b)(1+a)(1+a)=1+b+2a+aa+2ba+baa$$

which equals the subpattern representation of *baa*.

Lemma 3.6. *The Magnus transform is a ring isomorphism.*

Proof. The homomorphism

$$N:A(\Sigma)\to A(\Sigma)$$

defined by $N(\sigma_i)=\sigma_i-1$ has the property that $MN(\sigma_i)=\sigma_i$ and $MN(\sigma_i)=\sigma_i$. The lemma follows. N is the inverse of M.

Next the subpattern representation of the homomorphic image of a word is characterized.

Notation 3.7. Let f be a homomorphism. Following Hall [2] we let f^M denote MfM^{-1}. f^M therefore takes the subpattern representation of a word to the subpattern representation of the homomorphic image of that word. f^M is shown in the following diagram.

$$\begin{tikzcd}
A(\Sigma) \arrow[r, dash, "M"] \arrow[d, "f"'] & A(\Sigma) \arrow[d, "f^M"] \\
A(\Sigma) \arrow[r, dash, "M"'] & A(\Sigma)
\end{tikzcd}$$

Lemma 3.8 $f^M(\sigma_i)=M(f(\sigma_i))-1;\ i=1,2,\ldots,k.$

Proof. $f^M(\sigma_i)=MfM^{-1}(\sigma_i)=Mf(\sigma_i-1)=M(f(\sigma_i)-1)=Mf(\sigma_i)-1.$

Example 3.9.

$$f=\begin{cases} a\to b \\ b\to ab \end{cases} \qquad f^M=\begin{cases} a\to b \\ b\to a+b+ab \end{cases}$$

$$\begin{tikzcd}
ba \arrow[r, dash, "M"] \arrow[d, "f"'] & 1+a+b+ba \arrow[d, "f^M"'] \\
abb \arrow[r, dash, "M"] & 1+a+2b+2ab+bb+abb
\end{tikzcd}$$

By computation one verifies that

$$\begin{aligned} f^M(1+a+b+ba) &= MfM^{-1}(1+a+b+ba) \\ &= 1+a+2b+2ab+bb+abb. \end{aligned}$$

Lemma 3.10. $f^M(X_N) \subset X_N$.

Proof. Let $\Sigma' = \{\sigma \in \Sigma : f(\sigma) = \lambda\}$. Notice that if $f(\sigma) = \lambda$, then $f^M(\sigma) = 0$ by Lemma 3.8; if $f(\sigma) \neq \lambda$, then $f^M(\sigma)$ contains only monomials of degree 1 or more. Let next w be a monomial of degree n. If w contains letters from Σ', then $f^M(w) = 0$ otherwise $f^M(w)$ contains only monomials of degree N or more.

Corollary 3.11. *For all N, f^M induces a linear mapping f_N^M in the quotient vectorspace $A_N(\Sigma)$.*

Let us look closer at the matrix of f_N^M expressed in the bases $\Sigma \cup \Sigma^{\otimes 2} \cup \ldots \cup \Sigma^{\otimes N}$

$$\begin{array}{cccccc} & \longleftarrow \Sigma \longrightarrow & \longleftarrow \Sigma^{\otimes 2} \longrightarrow & & \longleftarrow \Sigma^{\otimes N} \longrightarrow & \\ f_N^M = & \left(\begin{array}{c|c} \boxed{M_{1,1}} & \boxed{} \\ \boxed{M_{2,1}} & \boxed{M_{2,2}} \\ \vdots & \\ \boxed{M_{N,1}} & \boxed{M_{N,2}} \end{array}\right. & & & \left.\begin{array}{c} \text{all zeroes} \\ \\ \\ \boxed{M_{N,N}} \end{array}\right) & \end{array}$$

The column indices of $M_{i,j}$ are the tensors of degree j and the row indices are the tensors of degree i. $M_{i,j}$ is therefore a $|\Sigma|^i \times |\Sigma|^j$ matrix. Each column of $M_{i,j}$ is the coefficients of the i-subpatterns of the image under f^M of the tensor of degree j which is the index of that column. By Lemma 3.10 f_N^M is zero above the box-diagonal. $M_{N,1}$ is all zeroes if N exceeds the maximal length of $f(\sigma)$ for all $\sigma \in \Sigma$. For convenience let $M_{i,1}$ be denoted by M_i. $M_{i,j}$ for $j > 1$ is uniquely determined by the first column of the matrices $M_{i,1}$. The next lemma states the explicit dependence and Corollary 3.14 yields a recursion formula for the computation of $M_{i,j}$.

Lemma 3.12.

$$M_{i,j} = \sum_{\substack{i_1+\ldots+i_j=1 \\ i_1>0,\ldots,i_j>0}} M_{i_1} \otimes M_{1_2} \otimes \cdots \otimes M_{i_j}.$$

Proof. The lemma is proved by induction in j. For $j=1$ the lemma states that $M_{i,1}=M_i$. Assume the lemma true for j and consider $M_{i,j+1}(w\sigma)$ where length $(w\sigma)=j+1$. $M_{i,j+1}(w\sigma)$ equals the degree i component of $f^M(w\sigma)$. Since $f^M(w\sigma)=f^M(w)f^M(\sigma)$,

$$M_{i,j+1}(w\sigma)=M_{i-1,j}(w)M_1(\sigma)+M_{i-2,j}(w)M_2(\sigma)+\cdots+M_{1,j}(w)M_{i-1}(\sigma).$$

The induction hypothesis finishes the proof.

Corollary 3.13. $M_{i,1}=M_1^{\otimes i}$.

Corollary 3.14. *The characteristic polynomial of f_N^M is the product of the characteristic polynomials of $M_1^{\otimes i}$, $i=1,2,\ldots,N$.*

Corollary 3.15.

$$M_{i,1}=0 \text{ for } i>\max_{\sigma\in\Sigma}\{|f(\sigma)|\}$$

$$M_{i,j}=M_{i-1,j-1}\otimes M_{1-2,j-1}\otimes M_2+\cdots+M_{j-1,j-1}\otimes M_{i-j+1}$$

Proof. Follows from the proof of Lemma 3.12.

4. Application to iterated homomorphisms

In this section a recursion formula is found for the generating function of the number of subpatterns in a sequence of iterated images by a homomorphism. First a rational expression for the generating function is found in Theorem 4.1. Then the N-subpattern equivalence problem is shown decidable. Finally a recursion formula is found making use of the lower diagonal block form of f_N^M. This result is formulated as Theorem 4.3. An example finishes the section. It illustrates how the generating functions can be computed knowing the characteristic vectors of f_1^M.

Let us proceed to find the generating function for the number of subpatterns of length N. Let $w\in\Sigma^*$ and let u_N be the canonical image of $M(w)$ in $A_N(\Sigma)$. Hence u_N is the formal sum of all subpatterns of w of length at most N, each subpattern v with coefficient $\binom{w}{v}$. Let U_N be the sequence of elements from $A_N(\Sigma)$

$$U_N=u_N, f_N^M u_N, \ldots, (f_N^M)^p u_N, \ldots.$$

$(f_N^M)^p u_N$ is the formal sum of all subpatterns of $f^p(w)$, each with its subpattern multiplicity as coefficient.

By the generating function of a sequence of vectors is simply meant the vector of generating functions of the elements of the vector. With this understanding, the generating function of U_N is the formal sum

$$u_N+f_N^M u_N y+\cdots+(f_N^M)^p u_N y^p+\cdots.$$

Let this formal sum be called $F_N(y)$.

Let E be the identity matrix and let A be any square matrix. Then the following identity holds

$$(E-Ay)^{-1}=E+Ay+A^2y^2+\cdots+A^py^p+\cdots.$$

Applying the identity using $A=f_N^M$ permits us to state the next theorem.

Theorem 4.1. *Let $F_N(y)$ be the generating function of the sequence of vectors of subpatterns of length at most N for the sequence*

$$w, f(w), \ldots, f^p(w), \ldots.$$

Then $F_N(y)=(E-f_N^My)^{-1}u_N$ where u_N is the image of $M(w)$ in $A_N(\Sigma)$.
In particular the generating function is rational.

Let f and g be two homomorphisms of Σ^* into Σ^*. Let $w \in \Sigma^*$. Let us define the *N-subpattern equivalence problem*: Consider the sequences

$$w, f(w), \ldots, f^p(w), \ldots,$$

$$w, g(w), \ldots, g^p(w), \ldots.$$

Are the number of occurrences of all subpatterns of length at most N identical for the two sequences?

Corollary 4.2. *The N-subpattern equivalence problem is decidable.*

Proof. The procedure simply is to find the generating functions and to decide if

$$(E-f_N^My)^{-1}u_N=(E-g_N^My)^{-1}u_N.$$

This can be done since both are rational: each is a quotient between two polynomials.

$$\frac{P(y)}{Q(y)}=\frac{P'(y)}{Q'(y)} \quad \text{if and only if} \quad P(y)Q'(y)=P'(y)Q(y).$$

The problem then reduces to the identity of two polynomials, which is a decidable problem.

This result is a generalization of the result by Paz and Salomaa [5] on the decidability of the growth equivalence problem for D0L sequences.

Next a recursion formula for $F_N(y)$ is found by exploring the particular simple form of the matrix for f_N^M.

First we observe that if a matrix can be written as a block matrix

$$A=\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

with all zeroes in the upper right hand corner, then the inverse matrix is found by the block matrix

$$A^{-1}=\left(\begin{array}{c|c} B^{-1} & 0 \\ \hline -D^{-1}CB^{-1} & D^{-1}\end{array}\right).$$

Applied to $M(f)_{N+1}$ we may write

$$M(f)_{N+1}=\left(\begin{array}{c|c} f_N^M & 0 \\ \hline M_{N+1,1}|M_{N+1,2}|\cdots|M_{N+1,N} & M_{N+1,N+1}\end{array}\right).$$

Here the vertical bar has been used to denote horizontal juxtaposition of matrices.

$$\begin{aligned}F_{N+1}(y)&=(E-f_{N+1}^My)^{-1}u_{N+1}\\ &=\left(\begin{array}{c|c} E-f_N^My & 0 \\ \hline -(M_{N+1,1}|\cdots|M_{N+1,N})y & (E-M_{N+1,N+1}y)\end{array}\right)^{-1}u_{N+1}\\ &=\left(\begin{array}{c|c} (E-f_N^My)^{-1} & 0 \\ \hline (E-M_{N+1,N+1}y)^{-1}(M_{N+1,1}|\cdots|M_{N\times 1,N})y(E-f_N^My)^{-1} & (E-M_{N+1,N+1}y)^{-1}\end{array}\right)u_{N+1}.\end{aligned}$$

We next define $v_1=u_1$ and for $N\geq 1$, v_{N+1} is the vector in $|\Sigma|^{\otimes N+1}$ such that $u_{N+1}=(u_N, v_{N+1})$. In this way the vector u_{N+1} is split into the vector of homogeneous components

$$u_N^{+1}=(v_1,\ldots,v_{N+1}).$$

Substituting $u_{N+1}=(u_N, v_{N+1})$ we get

$$F_{N+1}(y)=\begin{pmatrix}(E-f_N^My)^{-1}u_N\\ (E-M_{N+1,N+1}y)^{-1}(M_{N+1,1}|\cdots|M_{N+1,N})y(E-f_N^My)^{-1}u_N\\ \qquad+(E-M_{N+1,N+1}y)^{-1}v_{N+1}\end{pmatrix}$$

$$F_{N+1}(y)=\begin{pmatrix}F_N(y)\\ (E-M_1^{\otimes N+1}y)^{-1}(M_{N+1,1}|\cdots M_{N+1,N})yF_N(y)+(E-M_1^{\otimes N+1}y)^{-1}v_{N+1}\end{pmatrix}.$$

An alternative way to express the recursion is found by splitting the vector $F_{N+1}(y)$ of generating functions into the vectors $g_i(y)$ of generating functions for subpatterns of degree i.

$$F_{N+1}(y)=(g_1(y),\ldots,g_{N+1}(y)).$$

This result is formulated as a theorem:

Theorem 4.3. *Let $g_i(y)$ be the generating function of the sequence of vectors of subpatterns of length i for the sequence*

$$w, f(w),\ldots,f^i(w),\ldots.$$

Let v_i be the vector of subpatterns of length i of $M(w)$. Then

$$g_{N+1}(y)=(E-M_1^{\otimes N+1}y)^{-1}(M_{N+1,1}yg_1(y)+\cdots+M_{N+1,N}yg_N(y)+v_{N+1}).$$

So far the basis, in which f_N^M has been expressed, has been $\Sigma\cup\Sigma^{\otimes 2}\cup\cdots\cup\Sigma^{\otimes n}$. For the computation of the generating functions it is, however, more convenient to use another basis in order to make the matrix inversion as easy as possible. Let therefore B be the basis in which f_1^M is in rational canonical form. As a special case, if f_1^M has distinct characteristic values, then B can be chosen as the set of associated characteristic vectors. If f_N^M is expressed in the basis $B\cup B^{\otimes 2}\cup\cdots\cup B^{\otimes N}$, then the matrices $(E-M_1^{\otimes n+1}y)$ are much easier to invert. In the mentioned special case, all matrices are diagonal matrices.

Example 4.4. Let us consider the computation of the generating functions for subpatterns of length 1 and 2 for the homomorphism $f=\begin{cases} a\to b\\ b\to ab\end{cases}$ and $w=ba$. f_2^M is the homomorphism of Example 2.1 from which M_1, $M_{2,1}$ and $M_{2,2}$ is read. To choose a basis, let us compute the characteristic values of f_1^M.

$$\left|zE-f_1^M\right|=\begin{vmatrix} z & -1\\ -1 & z-1\end{vmatrix}=z(z-1)-1=z^2-z-1$$

$$=(z-\zeta)(z-\mu)\quad\text{where } \zeta,\mu=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{5}.$$

$e_1=(1,\zeta)$ and $e_2=(1,\mu)$ are characteristic vectors and constitute B.

$$Q=\begin{pmatrix}1 & 1\\ \zeta & \mu\end{pmatrix},\qquad Q^{-1}=\frac{1}{\mu-\zeta}\begin{pmatrix}\mu & 1\\ -\zeta & 1\end{pmatrix},$$

$$g_1(y)=-(E-M_1y)^{-1}v_1.$$

In basis B, v_1 has the coordinates $Q^{-1}\begin{pmatrix}1\\1\end{pmatrix}$ and M_1 is a diagonal matrix with diagonal entries ζ and μ.

$$g_i'(y)=\frac{-1}{(1-\zeta y)(1-\mu y)}\begin{pmatrix}\dfrac{1}{1-\zeta y} & 0\\ 0 & \dfrac{1}{1-\mu y}\end{pmatrix}\frac{1}{\mu-\zeta}\begin{pmatrix}1+\mu\\ 1-\zeta\end{pmatrix}$$

$$=\frac{-1}{(\mu-\zeta)}\cdot\frac{1}{(1-\zeta y)(1-\mu y)}\begin{pmatrix}\dfrac{1+\mu}{1-\zeta y}\\ \dfrac{1-\zeta}{1-\mu y}\end{pmatrix}.$$

In order to get the function in base Σ one just has to postmultiply by Q.

Written formally we have

$$g_1(y) = -(E - M_1 y)^{-1} v_1 = -Q \cdot Q^{-1}(E - M_1 y)^{-1} Q \cdot Q^{-1} v_1$$
$$= -Q \cdot (E - Q^{-1} M_1 Q y)^{-1} Q^{-1} v_1.$$

If we include tensors of degree 2 the basis becomes $B \cup B \otimes B$. The coordinate transformations are

$$P = \begin{pmatrix} Q & 0 \\ O & Q \otimes Q \end{pmatrix} \quad \text{and} \quad P^{-1} = \begin{pmatrix} Q^{-1} & 0 \\ 0 & (Q \otimes Q)^{-1} \end{pmatrix}.$$

The transformed homomorphism is

$$P^{-1} f_2^M P = \begin{pmatrix} Q^{-1} M_1 Q & 0 \\ (Q \otimes Q)^{-1} M_2 Q & (Q \otimes Q)^{-1}(M_2 \otimes M_1)(Q \otimes Q) \end{pmatrix},$$

$$g_2(y) = (E - M_1 \otimes M_1 y)^{-1} (M_{2,1} y g_1(y) - v_2).$$

In order to avoid the inversion we transform as before:

$$g_2(y) = (Q \otimes Q) \cdot (E - (Q \otimes Q)^{-1}(M_1 \otimes M_1)$$
$$\times (Q \otimes Q))^{-1} \cdot ((Q \otimes Q)^{-1} M_{2,1} y Q Q^{-1} g_1(y) - (Q \otimes Q)^{-1} v_2).$$

Here we only have to invert a diagonal matrix as well as $Q \otimes Q$. By the properties of a tensor product $(Q \otimes Q)^{-1} = Q^{-1} \otimes Q^{-1}$. $g_2(y)$ can now be found using tensor product, matrix multiplication and difference. The calculations are straightforward and are not shown.

5. Conclusion

The transform of a homomorphism induced by the Magnus transform is introduced. The first basic observation of this paper is expressed by Lemma 3.10 which says that the transform of a homomorphism never decreases the length of a word except possibly to zero. A consequence is that the transform can be described by a lower diagonal block matrix. Using matrix techniques the structure of this matrix is exhibited in Lemma 3.12 and Corollary 3.15. The second basic observation is the exploitation of the lower diagonal block matrix form to deduce the recursion formula of Theorem 4.3 for the generating function of a subpattern.

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