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Schauder bases and approximation property in fuzzy normed spaces

Yılmaz Yılmaz

Inonu University, Faculty of Arts and Sciences, Department of Mathematics, 44280, Malatya, Turkey

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ABSTRACT

Our aim in this article is to introduce and study the notion of weak and strong Schauder bases in fuzzy normed spaces. Further, we introduce strong and weak fuzzy approximation properties and set a relationship between these two new notions which may provide an acceleration to the structural analysis of fuzzy normed spaces.

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1. Introduction

Let X be a Banach space and $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements of X. If for every $x \in X$, there exists a unique sequence $\{a_n\}_{n=1}^{\infty}$ of scalars such that

$$\lim_{n\to\infty} \left\| x - \sum_{k=1}^{n} a_k x_k \right\| = 0,$$

then $\{x_n\}_{n=1}^{\infty}$ is called a Schauder basis for X where $\|\cdot\|$ is the norm of X.

Schauder bases play an important role in the structural investigation of Banach spaces of infinite dimensions. It coincides with the classical algebraic (Hamel) basis in finite dimensions while they are different in infinite dimensions. A Banach space may not have a Schauder basis whereas it necessarily has a Hamel basis. However, Schauder bases are more important and functional than Hamel bases in almost all investigations of infinite dimensional Banach spaces and of the operators acting on them. They are especially important in applications to operator equations on Banach or Hilbert spaces modelling many abstract family of problems in science. The solution of these problems in general consists of finding inverse of a given function by means of an operator. This function, for instance, is usually known as the load function in differential problems. If the Banach space of such functions has a Schauder basis one can easily construct an approximation of the load function by using the element of the Schauder basis of the space. So, the solution of the problem in an operator equation is obtained as the norm limit of a sequence of approximations (see, for example, [1]), and also one can establish an estimation of error.

By modifying his own studies on fuzzy topological vector spaces, Katsaras [2] first introduced the notion of fuzzy seminorm and norm on a vector space. Then, Felbin [3] gave the concept of a fuzzy normed space (FNS, for short) by applying the notion of fuzzy distance of Kaleva and Seikala [4] on vector spaces. Independently, Cheng and Mordeson [5] considered a fuzzy norm on a linear space whose associated metric is of Kramosil and Michalek type [6]. Further, Xiao and Zhu [7] improved slightly Felbin's definition of fuzzy norm of a linear operator between FNSs. Recently, Bag and Samanta [8] has given another notion of boundedness in FNS and introduced another type of boundedness of operators. With the novelty of

E-mail address: yyilmaz44@gmail.com.

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their approach, they could introduce fuzzy dual spaces and some important analogues of fundamental theorems in classical functional analysis [9]. Another important contribution to these results has come from Lael and Nourouzi [10] when they have introduced and investigated fuzzy compact operators. We aim in this article to give a contribution to the studies on FNSs by introducing weak and strong fuzzy basis notions.

2. Basic definition and results

$\mathbb{C}$ and $\mathbb{R}$ will denote the set of all complex and real numbers, respectively, in the context.

Definition 1 (*[8]*). A fuzzy subset N of $X \times \mathbb{R}$ is called a fuzzy norm on X if the following conditions are satisfied for all $x, y \in X$ and $c \in \mathbb{R}$;

(N.1) $N(x, t) = 0$ for all non-positive $t \in \mathbb{R}$,
(N.2) $N(x, t) = 1$ for all $t \in \mathbb{R}^+$ if and only if $x = 0$,
(N.3) $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ for all $t \in \mathbb{R}^+$ and $c \neq 0$,
(N.4) $N(x + y, t + s) \geq \min\{N(x, t), N(y, s)\}$ for all $s, t \in \mathbb{R}$,
(N.5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$, and $\sup_{t\in\mathbb{R}} N(x, t) = 1$.

The pair (X, N) will be referred to as a fuzzy normed space (FNS).

For some normed space $(X, \|\cdot\|)$, the functions

$$N_1(x, t) = \begin{cases} \dfrac{t}{t + \|x\|}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases} \quad \text{and} \quad N_2(x, t) = \begin{cases} 1, & \text{if } t > \|x\| \\ 0, & \text{if } t \leq \|x\| \end{cases}$$

are fuzzy norms on X.

Definition 2 (*[8]*). Let (X, N) be a FNS and $U \subset X$. U is said to be fuzzy open if for each $x \in U$ there exist some $t > 0$ and some $\alpha \in (0, 1)$ such that $B(x, \alpha, t) \subseteq U$ where $B(x, \alpha, t) = \{y : N(x - y, t) > 1 - \alpha\}$.

Theorem 1 (*[8]*). *Suppose (X, N) is a FNS with the condition*

(N.6) $N(x, t) > 0$ *for all $t \in \mathbb{R}^+$ implies that $x = 0$.*

Let $\|x\|_\alpha = \inf\{t > 0 : N(x, t) \geq \alpha\}$, for each $\alpha \in (0, 1)$. Then $\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of norms on X. These norms are called α-norms on X corresponding to fuzzy norm N.

Proposition 1 (*[10]*). *Let (X, N) be a FNS satisfying* (N.6) *and $\{x_n\}$ be a sequence in X. Then $\lim N(x_n - x, t) = 1$ if and only if $\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$ for all $\alpha \in (0, 1)$.*

Definition 3 (*[9]*). Let (X, N_1) and (Y, N_2) be two FNSs and $f : X \to Y$ be a mapping.

(1) f is called weakly fuzzy continuous at $x_0 \in X$ if for given $\varepsilon > 0$ and $\alpha \in (0, 1)$, there exists some $\delta = \delta(\varepsilon, \alpha) > 0$ such that, for all $x \in X$,

$$N_1(x - x_0, \delta) \geq \alpha \text{ implies } N_2(f(x) - f(x_0), \varepsilon) \geq \alpha.$$

(2) f is called strongly fuzzy continuous at $x_0 \in X$ if, for given $\varepsilon > 0$, there exists some $\delta = \delta(\varepsilon) > 0$ such that, for all $x \in X$,

$$N_2(f(x) - f(x_0), \varepsilon) \geq N_1(x - x_0, \delta).$$

(3) Let f be linear. f is called weakly fuzzy bounded on X if for every $\alpha \in (0, 1)$, there exists some $m_\alpha > 0$ such that, for all $x \in X$,

$$N_1\left(x, \frac{t}{m_\alpha}\right) \geq \alpha \text{ implies } N_2(f(x), t) \geq \alpha, \quad \forall t > 0.$$

The set of all these operators is denoted by $F'(X, Y)$ and it is a vector space.

(4) Let f be linear. f is called strongly fuzzy bounded on X if for every $\alpha \in (0, 1)$, there exists some $M > 0$ such that, for all $x \in X$,

$$N_2(f(x), t) \geq N_1\left(x, \frac{t}{M}\right), \quad \forall t > 0.$$

The set of all these operators is denoted by $F(X, Y)$ and it is a vector space.

Theorem 2 (*[9]*). *Let (X, N_1) and (Y, N_2) be two FNSs and $f : X \to Y$ be a linear mapping. Then f is strongly (weakly) fuzzy continuous if and only if it is strongly (weakly) fuzzy bounded.*

Following condition will be used in the context;

(N.7) For $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$ and strictly increasing on $\{t : 0 < N(x, t) < 1\}$.

Theorem 3 ([9]). *Let (X, N_1) and (Y, N_2) be two FNSs satisfying (N.6) and (N.7) and $f : X \to Y$ be a linear mapping. Then*

(1) *f is weakly fuzzy bounded if and only if it is bounded w.r.t. α-norms of N_1 and N_2, for each $\alpha \in (0, 1)$.*
(2) *f is strongly fuzzy bounded if and only if it is uniformly bounded w.r.t. α-norms of N_1 and N_2. That is, there exists some $M > 0$ (independent of α) such that $\|f(x)\|_\alpha \le M\|x\|_\alpha$, for all $\alpha \in (0, 1)$.*

3. Convergence and closure in FNSs

Definition 4. Let $\{x_n\}$ be a sequence in an FNS (X, N). Then

(1) It is said to be weakly fuzzy convergent to $x \in X$ and denoted by $\xrightarrow{wf} x$ iff, for every $\alpha \in (0, 1)$ and $\varepsilon > 0$, there exists some $n_0 = n_0(\alpha, \varepsilon)$ such that $n \ge n_0$ implies $N(x_n - x, \varepsilon) \ge 1 - \alpha$.
(2) It is said to be strongly fuzzy convergent to $x \in X$ and denoted by $x_n \xrightarrow{sf} x$ iff, for every $\alpha \in (0, 1)$, there exists some $n_0 = n_0(\alpha)$ such that $n \ge n_0$ implies $N(x_n - x, t) \ge 1 - \alpha$, for all $t > 0$.

Hence, the definitions of a sf(wf)-Cauchy sequence, sf(wf)-closure of a subset and a sf(wf)-complete FNS can be given in a similar way as in classical normed spaces.

Proposition 2 ([11]). *Let $\{x_n\}$ be a sequence in the FNS (X, N) satisfying (N.6). Then*

(1) $x_n \xrightarrow{wf} x$ *iff, for each $\alpha \in (0, 1)$,*

$$\lim_{n\to\infty} \|x_n - x\|_\alpha = 0.$$

(2) $x_n \xrightarrow{sf} x$ *iff*

$$\lim_{n\to\infty} \|x_n - x\|_\alpha = 0 \quad \textit{uniformly in } \alpha$$

where $\|\cdot\|_\alpha$ are α-norms of N.

It is obvious that, if a sequence is sf-convergent then it is wf-convergent to the same point, but not conversely.

Example 1. Let $X = \mathbb{C}$ and consider the fuzzy norm

$$N(z, t) = \begin{cases} \dfrac{t - |z|}{t + |z|}, & \text{if } t > |z| \\ 0, & \text{if } t \le |z| \end{cases}$$

on X. We can find α-norms of N since it satisfies (N.6) condition. Thus,

$$N(z, t) \ge \alpha \Leftrightarrow \frac{t - |z|}{t + |z|} \ge \alpha \Leftrightarrow \frac{1+\alpha}{1-\alpha}|z| \le t.$$

This shows that $\|z\|_\alpha = \inf\{t > 0 : N(z, t) \ge \alpha\} = \frac{1+\alpha}{1-\alpha}|z|$. We now show that the sequence $\{z_n\} = \left\{\frac{1}{n}\right\}$ is wf-convergent but not sf. Since each $\|\cdot\|_\alpha$ is equivalent to $|\cdot|$, obviously, $\{z_n\}$ is wf-convergent to 0. However, this convergence is not uniform in α. Indeed; for given $\varepsilon > 0$,

$$\|z_n\|_\alpha = \frac{1+\alpha}{1-\alpha}|z_n| < \varepsilon \Leftrightarrow \frac{1+\alpha}{(1-\alpha)\varepsilon} < n.$$

We cannot find desired n_0 since $\frac{1+\alpha}{(1-\alpha)\varepsilon} \to \infty$ as $\alpha \to 1$.

Definition 5. The sf(wf)-closure of a subset B in a FNS (X, N) is denoted by $\overset{-s}{B}$ $(\overset{-w}{B})$ and defined by the set of all $x \in X$ such that there exists a sequence $\{x_n\}$ in B such that $x_n \xrightarrow{sf(wf)} x$. We say that B is sf(wf)-closed whenever $\overset{-s}{B}$ $(\overset{-w}{B}) = B$.

It is easy to see that $\overset{-s}{B} \subseteq \overset{-w}{B}$. Let us present an example showing that this inclusion may be strict.

Example 2. Let X be a normed space. Again consider the FNS in the Example 1. Let $U_X = \{x \in X : \|x\| < 1\}$. Then $\overset{-w}{U_X} = B_X = \{x \in X : |x| \leq 1\}$. Let us show this. For every $x \in B_X$ we must find a sequence $\{x_n\}_{n=1}^{\infty} \subset U_X$ such that $\|x_n - x\|_\alpha \to 0$, as $n \to \infty$, for each $\alpha \in (0, 1)$. This is accomplished by taking $x_n = \left(1 - \frac{1}{n+1}\right) x$ since each $x_n \in U_X$ and

$$\begin{aligned} \|x_n - x\|_\alpha &= \left(\frac{1+\alpha}{1-\alpha}\right) \|x_n - x\| \\ &= \left(\frac{1+\alpha}{1-\alpha}\right) \frac{\|x\|}{n+1} \to 0, \quad \text{as } n \to \infty, \text{ for each } \alpha \in (0, 1). \end{aligned}$$

However, $\overset{-s}{U_X} = U_X$. Indeed; if $x \in \overset{-s}{U_X}$ then there exists $\{x_n\}_{n=1}^{\infty} \subset U_X$ such that $\|x_n - x\|_\alpha \to 0$ uniformly in α as $n \to \infty$. This means, given $\varepsilon > 0$, there exists an integer $n_0(\varepsilon) > 0$ such that for $n \geq n_0$ and for every $\alpha \in (0, 1)$,

$$\|x_n - x\|_\alpha < \varepsilon.$$

On the other hand,

$$\begin{aligned} \|x\| &\leq \|x_n - x\| + \|x_n\| < \|x_n - x\| + 1 \\ &= \left(\frac{1-\alpha}{1+\alpha}\right) \|x_n - x\|_\alpha + 1 \\ &< \left(\frac{1+\alpha}{1-\alpha}\right) \varepsilon + 1, \quad \text{for } n \geq n_0, \text{ and for every } \alpha \in (0, 1). \end{aligned}$$

By letting $\varepsilon \to 0$ we get $\|x\| < 1$. Note that, there is no danger of $\alpha \to 1$ as $\varepsilon \to 0$ since changes on ε (via n_0) does not affect α. Hence, $\overset{-s}{U_X} \subseteq U_X$.

Definition 6 ([11]). A subset B in a fuzzy normed space (X, N) is called sf(wf)-compact if each sequence of elements of B has a sf(wf)-convergent subsequence.

Definition 7 ([11]). Let (X, N_1) and (Y, N_2) be two FNSs and $f : X \to Y$ be a mapping. Then f is called sf(wf)-compact if for every fuzzy bounded subset B of X the subset $f(B)$ is relatively sf(wf)-compact, that is, sf(wf)-closure of $f(B)$ is sf(wf)-compact.

Remark 1. We should note that weakly fuzzy compact operators or weakly fuzzy convergent sequences do *not* fuzzy generalization of weakly compact operators or weakly convergent sequences in classical functional analysis. They have absolutely different stories.

4. Strong and weak fuzzy bases

Definition 8. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in an FNS (X, N). Then

(1) It is said to be *weak fuzzy (Schauder) basis* (wf-basis, for short) of X iff, for every $x \in X$, there exists a unique sequence $\{a_n\}_{n=1}^{\infty}$ of scalars such that

$$\sum_{k=1}^{n} a_k x_k \xrightarrow{wf} x.$$

This means, for each $\alpha \in (0, 1)$ and $\varepsilon > 0$, there exists some $n_0 = n_0(\alpha, \varepsilon)$ such that $n \geq n_0$ implies

$$N\left(x - \sum_{k=1}^{n} a_k x_k, \varepsilon\right) \geq 1 - \alpha.$$

In this case, it is called x has weak fuzzy representation

$$x = \sum_{k=1}^{\infty} a_k x_k.$$

(2) It is said to be *strong fuzzy (Schauder) basis* (sf-basis, for short) of X iff, for every $x \in X$, there exists a unique sequence $\{a_n\}_{n=1}^{\infty}$ of scalars such that

$$\sum_{k=1}^{n} a_k x_k \xrightarrow{sf} x.$$

This means, for each $\alpha \in (0, 1)$, there exists some $n_0 = n_0(\alpha)$ such that $n \geq n_0$ implies

$$N\left(x - \sum_{k=1}^{n} a_k x_k, t\right) \geq 1 - \alpha, \quad \text{for all } t > 0.$$

In this case, it is called x has the strong fuzzy representation

$$x = \sum_{k=1}^{\infty} a_k x_k.$$

We can better understand the definition whenever the fuzzy norm satisfies the condition (N.6) as the next proposition shows. The proof is similar to that of Proposition 2.

Proposition 3. *Let $\{x_n\}$ be a sequence in an FNS (X, N) satisfying* (N.6). *Then*

(1) $\{x_n\}_{n=1}^{\infty}$ *is a wf-basis of X iff, for every $x \in X$, there exists a unique sequence $\{a_n\}_{n=1}^{\infty}$ of scalars such that, for each $\alpha \in (0, 1)$,*

$$\lim_{n\to\infty} \left\| x - \sum_{k=1}^{n} a_k x_k \right\|_{\alpha} = 0.$$

(2) $\{x_n\}_{n=1}^{\infty}$ *is a sf-basis of X iff, for every $x \in X$, there exists a unique sequence $\{a_n\}_{n=1}^{\infty}$ of scalars such that,*

$$\lim_{n\to\infty} \left\| x - \sum_{k=1}^{n} a_k x_k \right\|_{\alpha} = 0 \quad \text{uniformly in } \alpha$$

where $\|\cdot\|_{\alpha}$ are α-norms of N.

Definition 9. By the notations in the Definition 8, a mapping

$$f_n : X \to \mathbb{R}, \qquad f_n(x) = f_n\left(\sum_{k=1}^{\infty} a_k x_k\right) = a_n$$

and

$$P_n : X \to X, \qquad P_n(x) = P_n\left(\sum_{k=1}^{\infty} a_k x_k\right) = \sum_{k=1}^{n} a_k x_k, \quad n = 1, 2, \ldots$$

are called coordinate functionals and natural projections, respectively, associated with the wf(sf)-basis $\{x_n\}$ in X.

Proposition 4. *Let $\{x_n\}$ be a basis in a wf-complete FNS (X, N) satisfying* (N.6). *Then each f_n and P_n is wf-continuous.*

Proof. By Proposition 3, $\{x_n\}$ is also a Schauder basis in the Banach space $(X, \|\cdot\|_{\alpha})$ for each $\alpha \in (0, 1)$. So,

$$f_n : (X, \|\cdot\|_{\alpha}) \to \mathbb{R}, \qquad f_n(x) = f_n\left(\sum_{k=1}^{\infty} a_k x_k\right) = a_n \quad \text{and}$$

$$P_n : (X, \|\cdot\|_{\alpha}) \to (X, \|\cdot\|_{\alpha}), \qquad P_n(x) = P_n\left(\sum_{k=1}^{\infty} a_k x_k\right) = \sum_{k=1}^{n} a_k x_k$$

are continuous. This means the mappings are wf-continuous for each n. $\square$

It is obvious that, if $\{x_n\}_{n=1}^{\infty}$ is a sf-basis of X then it is wf-basis of X, but not conversely.

Example 3. Consider classical Banach space c_0 with the norm $\|x\|_{\infty} = \sup \|x_n\|$ where $x = \{x_n\}$ and define

$$N(x, t) = \begin{cases} \dfrac{t - \|x\|_{\infty}}{t + \|x\|_{\infty}}, & \text{if } t > \|x\|_{\infty} \\ 0, & \text{if } t \leq \|x\|_{\infty} \end{cases}$$

on c_0 as a fuzzy norm. We know from the former example that α-norms of N are $\|x\|_{\alpha} = \frac{1+\alpha}{1-\alpha}\|x\|_{\infty}$. The sequence $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots)$, $\ldots$ is a wf-basis for the FNS (c_0, N). Let us show this. In fact, this is obvious by Proposition 3 since

$$\lim_{n\to\infty} \left\| x - \sum_{k=1}^{n} a_n x_n \right\|_{\alpha} = \frac{1+\alpha}{1-\alpha} \lim_{n\to\infty} \left\| x - \sum_{k=1}^{n} a_n x_n \right\|_{\infty} = 0,$$

for each $\alpha \in (0, 1)$. However, this convergence is not uniform in α since

$$\frac{1+\alpha}{(1-\alpha)\,\varepsilon} \to \infty \quad \text{as } \alpha \to 1.$$

This actually proves that no sequence in (c_0, N) can be a sf-basis.

However if we put

$$N_1(x,t) = \begin{cases} 1, & \text{if } t > \|x\|_\infty \\ 0, & \text{if } t \le \|x\|_\infty \end{cases}$$

on c_0, then (c_0, N_1) is a FNS satisfying (N.6) and $\{e_n\}_{n=1}^{\infty}$ is a sf-basis for c_0 since $\|x\|_\alpha = \|x\|_\infty$ for each $\alpha \in (0, 1)$.

Remark 2. In finite dimensional FNSs the definition of a basis is independent of the fuzzy norm and hence coincides with the classical definition of a basis (Hamel basis) in vector spaces.

It is classical in the basis theory that a normed space having a basis is separable. Let us now investigate fuzzy analogue of this result. We know that every fuzzy normed space induces a topology τ such that, for some $A \subset X$, $A \in \tau$ if and only if for every $x \in A$ there exist $t > 0$ and $0 < \alpha < 1$ such that $B(x, \alpha, t) \subset A$ where $B(x, \alpha, t) = \{y : N(x - y, t) \ge 1 - \alpha\}$ [9].

Proposition 5. *τ is a vector topology for X, that is, the vector space operations are continuous in this topology.*

Proof. Since the family $\left\{B\left(x, \frac{1}{n}, \frac{1}{n}\right) : n = 1, 2, \ldots\right\}$ is a countable local basis at x, τ is a first countable topology for X. Hence it is sufficient to show only that the vector space operations are sequentially continuous in τ. Suppose $x_n \to x$ and $y_n \to y$ in the topological space (X, τ). This means $\mu(x_n - x, t/2) \to 1$ and $\mu(y_n - y, t/2) \to 1$ as $n \to \infty$, for every $t > 0$.

Now

$$\begin{aligned} \mu\left(x_n + y_n - (x+y), t\right) &\ge \min\left\{\mu\left(x_n - x, t/2\right), \mu(y_n - y, t/2)\right\} \\ &\to \min\{1, 1\} = 1, \quad \text{as } n \to \infty. \end{aligned}$$

Further, if $\lambda_n \to \lambda$ in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the scalar field of X, then

$$\begin{aligned} \mu\left(\lambda_n x_n - \lambda x, t\right) &= \mu\left(\lambda_n x_n - \lambda x_n + \lambda x_n - \lambda x, t\right) \\ &= \mu\left(\left(\lambda_n - \lambda\right) x_n + \lambda\left(x_n - x\right), t\right) \\ &\ge \min\left\{\mu\left(x_n, \frac{t}{2\left|\lambda_n - \lambda\right|}\right), \mu\left(x_n - x, \frac{t}{2\left|\lambda\right|}\right)\right\} \\ &\to \min\{1, 1\} = 1, \quad \text{as } n \to \infty. \end{aligned}$$

Here $\mu\left(x_n, \frac{t}{2|\lambda_n - \lambda|}\right) \to 1$ since $\frac{t}{2|\lambda_n - \lambda|} \to \infty$ as $n \to \infty$ by the last condition on μ. This completes the proof. □

Theorem 4. *Let (X, N) be an FNS having a wf-basis $\{x_n\}$. Then the topological space (X, τ) is separable.*

Proof. Let E denotes the set of all finite linear combinations $\sum_{k=1}^{n} b_k x_k$ where each b_k is (real or complex) rational number. Obviously, E is countable and let us show that it is dense in τ. Suppose $x \in X$ is arbitrary. There exists a unique sequence $\{a_n\}_{n=1}^{\infty}$ of scalars such that, for each $\alpha \in (0, 1)$ and $\varepsilon > 0$, we can find some integer $n_0 = n_0(\alpha, \varepsilon)$ such that $n \ge n_0$ implies

$$N\left(x - \sum_{k=1}^{n} a_k x_k, \varepsilon\right) \ge 1 - \alpha.$$

That is, for all $n \ge n_0$,

$$\sum_{k=1}^{n} a_k x_k \in B\left(x, \alpha, \varepsilon\right).$$

On the other hand, one can constitute a sequence $\left(b_k^i\right)_{i=1}^{\infty}$ of scalars converging to a_k for each k. Hence the sequence $\left(\sum_{k=1}^{n} b_k^i x_k\right)_{i=1}^{\infty}$ converges to $\sum_{k=1}^{n} a_k x_k$ in τ by the continuity of vector space operations. This implies every x-centered τ-open sphere $B\left(x, \alpha, \varepsilon\right)$ includes an element $\sum_{k=1}^{n} b_k^i x_k$ of E. □

Theorem 5. *Let $(X, \|\cdot\|)$ be a normed space and $\{x_n\}$ be a basis in X. Then $\{x_n\}$ is a wf-basis for FNS (X, N) where*

$$N(x,t) = \begin{cases} \dfrac{t}{t + \|x\|}, & \text{if } t > 0 \\ 0, & \text{if } t \le 0. \end{cases}$$

Proof. By the hypothesis, for each $x \in X$, there exists a unique sequence $\{a_n\}$ of scalars with $\sum_{k=1}^{n} a_k x_k \to 0$ in the norm topology as $n \to \infty$. Explicitly, for each $\delta > 0$, there exists some integer $n_0 = n_0(\delta)$ such that $n \geq n_0$ implies

$$\left\| x - \sum_{k=1}^{n} a_n x_n \right\| \leq \delta.$$

Now, for each $\alpha \in (0, 1)$ and $\varepsilon > 0$, take $\delta = \frac{\alpha\varepsilon}{1-\alpha}$ in this inequality. So, there exists some integer $n_0 = n_0(\delta) = n_0(\alpha, \varepsilon)$ such that $n \geq n_0$ implies

$$\left\| x - \sum_{k=1}^{n} a_n x_n \right\| \leq \frac{\alpha\varepsilon}{1-\alpha}$$

if and only if

$$N\left(x - \sum_{k=1}^{n} a_k x_k, \varepsilon\right) = \frac{\varepsilon}{\varepsilon + \left\| x - \sum_{k=1}^{n} a_n x_n \right\|} \geq 1 - \alpha. \quad \square$$

An important topics in the structural investigation of classical normed spaces, especially, of Banach spaces is approximation property. We want to introduce this fundamental topics in FNSs.

Definition 10. (1) A wf-complete FNS (X, N) is said to have weak fuzzy approximation property, briefly wf-AP, if for every wf-compact set K in X and for each $\alpha \in (0, 1)$ and $\varepsilon > 0$, there exists an operator $T_\alpha : X \to X$ of finite rank such that

$$N\left(T_\alpha(x) - x, \varepsilon\right) \geq 1 - \alpha$$

for every $x \in K$.

(2) A sf-complete FNS (X, N) is said to have strong fuzzy approximation property, briefly sf-AP, if for every sf-compact set K in X and for each $\alpha \in (0, 1)$, there exists an operator $T : X \to X$ of finite rank such that

$$N\left(T(x) - x, t\right) \geq 1 - \alpha, \quad \forall t > 0,$$

for every $x \in K$.

Remark 3. The operator T in wf-AP depends both on $\alpha \in (0, 1)$ and $\varepsilon > 0$ whereas it depends only on $\varepsilon > 0$ in sf-AP. Of course, T depends on the set K in both situation.

Proof of the following Proposition can be derived as similar to that of Proposition 2.

Proposition 6. (1) *A wf-complete FNS (X, N) satisfying (N.6) has wf-AP if and only if for every wf-compact set K in X and for each $\alpha \in (0, 1)$ and $\varepsilon > 0$, there exists an operator $T_\alpha : X \to X$ of finite rank such that*

$$\|T_\alpha(x) - x\|_\alpha < \varepsilon$$

for every $x \in K$.

(2) *A sf-complete FNS (X, N) satisfying (N.6) has sf-AP if and only if for every sf-compact set K in X and for each $\varepsilon > 0$, there exists an operator $T : X \to X$ of finite rank, independent of $\alpha \in (0, 1)$, such that*

$$\|T(x) - x\|_\alpha < \varepsilon$$

for every $x \in K$.

Theorem 6. *Let (X, N) be an FNS possessing a wf-basis $\{x_n\}$. Then X has the wf-AP.*

Proof. Let $K \subset X$ be a wf-compact subset of X and $\alpha \in (0, 1)$ and $\varepsilon > 0$ be arbitrary. By the hypothesis, for some $x \in K$, there exists a unique sequence $\{a_n\}_{n=1}^{\infty}$ of scalars such that

$$P_n(x) = \sum_{k=1}^{n} a_k x_k \xrightarrow{wf} x, \quad \text{as } n \to \infty.$$

This means there exists some $n_0(\alpha, \varepsilon)$ such that

$$N\left(P_n(x) - x, \varepsilon\right) \geq 1 - \alpha$$

for $n \geq n_0$. Further, each P_n has finite rank in the linear space X since $\dim P_n(X) = n$. Hence, each P_n such that $n \geq n_0$ can be taken as desired finite rank operator in the definition. $\square$

One can show with a similar proof that every FNS having a sf-basis has the sf-AP.

The converse of the above theorem may not be true.

Example 4. Let us consider Banach space ℓ_∞ with its usual sup-norm $\|x\|_\infty = \sup_n |x_n|$. Also, $\|x\|_0 = \sup_n \left|\frac{x_n}{n}\right|$ is another norm on ℓ_∞. Now let us define

$$N(x,t) = \begin{cases} 1, & \text{if } t > \|x\|_\infty \\ 1/2, & \text{if } \|x\|_0 < t \leq \|x\|_\infty \\ 0, & \text{if } t \leq \|x\|_0. \end{cases}$$

It is proved in [9] that N is a fuzzy norm on ℓ_∞ satisfying the condition (N.6) and its α-norms are

$$\|x\|_\alpha = \|x\|_\infty \quad \text{for } 1 > \alpha > \frac{1}{2}$$

$$\|x\|_\alpha = \|x\|_0 \quad \text{for } 0 < \alpha \leq \frac{1}{2}.$$

(ℓ_∞, N) cannot have a wf (hence sf)-basis since the Banach space $(\ell_\infty, \|\cdot\|_\alpha)$, for $1 > \alpha > \frac{1}{2}$, is not separable. However, let us show that (ℓ_∞, N) has sf-AP. Recall that a partition of natural numbers $\mathbb{N}$ is a finite family $\mathrm{p} = (\beta_1, \beta_2, \dots \beta_n)$ of subsets of $\mathbb{N}$ together with a distinguished point $h_i \in \beta_i$ if $\beta_i \neq \varnothing$, where $1 \leq j \leq n$, such that $\beta_i \cap \beta_j = \varnothing (i \neq j)$ and $\bigcup_{j=1}^n \beta_i = \mathbb{N}$. The set D of all partitions of $\mathbb{N}$ is a directed set by the relation $\mathrm{p} \ll \mathrm{p}'$ which means each $\beta_i \in \mathrm{p}$ included in some $\beta_j' \in \mathrm{p}'$, where $\mathrm{p} = (\beta_1, \beta_2, \dots \beta_n)$ and $\mathrm{p}' = (\beta_1', \beta_2', \dots \beta_n')$ are two arbitrary partitions of $\mathbb{N}$. Now, for each $\mathrm{p} = (\beta_1, \beta_2, \dots \beta_n) \in D$, write

$$\Lambda_{\mathrm{p}}(x) = \sum_{i=1}^{n} x_{h_i} . \chi_{\beta_i}, \quad \text{for } x \in \ell_\infty$$

where h_i is the distinguished point in β_i and χ_{β_i} is the characteristic function of β_i for $1 \leq i \leq n$. Then Λ_{p} is a projection on ℓ_∞ of finite rank. It is well known that the net $(\Lambda_{\mathrm{p}}(x), D)$ converges to x in the Banach space $(\ell_\infty, \|\cdot\|_\infty)$ (see, [12, pp. 25, prob. 117]). Let $K \subset \ell_\infty$ be sf-compact, $x \in K$ and $\varepsilon > 0$ be given. Then there exists a partition $\mathrm{p}_0(\varepsilon)$ such that, for $\mathrm{p}_0(\varepsilon) \ll \mathrm{p}$

$$\left\| \Lambda_{\mathrm{p}}(x) - x \right\|_\infty < \varepsilon.$$

But, since $\|x\|_0 \leq \|x\|_\infty$ for every $x \in \ell_\infty$,

$$\left\| \Lambda_{\mathrm{p}}(x) - x \right\|_0 < \varepsilon$$

for $\mathrm{p}_0(\varepsilon) \ll \mathrm{p}$. That is; for every $\alpha \in (0, 1)$,

$$\left\| \Lambda_{\mathrm{p}}(x) - x \right\|_\alpha < \varepsilon, \quad (\mathrm{p}_0(\varepsilon) \ll \mathrm{p}).$$

Hence any projection operator Λ_{p} $(\mathrm{p}_0 \ll \mathrm{p})$ meets all requirements for sf-AP by Proposition 6.

5. Conclusion

We introduce a useful property in applications, namely approximation property for FNSs and a basis notion depending on the fuzzy norm for infinite dimensional FNSs. We hope that wf(sf)-bases may provide some necessary tools for structural analysis of FNSs. One can also use to determine strong and weak fuzzy continuous dual spaces of some important infinite dimensional FNSs by using wf(sf)-bases. The representation of functionals and operators between FNSs has a key role in fuzzy analysis. It is performed by a basis and gives exact resolution of the operator. Further, wf(sf)-bases give countable finite dimensional decomposition of the FNS so that one can approximate to the space by some finite dimensional space which is more understandable. Many kinds of approximation techniques can be introduced, in two ways as strong and weak, into families of equations of these operators.

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