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Maximal element theorems in product FC-spaces and generalized games ☆

Xie Ping Ding

College of Mathematics and Software Science, Sichuan Normal University, Chengdu 610066, PR China

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Abstract

Let I be a finite or infinite index set, X be a topological space and $(Y_i, \{\varphi_{N_i}\})_{i\in I}$ be a family of finitely continuous topological spaces (in short, FC-space). For each $i \in I$, let $A_i : X \to 2^{Y_i}$ be a set-valued mapping. Some existence theorems of maximal elements for the family $\{A_i\}_{i\in I}$ are established under noncompact setting of FC-spaces. As applications, some equilibrium existence theorems for generalized games with fuzzy constraint correspondences are proved in noncompact FC-spaces. These theorems improve, unify and generalize many important results in recent literature.

Keywords: Maximal element; Generalized game; Fuzzy constraint; Equilibrium; FC-space

1. Introduction and preliminaries

It is well known that many existence theorems of maximal elements for set-valued mappings have been established in topological vector spaces, H-spaces and G-convex spaces by many authors. Their important applications to mathematical economies and generalized games have been studied extensively by many authors. For existence results of maximal elements of various classes of set-valued mappings and their applications to mathematical

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E-mail address: dingxip@sicnu.edu.cn.

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economies, generalized games and other branches of mathematics, the reader may consult [3,5–12,16–25,27,33–45] and the references therein.

Let X be a nonempty set. We denote by 2^X and $\langle X\rangle$ the family of all subsets of X and the family of all nonempty finite subsets of X respectively. Let Δ_n be the standard n-dimensional simplex with vertices $e_0, e_1, \ldots, e_n$. If J is a nonempty subset of $\{0, 1, \ldots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j\colon j \in J\}$.

The following notions was introduced by Ding in [13,14].

Let X and Y be topological spaces. A subset A of X is said to be compactly open (respectively, compactly closed) if for each nonempty compact subset K of X, $A \cap K$ is open (respectively, closed) in K. The compact interior and the compact closure of A are defined by

$$\operatorname{cint} A = \bigcup\{B \subset X\colon\ B \subset A \text{ and } B \text{ is compactly open in } X\}, \quad \text{and}$$
$$\operatorname{ccl} A = \bigcap\{B \subset X\colon\ A \subset B \text{ and } B \text{ is compactly closed}\}.$$

Clearly, we have $X \setminus \operatorname{cint} A = \operatorname{ccl}(X \setminus A)$ and $X \setminus \operatorname{ccl} A = \operatorname{cint}(X \setminus A)$. For any compact subset K of X, we have $\operatorname{cint} A \cap K = \operatorname{int}_K(A \cap K)$ and $\operatorname{ccl} A \cap K = \operatorname{cl}_K(A \cap K)$.

A set-valued mapping $T : X \to 2^Y$ is said to be transfer compactly open-valued if for $x \in X$ and for each compact subset K of Y, $y \in T(x) \cap K$ implies that there exists $x' \in X$ such that $y \in \operatorname{int}_K(T(x') \cap K)$.

The following notion of a finitely continuous topological space (in short, FC-space) was introduced by Ding in [15].

Definition 1.1. $(Y, \{\varphi_N\})$ is said to be a FC-space if Y is a topological space and for each $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. A subset D of $(Y, \{\varphi_N\})$ is said to be a FC-subspace of Y if for each $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ and for each $\{y_{i_0}, \ldots, y_{i_k}\} \subset N \cap D$, $\varphi_N(\Delta_k) \subset D$ where $\Delta_k = \operatorname{co}(\{e_{i_j}\colon\ j = 0, \ldots, k\})$.

Clearly, each FC-subspace D of a FC-space $(Y, \{\varphi_N\})$ is also a FC-space.

The following notion of generalized convex (in short, G-convex) spaces was introduced by Park and Kim in [31] and Park in [30].

Definition 1.2. (Y, Γ) is said to be a G-convex space if Y is a topological space and $\Gamma : \langle Y\rangle \to 2^Y \setminus \{\emptyset\}$ such that for each $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$, there exists a continuous mapping $\varphi_N : \Delta_n \to \Gamma(N)$ satisfying that for each $B = \{y_{i_0}, \ldots, y_{i_k}\} \subset N$, $\varphi_N(\Delta_k) \subset \Gamma(B)$ where $\Delta_k = \operatorname{co}\{e_{i_j}\colon\ j = 0, \ldots, k\}$. A subset D of Y is said to be G-convex if for any $N \in \langle D\rangle$, $\Gamma(N) \subset D$.

It is clear that the class of G-convex spaces is a true subclass of FC-spaces. We emphasis that FC-space is a topological space without any convexity structure. Major examples of FC-space is convex subsets of topological vector spaces, Lassonde's convex spaces in [26], C-spaces (or H-spaces) due to Horvath in [24], G-convex space due to Park and Kim in [30,31] and many topological spaces with abstract convexity structure, see [30,31].

Let X be a topological space and $(Y, \{\varphi_N\})$ be a FC-space. The class $\mathcal{B}(Y, X)$ of better admissible mappings was introduced as follows: $F \in \mathcal{B}(Y, X) \Leftrightarrow F : Y \to 2^X$ is a upper semicontinuous set-valued mapping with compact values such that for any $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ and any continuous mapping $\psi : F(\varphi_N(\Delta_n)) \to \Delta_n$, the composition mapping $\psi \circ F|_{\varphi_N(\Delta_n)} \circ \varphi_N : \Delta_n \to 2^{\Delta_n}$ has a fixed point.

If $(Y, \{\varphi_N\})$ be a G-convex space, the class $\mathcal{B}(Y, X)$ was introduced by Park in [28]. If Y is a nonempty convex subset of a vector space E, the class $\mathcal{B}(Y, X)$ is introduced and studied by Park in [29]. The class $\mathcal{B}(Y, X)$ of better admissible set-valued mappings includes many important classes of set-valued mappings, for example, $\mathcal{U}_c^k(Y, X)$ in [31], $KKM(Y, X)$ in [4] and $\mathbf{A}(Y, X)$ in [2] and so on as proper subclasses, see [28].

Lemma 1.1. *Let I be any index set. For each $i \in I$, let $(Y_i, \{\varphi_{N_i}\})$ be a FC-space. Let $Y = \prod_{i\in I} Y_i$ and $\varphi_N = \prod_{i\in I} \varphi_{N_i}$. Then $(Y, \{\varphi_N\})$ is also a FC-space.*

Proof. Let Y be equipped with the product topology and for $i \in I$, let $\pi_i : Y \to Y_i$ be the projective mapping from Y to Y_i. For any given $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$, let $N_i = \pi_i(N) = \{\pi_i(y_0), \dots, \pi_i(y_n)\} \in \langle Y_i \rangle$. Since Y_i is a FC-space, there exists a continuous mapping $\varphi_{N_i} : \Delta_n \to Y_i$. Define a mapping $\varphi_N : \Delta_n \to Y$ by

$$\varphi_N(\alpha) = \prod_{i\in I} \varphi_{N_i}(\alpha), \quad \forall \alpha \in \Delta_n.$$

Then φ_N is continuous and hence $(Y, \{\varphi_N\})$ is also a FC-space. □

By the definition of FC-subspace of a FC-space and Lemma 1.1, we can prove that if for each $i \in I$, D_i is a FC-subspace of FC-space $(Y_i, \{\varphi_{N_i}\})$, then $D = \prod_{i\in I} D_i$ is also a FC-subspace of the FC-space $(Y, \{\varphi_N\})$ defined in Lemma 1.1.

Lemma 1.2 [13]. *Let X and Y be topological spaces, $T : X \to 2^Y$ be a set-valued mapping with nonempty values. Then the following conditions are equivalent*:

(1) *T has the compactly local intersection property,*
(2) *for each compact subset K of X and for each $y \in Y$, there exists an open subset O_y of X (which may be empty) such that $O_y \cap K \subset T^{-1}(y)$ and $K = \bigcup_{y\in Y}(O_y \cap K)$,*
(3) *for each compact subset K of X, there exists a set-valued mapping $F : X \to 2^Y$ such that for each $y \in Y$, $F^{-1}(y)$ is open or empty in X, and $F^{-1}(y) \cap K \subset T^{-1}(y)$ for each $y \in Y$ and $K = \bigcup_{y\in Y}(F^{-1}(y) \cap K)$,*
(4) *for each compact subset K of X and for each $x \in K$, there exists $x \in \operatorname{cint} T^{-1}(y) \cap K$, i.e.,*

$$K = \bigcup_{y\in Y}\big(\operatorname{cint} T^{-1}(y) \cap K\big) = \bigcup_{y\in Y}\big(T^{-1}(y) \cap K\big),$$

(5) *$T^{-1} : Y \to 2^X$ is transfer compactly open-valued on Y.*

2. Existence of maximal elements

In this section, we shall show several existence theorems of maximal elements for a set-valued mapping and for a family of set-valued mappings involving a better admissible set-valued mapping.

Theorem 2.1. *Let X be a topological space, $(Y, \{\varphi_N\})$ be a FC-space, $F \in \mathcal{B}(Y, X)$ and $A : X \to 2^Y$ such that,*

(i) *for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ and for each $\{y_{i_0}, \dots, y_{i_k}\} \subset N$,*

$$F\big(\varphi_N(\Delta_k)\big) \cap \left(\bigcap_{j=0}^{k} \operatorname{cint} A^{-1}(y_{i_j})\right) = \emptyset,$$

(ii) *$A^{-1} : Y \to 2^X$ is transfer compactly open-valued,*

(iii) *there exists a nonempty set $Y_0 \subset Y$ and for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$, there exists a compact FC-subspace L_N of Y containing $Y_0 \cup N$ such that $K = \bigcap_{y \in Y_0} (\operatorname{cint} A^{-1}(y))^c$ is empty or compact in X where $(\operatorname{cint} A^{-1}(y))^c$ denotes the complement of $\operatorname{cint} A^{-1}(y)$.*

Then there exists a point $\hat{x} \in X$ such that $A(\hat{x}) = \emptyset$.

Proof. Suppose the conclusion is false, then $A(x) \neq \emptyset$ for each $x \in X$. If K is empty, then we have

$$X = X \setminus \bigcap_{y \in Y_0} \big(\operatorname{cint} A^{-1}(y)\big)^c = \bigcup_{y \in Y_0} \operatorname{cint} A^{-1}(y). \tag{1}$$

If K is nonempty and compact, by (ii) and Lemma 1.2, we have

$$K = \bigcup_{y \in Y} \big(\operatorname{cint} A^{-1}(y) \cap K\big).$$

Since K is compact, there exists $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ such that

$$K = \bigcup_{i=0}^{n} \big(\operatorname{cint} A^{-1}(y_i) \cap K\big) \subset \bigcup_{i=0}^{n} \operatorname{cint} A^{-1}(y_i).$$

It follows that

$$X \setminus \bigcup_{y \in Y_0} \operatorname{cint} A^{-1}(y) = \bigcap_{y \in Y_0} \big(\operatorname{cint} A^{-1}(x)\big)^c = K \subset \bigcup_{y \in N} \operatorname{cint} A^{-1}(y).$$

Hence we obtain

$$X = \bigcup_{y \in Y_0 \cup N} \operatorname{cint} A^{-1}(y). \tag{2}$$

Therefore, in both cases, there exists $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ such that (2) holds. By condition (iii), there exists a compact FC-subspace L_N of Y containing $Y_0 \cup N$. Since F is upper semicontinuous with compact values and L_N is compact, it follows from Proposition 3.1.11 of Aubin and Ekeland in [1] that $F(L_N)$ is compact in X. By (2), we have

$$F(L_N)=\bigcup_{y\in L_N}\big(\operatorname{cint}A^{-1}(y)\cap F(L_N)\big). \tag{3}$$

Hence there exists a finite set $M=\{z_0,\dots,z_m\}\in\langle L_N\rangle\subset\langle Y\rangle$ such that

$$F(L_N)=\bigcup_{i=0}^{m}\big(\operatorname{cint}A^{-1}(z_i)\cap F(L_N)\big). \tag{4}$$

Since L_N is a FC-subspace of Y, we have

$$\varphi_M(\Delta_r)\subset L_N,\quad \forall\{z_{i_0},\dots,z_{i_r}\}\subset M, \tag{5}$$

where $\Delta_r=\operatorname{co}(\{e_{i_j}\colon\ j=0,\dots,r\})$.

Let $\{\psi_i\}_{i=0}^m$ is the continuous partition of unity subordinated to the open covering $\{\operatorname{cint}A^{-1}(z_i)\cap F(L_N)\}_{i=0}^m$. Then for each $i\in\{0,1,\dots,m\}$ and $x\in F(L_N)$,

$$\psi_i(x)\neq 0\quad\Leftrightarrow\quad x\in\operatorname{cint}A^{-1}(z_i)\cap F(L_N)\subset\operatorname{cint}A^{-1}(z_i). \tag{6}$$

Define a mapping $\psi: F(L_N)\to\Delta_m$ by

$$\psi(x)=\sum_{i=0}^{m}\psi_i(x)e_i,\quad \forall x\in F(L_N). \tag{7}$$

Hence ψ is continuous and

$$\psi(x)=\sum_{j\in J(x)}\psi_j(x)e_j\in\Delta_{J(x)},\quad \forall x\in F(L_N), \tag{8}$$

where $J(x)=\{j\in\{0,1,\dots,m\}\colon\ \psi_j(x)\neq 0\}$. Note that $M\subset L_N$ and L_N is FC-subspace of Y, we have $F(\varphi_M(\Delta_m))\subset F(L_N)$. Since $F\in\mathcal{B}(Y,X)$, it follows from (5) and (7) that the function $\psi\circ F|_{\varphi_M(\Delta_m)}\circ\varphi_M:\Delta_m\to\Delta_m$ has a fixed point $z\in\Delta_m$, i.e., $z\in\psi\circ F|_{\varphi_M(\Delta_m)}\circ\varphi_M(z)$. Hence there exists $\bar{x}\in F|_{\varphi_M(\Delta_m)}\circ\varphi_M(z)$ such that

$$z=\psi(\bar{x})=\sum_{j\in J(\bar{x})}\psi_j(\bar{x})e_j\in\Delta_{J(\bar{x})},$$

where $J(\bar{x})=\{j\in\{0,\dots,m\}\colon\ \psi_j(\bar{x})\neq 0\}$. It follows from (i) that

$$\bar{x}\in F|_{\varphi_M(\Delta_M)}\circ\varphi_M(z)\subset F\big(\varphi_M(\Delta_{J(\bar{x})})\big)\subset\bigcup_{j\in J(\bar{x})}\big(X\setminus\operatorname{cint}A^{-1}(z_j)\big).$$

Therefore there exists $j_0\in J(\bar{x})$ such that $\bar{x}\notin\operatorname{cint}A^{-1}(z_{j_0})$. On the other hand, by the definition of $J(\bar{x})$, we have $\psi_{j_0}(\bar{x})\neq 0$. It follows from (6) that $\bar{x}\in\operatorname{cint}A^{-1}(z_{j_0})$ which is a contradiction. Hence there must exists $\hat{x}\in X$ such that $A(\hat{x})=\emptyset$. □

Remark 2.1. Theorem 2.1 generalizes Theorem 2.1 of Ding in [11] from G-convex spaces to FC-space without any convexity structure.

Theorem 2.2. *Let X be a topological space, K be a nonempty compact subset of X, $(Y, \{\varphi_N\})$ be a FC-space, $F \in \mathcal{B}(Y, X)$ and $A : X \to 2^Y$ such that,*

(i) *for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and for each $\{y_{i_0}, \ldots, y_{i_k}\} \subset N$,*

$$F\big(\varphi_N(\Delta_k)\big) \cap \left(\bigcap_{j=0}^{k} \operatorname{cint} A^{-1}(y_{i_j})\right) = \emptyset,$$

(ii) *$A^{-1} : Y \to 2^X$ is transfer compactly open-valued,*

(iii) *for each $N \in \langle Y \rangle$, there exists a compact FC-subspace L_N of Y containing N such that*

$$F(L_N) \setminus K \subset \bigcup_{y \in L_N} \operatorname{cint} A^{-1}(y).$$

Then there exists a point $\hat{x} \in K$ such that $A(\hat{x}) = \emptyset$.

Proof. Suppose the conclusion is false, then $A(x) \neq \emptyset$ for each $x \in X$. By (ii) and Lemma 1.2, we have

$$K = \bigcup_{y \in Y} \big(\operatorname{cint} A^{-1}(y) \cap K\big).$$

Since K is compact, there exists $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ such that

$$K = \bigcup_{i=0}^{n} \big(\operatorname{cint} A^{-1}(y_i) \cap K\big).$$

By (iii) and $F \in \mathcal{B}(Y, X)$, there exists a compact FC-subspace L_N of Y containing N and $F(L_N)$ is compact in X and hence we have

$$F(L_N) = \bigcup_{y \in L_N} \big(\operatorname{cint} A^{-1}(y) \cap F(L_N)\big).$$

By using similar argument as in the proof of Theorem 2.1, we can show that there exists $\hat{x} \in X$ such that $A(\hat{x}) = \emptyset$. The condition (iii) implies that $\hat{x}$ must be in K. □

Remark 2.2. Theorem 2.2 generalizes Theorem 2.2 of Ding in [11] from G-convex space to FC-space without any convexity structure.

Corollary 2.1. *Let $(X, \{\varphi_N\})$ be a FC-space and K be a nonempty compact subset of X. Let $F \in \mathcal{B}(X, X)$ and $A : X \to 2^X$ be such that*

(i) *for each $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ and for each $\{x_{i_0}, \ldots, x_{i_k}\} \subset N$,*

$$F\big(\varphi_N(\Delta_k)\big) \cap \left(\bigcap_{j=0}^{k} \operatorname{cint} A^{-1}(x_{i_j})\right) = \emptyset,$$

(ii) *$A^{-1}: X \to 2^X$ is transfer compactly open-valued,*
(iii) *for each $N \in \langle X \rangle$ there exists a compact FC-subspace L_N of X containing N such that*

$$F(L_N) \setminus K \subset \bigcup_{y \in L_N} \operatorname{cint} A^{-1}(y).$$

Then A has a maximal element $\hat{x} \in K$, i.e., $A(\hat{x}) = \emptyset$.

Proof. The conclusion of Corollary 2.1 follows from Theorem 2.2 with $X = (Y, \{\varphi_N\})$. □

If F is the identity mapping in Corollary 2.1, then we have obtain the following result.

Corollary 2.2. *Let $(X, \{\varphi_N\})$ be a FC-space and K be a nonempty compact subset of X. Let $A: X \to 2^X$ be such that*

(i) *for each $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ and for each $\{x_{i_0}, \ldots, x_{i_k}\} \subset N$,*

$$\varphi_N(\Delta_k) \cap \left(\bigcap_{j=0}^{k} \operatorname{cint} A^{-1}(x_{i_j}) \right) = \emptyset,$$

(ii) *$A^{-1}: X \to 2^X$ is transfer compactly open-valued,*
(iii) *for each $N \in \langle X \rangle$ there exists a compact FC-subspace L_N of X containing N such that*

$$L_N \setminus K \subset \bigcup_{y \in L_N} \operatorname{cint} A^{-1}(y).$$

Then A has a maximal element $\hat{x} \in K$, i.e., $A(\hat{x}) = \emptyset$.

Remark 2.3. We note that the coercive condition (iii) of Theorem 2.1 and the coercive condition (iii) of Theorem 2.2 are not equivalent. Hence they are different results. Corollary 2.1 generalizes Corollary 2.1 of Ding in [11] from G-convex space to FC-space. Corollaries 2.1 and 2.2, in turn, generalizes Theorem 2.1 of Shen in [32], Theorem 1 of Ding and Tan in [19], Theorem 1 of Ding et al. in [17], Theorem 2 of Tulcea in [40], Theorem 2.2 of Toussaint in [39], Theorem 5.1 of Yannelis and Prabhakar in [42] and Corollary 1 of Borglin and Keiding in [3] in many aspects.

Theorem 2.3. *Let X be a topological space and I be an any index set. For each $i \in I$, let $(Y_i, \{\varphi_{N_i}\})$ be a FC-space and let $Y = \prod_{i \in I} Y_i$ such that $(Y, \{\varphi_N\})$ is a FC-space defined as in Lemma* 1.1. *Let $F \in \mathcal{B}(Y, X)$ and for each $i \in I$, $A_i: X \to 2^{Y_i}$ such that,*

(i) *for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and for each $\{y_{i_0}, \ldots, y_{i_k}\} \subset N$,*

$$F\big(\varphi_N(\Delta_k)\big) \cap \left(\bigcap_{j=0}^{k} \operatorname{cint} A_i^{-1}\big(\pi_i(y_{i_j})\big) \right) = \emptyset,$$

where π_i is the projection from Y to Y_i,

(ii) *$A_i^{-1}: Y_i \to 2^X$ is transfer compactly open-valued,*

(iii) *for each $x \in X$, $I(x) = \{i \in I\colon A_i(x) \neq \emptyset\}$ is finite,*

(iv) *there exists a nonempty set $Y_0 \subset Y$ and for each $N = \{y_0, \dots, y_n\} \in \langle Y\rangle$, there exists a compact FC-subspace L_N of Y containing $Y_0 \cup N$ such that*

$$K = \bigcap_{y \in Y_0} \operatorname{ccl}\bigl\{x \in X\colon \exists i \in I(x),\ \pi_i(y) \notin A_i(x)\bigr\}$$

is empty or compact in X.

Then there exists $\hat{x} \in X$ such that $A_i(\hat{x}) = \emptyset$ for each $i \in I$.

Proof. Define $A: X \to 2^Y$ by

$$A(x) = \begin{cases} \bigcap_{i \in I(x)} \pi_i^{-1}(A_i(x)), & \text{if } I(x) \neq \emptyset, \\ \emptyset, & \text{if } I(x) = \emptyset. \end{cases}$$

Then for each $x \in X$, $A(x) \neq \emptyset$ if and only if $I(x) \neq \emptyset$. Let $x \in X$ with $A(x) \neq \emptyset$, then there exists an $i_0 \in I(x)$ such that $A_{i_0}(x) \neq \emptyset$. For each $y \in Y$, we have

$$\begin{aligned} A^{-1}(y) &= \bigl\{x \in X\colon y \in A(x)\bigr\} = \Bigl\{x \in X\colon y \in \bigcap_{i \in I(x)} \pi_i^{-1}\bigl(A_i(x)\bigr)\Bigr\} \\ &= \bigl\{x \in X\colon \pi_i(y) \in A_i(x),\ \forall i \in I(x)\bigr\} \\ &\subset \bigl\{x \in X\colon x \in A_{i_0}^{-1}\bigl(\pi_i(y)\bigr)\bigr\} = A_{i_0}^{-1}\bigl(\pi_i(y)\bigr). \end{aligned}$$

For each $N = \{y_0, \dots, y_n\} \in \langle Y\rangle$ and for each $\{y_{i_0}, \dots, y_{i_k}\} \subset N$, if $u \in \bigcap_{j=0}^k \operatorname{cint} A^{-1}(y_{i_j})$, then $u \in \bigcap_{j=0}^k \operatorname{cint} A_{i_0}^{-1}(\pi_i(y_{i_j}))$. By (i), $u \notin F(\varphi_N(\Delta_k))$. It follows that

$$F\bigl(\varphi_N(\Delta_k)\bigr) \cap \left(\bigcap_{j=0}^k \operatorname{cint} A^{-1}(y_{i_j})\right) = \emptyset.$$

The condition (i) of Theorem 2.1 is satisfied.

For any compact subset D of X, if $x \in A^{-1}(y) \cap D$, then for each $i \in I(x)$, $x \in A_i^{-1}(\pi_i(y)) \cap D$. By (ii), each A_i^{-1} is transfer compactly open-valued and hence there exists $\bar{y}_i \in Y_i$ such that $x \in \operatorname{int}_D(A_i^{-1}(\bar{y}_i) \cap D)$. Note that $I(x)$ is finite by (iii), we have

$$x \in \bigcap_{i \in I(x)} \operatorname{int}_D\bigl(A_i^{-1}(\bar{y}_i) \cap D\bigr) \subset \operatorname{int}_D\left(\bigcap_{i \in I(x)} \bigl(A_i^{-1}(\bar{y}_i) \cap D\bigr)\right).$$

Let $\bar{y} = \prod_{i \in I(x)} \bar{y}_i \otimes \prod_{j \in I \setminus I(x)} (y_j)$ where $y_j \in Y_j$ is an any fixed element for each $j \in I \setminus I(x)$. Hence there exists $\bar{y} \in Y$ such that

$$x \in \operatorname{int}_D\left(\bigcap_{i \in I(x)} A_i^{-1}\bigl(\pi_i(\bar{y})\bigr) \cap D\right) = \operatorname{int}_D\bigl(A^{-1}(\bar{y}) \cap D\bigr).$$

Hence A^{-1} is transfer compactly open-valued, the condition (ii) of Theorem 2.1 is satisfied. By the definition of A, for each $y \in Y$ we have

$$A^{-1}(y) = \{x \in X\colon \pi_i(y) \in A_i(x),\ \forall i \in I(x)\}.$$

It follows from (iv) that

$$K = \bigcap_{y \in Y_0} \left(\operatorname{cint} A^{-1}(y)\right)^c = \bigcap_{y \in Y_0} \operatorname{ccl}\{x \in X\colon \exists i \in I(x),\ \pi_i(y) \notin A_i(x)\}$$

is empty or compact and hence the condition (iii) of Theorem 2.1 is satisfied. By Theorem 2.1, there exists $\hat{x} \in X$ such that $A(\hat{x}) = \emptyset$ which implies $I(\hat{x}) = \emptyset$, i.e., $A_i(\hat{x}) = \emptyset$ for all $i \in I$. This completes the proof. □

Let $X = Y = \prod_{i \in I} Y_i$ and F be an identity mapping on Y, then, by Theorem 2.3, we have the following result.

Corollary 2.3. *Let I be an any index set. For each $i \in I$, let $(X_i, \{\varphi_{N_i}\})$ be a FC-space and let $X = \prod_{i \in I} X_i$ such that $(X, \{\varphi_N\})$ is a FC-space defined as in Lemma 1.1. For each $i \in I$, let $A_i : X \to 2^{X_i}$ such that*

(i) *for each $N = \{x_0, \dots, x_n\} \in \langle X\rangle$ and for each $\{x_{i_0}, \dots, x_{i_k}\} \subset N$,*

$$\varphi_N(\Delta_k) \cap \left(\bigcap_{j=0}^{k} \operatorname{cint} A_i^{-1}\left(\pi_i(x_{i_j})\right)\right) = \emptyset,$$

(ii) *$A_i^{-1} : X_i \to 2^X$ is transfer compactly open-valued,*
(iii) *for each $x \in X$, $I(x) = \{i \in I\colon A_i(x) \neq \emptyset\}$ is finite,*
(iv) *there exists a nonempty set $X_0 \subset X$ and for each $N = \{x_0, \dots, x_n\} \in \langle X\rangle$, there exists a compact FC-subspace L_N of X containing $X_0 \cup N$ such that $K = \bigcap_{y \in X_0}\{x \in X\colon \exists i \in I(x),\ \pi_i(y) \notin A_i(x)\}$ is empty or compact.*

Then there exists $\hat{x} \in X$ such that $A_i(\hat{x}) = \emptyset$ for each $i \in I$.

Theorem 2.4. *Let X be a topological space and I be an any index set. For each $i \in I$, let $(Y_i, \{\varphi_{N_i}\})$ be a FC-space and let $Y = \prod_{i \in I} Y_i$ such that $(Y, \{\varphi_N\})$ is a FC-space defined as in Lemma 1.1. Let $F \in \mathcal{B}(Y, X)$ and for each $i \in I$, $A_i : X \to 2^{Y_i}$ such that*

(i) *for each $N = \{y_0, \dots, y_n\} \in \langle Y\rangle$ and for each $\{y_{i_0}, \dots, y_{i_k}\} \subset N$,*

$$F\left(\varphi_N(\Delta_k)\right) \cap \left(\bigcap_{j=0}^{k} \operatorname{cint} A_i^{-1}\left(\pi_i(y_{i_j})\right)\right) = \emptyset,$$

(ii) *$A_i^{-1} : Y_i \to 2^X$ is transfer compactly open-valued,*
(iii) *for each $x \in X$, $I(x) = \{i \in I\colon A_i(x) \neq \emptyset\}$ is finite,*
(iv) *there exists a compact subset K of X and for each $i \in I$ and $N_i \in \langle Y_i\rangle$, there exists a nonempty compact FC-subspace L_{N_i} of Y_i containing N_i such that for each $x \in X \setminus K$, there exists $y \in L_N = \prod_{i \in I} L_{N_i}$ such that for each $i \in I(x)$, $x \in \operatorname{cint} A_i^{-1}(\pi_i(y))$.*

Then there exists $\hat{x} \in K$ such that $A_i(\hat{x}) = \emptyset$ for each $i \in I$.

Proof. Define $A : X \to 2^Y$ by

$$A(x) = \begin{cases} \bigcap_{i \in I(x)} \pi_i^{-1}(A_i(x)), & \text{if } I(x) \neq \emptyset, \\ \emptyset, & \text{if } I(x) = \emptyset. \end{cases}$$

Then for each $x \in X$, $A(x) \neq \emptyset$ if and only if $I(x) \neq \emptyset$. By the conditions (i)–(iii) and the proof of Theorem 2.3, the conditions (i) and (ii) of Theorem 2.2 are satisfied. For each $N \in \langle Y \rangle$ and $i \in I$, let $N_i = \pi_i(N)$. By (iv), there exists compact FC-subspace L_{N_i} containing N_i. Let $L_N = \prod_{i \in I} L_{N_i}$, then L_N is a compact FC-subspace of Y and

$$L_N = \prod_{i \in I} L_{N_i} \supset \prod_{i \in I} \pi_i(N) \supset N.$$

By (iv) again, we have

$$\begin{aligned} F(L_N) \setminus K \subset X \setminus K &\subset \bigcup_{y \in L_N} \left(\bigcap_{i \in I(x)} \operatorname{cint} A_i^{-1}\big(\pi_i(y)\big) \right) \\ &\subset \bigcup_{y \in L_N} \operatorname{cint}\left(\bigcap_{i \in I(x)} A_i^{-1}\big(\pi_i(y)\big) \right) = \bigcup_{y \in L_N} \operatorname{cint} A^{-1}(y). \end{aligned}$$

The condition (iii) of Theorem 2.2 is satisfied. By Theorem 2.2, there exists $\hat{x} \in K$ such that $A(\hat{x}) = \emptyset$ which implies $I(\hat{x}) = \emptyset$, i.e., $A_i(\hat{x}) = \emptyset$ for all $i \in I$. This completes the proof. □

Let $X = Y = \prod_{i \in I} Y_i$ and F be the identity mapping on Y, then, by Theorem 2.4, we have the following result.

Corollary 2.4. *Let I be an any index set. For each $i \in I$, let $(X_i, \{\varphi_{N_i}\})$ be a FC-space and let $X = \prod_{i \in I} X_i$ such that $(X, \{\varphi_N\})$ is a FC-space defined as in Lemma* 1.1. *For each $i \in I$, let $A_i : X \to 2^{X_i}$ such that*

(i) *for each $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ and for each $\{x_{i_0}, \ldots, x_{i_k}\} \subset N$,*

$$\varphi_N(\Delta_k) \cap \left(\bigcap_{j=0}^{k} \operatorname{cint} A_i^{-1}\big(\pi_i(x_{i_j})\big) \right) = \emptyset,$$

(ii) *$A_i^{-1} : X_i \to 2^X$ is transfer compactly open-valued,*

(iii) *for each $x \in X$, $I(x) = \{i \in I:\ A_i(x) \neq \emptyset\}$ is finite,*

(iv) *there exists a compact subset K of X and for each $i \in I$ and $N_i \in \langle X_i \rangle$, there exists a nonempty compact FC-subspace L_{N_i} of X_i containing N_i such that for each $x \in X \setminus K$, there exists $x \in L_N = \prod_{i \in I} L_{N_i}$ such that for each $i \in I(x)$, $x \in \operatorname{cint} A_i^{-1}(\pi_i(x))$.*

Then there exists $\hat{x} \in K$ such that $A_i(\hat{x}) = \emptyset$ for each $i \in I$.

Corollary 2.5. *Let I be an any index set. For each $i \in I$, let $(X_i, \{\varphi_{N_i}\})$ be a FC-space and let $X = \prod_{i \in I} X_i$ such that $(X, \{\varphi_N\})$ is a FC-space defined as in Lemma* 1.1. *For each $i \in I$, let $A_i : X \to 2^{X_i}$ such that*

(i) *for each* $x \in X$, $A_i(x)$ *is a FC-subspace of* X_i,
(ii) *for each* $x \in X$, $x_i = \pi_i(x) \notin A_i(x)$ *and* $A_i^{-1} : X_i \to 2^X$ *is transfer compactly open-valued,*
(iii) *for each* $x \in X$, $I(x) = \{i \in I\colon A_i(x) \neq \emptyset\}$ *is finite,*
(iv) *there exists a compact subset* K *of* X *and for each* $i \in I$ *and* $N_i \in \langle X_i \rangle$, *there exists a nonempty compact FC-subspace* L_{N_i} *of* X_i *containing* N_i *such that for each* $x \in X \setminus K$, *there exists* $y \in L_N = \prod_{i\in I} L_{N_i}$ *such that for each* $i \in I(x)$, $x \in \operatorname{cint} A_i^{-1}(\pi_i(y))$.

Then there exists $\hat{x} \in K$ *such that* $A_i(\hat{x}) = \emptyset$ *for each* $i \in I$.

Proof. It is sufficient to show that the conditions (i) and $\pi_i(x) \notin A_i(x)$ for each $x \in X$ imply the condition (i) of Corollary 2.4 holds. Suppose that the condition (i) of Corollary 2.4 does not hold, then there exist $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ and $\{x_{i_0}, \ldots, x_{i_k}\} \subset N$ such that

$$\varphi_N(\Delta_k) \cap \left(\bigcap_{j=0}^{k} \operatorname{cint} A^{-1}\big(\pi_i(x_{i_j})\big) \right) \neq \emptyset.$$

Hence there exists $\hat{x} \in \varphi_N(\Delta_k)$ such that $\hat{x} \in \operatorname{cint} A_i^{-1}(\pi_i(x_{i_j})) \subset A_i^{-1}(\pi_i(x_{i_j}))$ for all $j = 0, \ldots, k$. It follow that $\{\pi_i(x_{i_j})\colon j = 0, \ldots, k\} \subset A_i(\hat{x})$. Since $A_i(\hat{x})$ is a FC-subspace of Y_i, we have

$$\hat{x}_i = \pi_i(\hat{x}) \in \pi_i\big(\varphi_N(\Delta_k)\big) = \varphi_{N_i}(\Delta_k) \subset A_i(\hat{x})$$

which contradicts the condition that for each $x \in X$, $x_i = \pi_i(x) \notin A_i(x)$. Hence the condition (i) of Corollary 2.4 hold. The conclusion follows from Corollary 2.4. □

Remark 2.4. Corollary 2.5 generalizes Theorem 4.1 of Lin, Yu, Ansari and Lai in [27] from convex subsets of topological vector space to FC-spaces without any convexity structure.

3. Equilibria of generalized games

In this section, by using the maximal element theorems obtained in the above section, we will establish a new existence theorems for equilibrium points of generalized games with fuzzy constraint correspondences in FC-spaces.

Because of the fuzziness of consumers' behavior or market situations, in a real market, any preference of a real agent would be unstable. Therefore Kim and Tan [25] introduced the following model of generalized games with fuzzy constraint correspondences.

Let I be a finite or infinite set of agents. For each $i \in I$, let X_i be a strategy set (or commodity space) of ith agent. A generalized game $\Gamma = (X_i, A_i, F_i, P_i)_{i\in I}$ is defined as a family of ordered quadruples (X_i, A_i, F_i, P_i), where $A_i : X = \prod_{i\in I} X_i \to 2^{X_i}$ is a constraint correspondence such that $A_i(x)$ is the state attainable for ith agent; $F_i : X \to 2^{X_i}$ is a fuzzy constraint correspondence such that $F_i(x)$ is the unstable state for ith agent, and $P_i : X \times X \to 2^{X_i}$ is a preference correspondence such that $P_i(x, y)$ is the state preference of ith agent at (x, y). An equilibrium for generalized game Γ is a point $(\hat{x}, \hat{y}) \in X \times X$ such that for each $i \in I$, $\hat{x}_i = \pi_i(\hat{x}) \in A_i(\hat{x})$, $\hat{y}_i = \pi_i(\hat{y}) \in F_i(\hat{y})$, and $A_i(\hat{x}) \cap P_i(\hat{x}, \hat{y}) = \emptyset$.

If for each $i \in I$, $F_i(x) = X_i$ and $P_i(x, y) = P_i(x)$ for all $(x, y) \in X \times X$, then the above definition of a generalized game Γ and an equilibrium point of Γ coincide with the usual definition of a generalized game in [3,5,7–10,16–21,23,25,32–45].

Theorem 3.1. *Let $\Gamma = ((X_i, \{\varphi_{N_i}\}), A_i, F_i, P_i)_{i\in I}$ be a generalized game and K be a nonempty compact subset of $X = \prod_{i\in I} X_i$ such that for each $i \in I$, the following conditions are satisfied*:

- (i) *for each $x \in X$, $A_i(x)$, $F_i(x)$ are nonempty FC-subspaces of X_i,*
- (ii) *for each $y_i \in X_i$, $A_i^{-1}(y_i)$, $F_i^{-1}(y_i)$, and $P_i^{-1}(y_i)$ are compactly open,*
- (iii) *for all $(x, y) \in X \times X$, $P_i(x, y)$ is a FC-subspace of X_i and $x_i = \pi_i(x) \notin P_i(x, y)$,*
- (iv) *the set $W_i = \{(x, y) \in X \times X\colon \pi_i(x) \in A_i(x) \text{ and } \pi_i(y) \in F_i(x)\}$ is compactly closed,*
- (v) *for each $(x, y) \in X \times X$, the set $I(x, y) = \{i \in I\colon\ A_i(x) \cap P_i(x, y) \neq \emptyset\}$ is finite,*
- (vi) *for each $N_i, M_i \in \langle X_i\rangle$, there exist compact FC-subspaces L_{N_i} and L_{M_i} of $\langle X_i\rangle$ containing N_i and M_i respectively, such that for each $(x, y) \in X \times X \setminus K \times K$, there exists $(u, v) \in L_N \times L_M$, where $L_N = \prod_{i\in I} L_{N_i}$ and $L_M = \prod_{i\in I} L_{M_i}$, such that for each $i \in I(x, y)$, $\pi_i(u) \in A_i(x) \cap P_i(x, y)$ and $\pi_i(v) \in F_i(x)$.*

Then there exists $(\hat{x}, \hat{y}) \in K \times K$ such that for each $i \in I$, $\hat{x}_i = \pi_i(\hat{x}) \in A_i(\hat{x})$, $\hat{y}_i = \pi_i(\hat{y}) \in F_i(\hat{y})$, and $A_i(\hat{x}) \cap P_i(\hat{x}, \hat{y}) = \emptyset$, i.e., $(\hat{x}, \hat{y})$ is an equilibrium point of Γ.

Proof. By Lemma 1.1, $(X \times X, \{\varphi_N\})$ is also a FC-space where $X \times X = \prod_{i\in I}(X_i \times X_i)$. For each $i \in I$, define $G_i : X \times X \to 2^{X_i\times X_i}$ by

$$G_i(x, y) = \begin{cases} [A_i(x) \cap P_i(x, y)] \times F_i(x), & \text{if } (x, y) \in W_i, \\ A_i(x) \times F_i(x), & \text{if } (x, y) \notin W_i. \end{cases}$$

Then, by (i) and (iii), for each $i \in I$ and for each $(x, y) \in X \times X$, $G_i(x, y)$ is a FC-space of X_i and so the condition (i) of Corollary 2.5 is satisfied. By (iii) and the definition of W_i, we have $(x_i, y_i) = (\pi_i(x), \pi_i(y)) \notin G_i(x, y)$ for each $i \in I$ and for any $(x, y) \in X \times X$. For each $i \in I$ and for any $(u_i, v_i) \in X_i \times X_i$, we have

$$\begin{aligned} G_i^{-1}(u_i, v_i) = {} & \big[P_i^{-1}(u_i) \cap \big(A_i^{-1}(u_i) \times X\big) \cap \big(F_i^{-1}(v_i) \times X\big)\big] \\ & \cup \big[\big((X \times X) \setminus W_i\big) \cap \big(A_i^{-1}(u_i) \times X\big) \cap \big(F^{-1}(v_i) \times X\big)\big]. \end{aligned}$$

By the conditions (ii) and (iv), $G_i^{-1}(u_i, v_i)$ is compactly open-valued and hence G_i^{-1} it transfer compactly open-valued on $X_i \times X_i$. The condition (ii) of Corollary 2.5 is satisfied. The condition (v) implies that the condition (iii) of Corollary 2.5 holds. Note that G_i^{-1} is compactly open-valued, from condition (vi), we have

$$\begin{aligned} (X \times X) \setminus (K \times K) & \subset \bigcup\big\{G_i^{-1}\big(\pi_i(u), \pi_i(v)\big)\colon\ (u, v) \in L_N \times L_M\big\} \\ & = \bigcup\big\{\operatorname{cint} G_i^{-1}\big(\pi_i(u), \pi_i(v)\big)\colon\ (u, v) \in L_N \times L_M\big\} \end{aligned}$$

and so the condition (iv) of Corollary 2.5 is satisfied. By Corollary 2.5, there exists $(\hat{x}, \hat{y}) \in X \times X$ such that $G_i(\hat{x}, \hat{y}) = \emptyset$ for all $i \in I$. If $(\hat{x}, \hat{y}) \notin W_j$ for some $j \in I$, then either $A_i(\hat{x}) = \emptyset$ or $F_i(\hat{x}) = \emptyset$ which contradicts the fact that $A_i(x)$ and $F_i(x)$ are both nonempty for each $x \in X$ and for any $i \in I$. Therefore we have $(\hat{x}, \hat{y}) \in W_i$ for all $i \in I$, and hence for each $i \in I$, $\hat{x}_i = \pi_i(\hat{x}) \in A_i(\hat{x})$, $\hat{y}_i = \pi_i(\hat{y}) \in F_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}, \hat{y}) = \emptyset$. This completes the proof. □

Remark 3.1. Theorem 3.1 generalizes Theorem 5.1 of Lin, Yu, Ansari and Lai [27] from convex subsets of topological vector spaces to *FC*-spaces without any convexity structure under much weaker assumptions.

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