Norms Concerning Subdivision Sequences and Their Applications in Wavelets

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Let \( a := (a(\alpha))_{\alpha \in \mathbb{Z}^s} \) be a finitely supported sequence of \( r \times r \) matrices and \( M \) be a dilation matrix. The subdivision sequence \( \{(a_n(\alpha))_{\alpha \in \mathbb{Z}^s} : n \in \mathbb{N}\} \) is defined by

\[
a_1 = a \quad \text{and} \quad a_{n+1}(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a_n(\beta) a(\alpha - M\beta), \quad \alpha \in \mathbb{Z}^s, \quad n \in \mathbb{N}.
\]

Let \( 1 \leq p \leq \infty \) and \( f = (f^1, \ldots, f^r)^T \) be a vector of compactly supported functions in \( L_p(\mathbb{R}^s) \). The stability is not assumed for \( f \). The purpose of this paper is to give a formula for the asymptotic behavior of the \( L_p \)-norms of the combinations of the shifts of \( f \) with the subdivision sequence coefficients:

\[
\left\| \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha) f(x - \alpha) \right\|_p.
\]

Such an asymptotic behavior plays an essential role in the investigation of wavelets and subdivision schemes. In this paper we show some applications in the convergence of cascade algorithms, construction of inhomogeneous multiresolution analyzes, and smoothness analysis of refinable functions. Some examples are provided to illustrate the method.

Key Words: subdivision sequence; joint spectral radius; cascade algorithm; inhomogeneous refinement equation; wavelets; smoothness.

1. INTRODUCTION

Let \( a := (a(\alpha))_{\alpha \in \mathbb{Z}^s} \) be a finitely supported sequence of \( r \times r \) complex matrices and \( M \) be a dilation matrix on \( \mathbb{R}^s \); that is, \( M \) is an \( s \times s \) integer matrix and \( \lim_{n \to \infty} M^{-n} = 0 \).

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The subdivision sequence \( \{(a_n(\alpha))_{a \in \mathbb{Z}^s} : n \in \mathbb{N}\} \) associated with \( a \) and \( M \) is defined by \( a_1 = a \) and
\[
a_{n+1}(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a_n(\beta) a(\alpha - M\beta), \quad \alpha \in \mathbb{Z}^s, \quad n \in \mathbb{N}.
\]
(1.1)

The subdivision sequence arose naturally in subdivision schemes and wavelet analysis. It often appears as coefficients of combinations of shifts \( \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha)f(x - \alpha) \). For some applications we need to measure the asymptotic behavior of the norms of these combinations. The first purpose of this paper is to give a formula to study this asymptotic behavior.

For \( 1 \leq p \leq \infty \), let \( (L_p(\mathbb{R}^s))^r \) denote the linear space of all vectors \( f = (f_1, \ldots, f_r)^T \) such that \( f_1, \ldots, f_r \in L_p(\mathbb{R}^s) \). The norm on \( (L_p(\mathbb{R}^s))^r \) is defined by
\[
\| f \|_p := \left( \sum_{j=1}^r \| f_j \|_p^p \right)^{1/p}, \quad f = (f_1, \ldots, f_r)^T \in (L_p(\mathbb{R}^s))^r.
\]

Suppose that \( f \in (L_p(\mathbb{R}^s))^r \) is compactly supported. Then the limit
\[
\lim_{n \to \infty} \left\| \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha)f(x - \alpha) \right\|_p^{1/n}
\]
(1.2)
exists and equals the \( p \)-norm joint spectral radius of a set of \( m = |\det(M)| \) linear operators restricted to a finite dimensional space. This is the main result of Section 2. Thus, this limit can be computed explicitly for \( p \) being even integers, and be estimated for other \( p \).

The above-mentioned formula for the asymptotic behavior of the norms of combinations has several essential applications in wavelet analysis and subdivision schemes. In this paper we present three: the convergence of cascade algorithms in Sobolev spaces (Section 3), convergence of cascade algorithms associated with inhomogeneous refinement equations (Section 4), and smoothness analysis of refinable functions (Section 7). The second has an application in constructing inhomogeneous multiresolution analyzes (Section 5). Some examples are provided in Section 6 to illustrate the general theory on the convergence of cascade algorithms.

2. A FORMULA FOR THE NORMS CONCERNING SUBDIVISION SEQUENCES

In this section we provide a formula for the limit (1.2) concerning the norms of combinations of shifts with subdivision sequence coefficients.

The norm \( \| \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha)f(x - \alpha) \|_p \) can be easily expressed up to a uniform constant if the shifts \( \{f_j(x - \alpha) : 1 \leq j \leq r, \alpha \in \mathbb{Z}^s\} \) are stable; that is, there exists a positive constant \( C \) such that
\[
\| c \|_p / C \leq \left\| \sum_{j=1}^r \sum_{\alpha \in \mathbb{Z}^s} c_j(\alpha)f_j(x - \alpha) \right\|_p \leq C\| c \|_p.
\]
(2.1)
Here for $c = (c_1, \ldots, c_r)^T \in (\ell_p(\mathbb{Z}^s))^r$, the linear space of vectors of $\ell_p(\mathbb{Z}^s)$ sequences, the norm $\|c\|_p$ is defined by

$$\|c\|_p := \left( \sum_{j=1}^r \|c_j\|_p^p \right)^{1/p}.$$  

In general, without assuming stability, there always exist $d \in \mathbb{N}$ and a vector of compactly supported functions $g = (g^1, \ldots, g^d)^T \in (L_p(\mathbb{R}^s))^d$ such that the shifts of $g$ are stable and

$$f(x) = \sum_{\alpha \in \mathbb{Z}^s} b(\alpha) g(x - \alpha), \quad (2.2)$$

where $b := (b(\alpha))_{\alpha \in \mathbb{Z}^s}$ is in $((\ell_0(\mathbb{Z}^s))^r)^d$, the space of all finitely supported sequences of $r \times d$ matrices. Such a vector $g$ is called a generator of the shift-invariant space

$$S(f) := \left\{ \sum_{j=1}^r \sum_{\alpha \in \mathbb{Z}^s} c_j(\alpha) f_j(x - \alpha) : c_j(\alpha) \in \mathbb{C} \right\}. \quad (2.3)$$

In fact, we may even choose generators $g$ with linearly independent shifts by taking a basis of $\text{span}\{f_j|_{\alpha+\{0,1\}^s} : 1 \leq j \leq r, \alpha \in \mathbb{Z}^s\}$. We denote $G(f)$ as the set of all generators of $S(f)$.

A generator $g$ of $S(f)$ is called perfect if $S(f) = S(g)$. In the univariate case $s = 1$, it was shown by de Boor and DeVore [1] ($p = \infty$, $r = 1$), Ron [20] ($r = 1$), and generally by Jia [11] that perfect generators with $d \leq r$ always exist. In the multivariate case $s > 1$, perfect generators may not exist. This can be seen from the examples of de Boor and Höllig [2] ($r = 2$), Ron [21] ($r = 1$), and the characterization of approximation order of finitely generated shift-invariant spaces (e.g., [12]).

Suppose that $g \in G(f)$ and (2.2) holds. Then

$$\left\| \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha) f(x - \alpha) \right\|_p = \left\| \sum_{\alpha \in \mathbb{Z}^s} a_n \ast b(\alpha) g(x - \alpha) \right\|_p,$$

where $a_n \ast b \in (\ell_0(\mathbb{Z}^s))^r$ is the convolution given by

$$a_n \ast b(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a_n(\beta) b(\alpha - \beta), \quad \alpha \in \mathbb{Z}^s.$$  

The $p$-norm on $(\ell_0(\mathbb{Z}^s))^r$ is defined by

$$\|c\|_p := \left( \sum_{j=1}^r \sum_{k=1}^d \|c_{jk}\|_p^p \right)^{1/p}, \quad c(\alpha) = (c_{jk}(\alpha))_{1 \leq j \leq r, 1 \leq k \leq d}, \alpha \in \mathbb{Z}^s.$$  

It follows from the stability assumption for $g$ that for all $n \in \mathbb{N},$

$$\|a_n \ast b\|_p / C \leq \left\| \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha) f(x - \alpha) \right\|_p \leq C \|a_n \ast b\|_p. \quad (2.4)$$
Thus, we only need to understand the norm of the sequence $a_n \ast b$. To this end, we need to introduce some linear operators.

For $\varepsilon \in \mathbb{Z}^s$, we define the linear operator $A_\varepsilon$ on $(\ell_0(\mathbb{Z}^s))^{r \times 1}$ as

$$A_\varepsilon v(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a(\varepsilon + M\alpha - \beta)v(\beta), \quad \alpha \in \mathbb{Z}^s, \ v \in (\ell_0(\mathbb{Z}^s))^{r \times 1}. \quad (2.5)$$

Then the relation between $a_n \ast b$ and the linear operator $\{A_\varepsilon : \varepsilon \in \mathbb{Z}^s\}$ can be seen from the following result. This result was proved by Goodman et al. [9] for $r = 1$, $M = 2I$, by Han and Jia [10] for $r = 1$ and the general dilation matrix $M$, and by Jia et al. [17] for $r > 1$, $s = 1$.

**Lemma 2.1.** Let $\alpha = \varepsilon_1 + M\varepsilon_2 + \cdots + M^{n-1}\varepsilon_n + M^n\gamma$ with $\varepsilon_1, \ldots, \varepsilon_n, \gamma \in \mathbb{Z}^s$. Then for $v \in (\ell_0(\mathbb{Z}^s))^{r \times 1}$, $n \in \mathbb{N}$,

$$a_n \ast v(\alpha) = A_{\varepsilon_n} \cdots A_{\varepsilon_2} A_{\varepsilon_1} v(\gamma).$$

**Proof.** The case $n = 1$ is trivial. Suppose the statement is true for $n$. Let $\alpha = \varepsilon_1 + M\alpha_1$. Then by the definition (1.1)

$$a_{n+1} \ast v(\alpha) = \sum_{\beta \in \mathbb{Z}^s} \sum_{\eta \in \mathbb{Z}^s} a_n(\eta) a(\varepsilon_1 + M\alpha_1 - \beta - M\eta)v(\beta)$$

$$= \sum_{\eta \in \mathbb{Z}^s} a_n(\alpha_1 - \eta) \sum_{\beta \in \mathbb{Z}^s} a(\varepsilon_1 + M\eta - \beta)v(\beta) = a_n \ast (A_{\varepsilon_1} v)(\alpha_1).$$

Set $\alpha_1 = \varepsilon_2 + \cdots + M^{n-1}\varepsilon_{n+1} + M^n\gamma$. The induction hypothesis tells us that

$$a_n \ast (A_{\varepsilon_1} v)(\alpha_1) = A_{\varepsilon_n} \cdots A_{\varepsilon_2} (A_{\varepsilon_1} v)(\gamma).$$

Therefore,

$$a_{n+1} \ast v(\alpha) = A_{\varepsilon_n} \cdots A_{\varepsilon_2} A_{\varepsilon_1} v(\gamma),$$

thereby completing the induction procedure. $\blacksquare$

Applying Lemma 2.1 we can reduce $\|a_n \ast b\|_p$ and then the limit (1.2) into the $p$-norm joint spectral radius.

Let $\mathcal{A}$ be a finite collection of linear operators on a vector space $V$, which is not necessarily finite dimensional. A subspace of $V$ is said to be $\mathcal{A}$-invariant if it is invariant under every operator in $\mathcal{A}$. For $v \in V$, we call the intersection of all $\mathcal{A}$-invariant subspaces of $V$ containing $v$ the minimal $\mathcal{A}$-invariant subspace generated by $v$, denoted as $V(v)$. If $W$ is an $\mathcal{A}$-invariant subspace of $V$ with $\dim W < \infty$ and $v \in W$, then $V(v)$ is spanned by

$$\{A_1 \cdots A_j v : A_1, \ldots, A_j \in \mathcal{A}, \ j = 0, 1, \ldots, \dim W - 1\}.$$

Suppose that $V(v)$ is finite dimensional. We choose an arbitrary norm $\| \cdot \|$ on $V(v)$. For a linear operator $A$ on $V(v)$,

$$\|A\| = \max\{\|Au\| : u \in V(v), \|u\| = 1\}.$$
Define
\[ \|A^n|_{V(v)}\|_p := \begin{cases} \left( \sum_{A_1, \ldots, A_n \in A} \|A_1 \cdots A_n|_{V(v)}\|_p \right)^{1/p}, & \text{if } 0 < p < \infty, \\ \max_{A_1, \ldots, A_n \in A} \|A_1 \cdots A_n|_{V(v)}\|, & \text{if } p = \infty. \end{cases} \]

Then the \( p \)-norm joint spectral radius of \( A|_{V(v)} \) is defined to be
\[ \rho_p(A|_{V(v)}) = \lim_{n \to \infty} \|A^n|_{V(v)}\|_p^{1/n}. \]

It is easily seen that \( \rho_p(A|_{V(v)}) \) is independent of the choice of the vector norm \( \| \cdot \| \) on \( V(v) \), and
\[ \lim_{n \to \infty} \|A^n|_{V(v)}\|_p^{1/n} = \inf_{n \in \mathbb{N}} \|A^n|_{V(v)}\|_p^{1/n}. \]

The uniform joint spectral radius (\( p = \infty \)) was introduced by Rota and Strang [22] and applied to the investigation of wavelets by Daubechies and Lagarias [7]. The mean joint spectral radius (\( p = 1 \)) was studied by Wang [27]. The \( p \)-norm joint spectral radius was introduced by Jia [13] for \( 1 \leq p \leq \infty \), while for \( 0 < p < 1 \) it appeared in [28].

The \( p \)-norm joint spectral radius is hard to compute if one uses the definition, since the limit in the definition is reached very slowly. An efficient formula provided by Zhou [28] is to compute the \( p \)-norm joint spectral radius in terms of the spectral radius of some finite matrix when \( p \) is an even integer. With this formula we can estimate the \( p \)-norm joint spectral radius for other \( p \) by the relation among the \( p \)-norm joint spectral radii presented by Strang and Zhou in [26].

Denote
\[ \|A^n_{v}\|_p := \begin{cases} \left( \sum_{A_1, \ldots, A_n \in A} \|A_1 \cdots A_n v\|_p \right)^{1/p}, & \text{if } 0 < p < \infty, \\ \max_{A_1, \ldots, A_n \in A} \|A_1 \cdots A_n v\|, & \text{if } p = \infty. \end{cases} \]

Then
\[ \rho_p(A|_{V(v)}) = \lim_{n \to \infty} \|A^n_{v}\|_p^{1/n}. \]

This relation was proved for \( 1 \leq p \leq \infty \) by Han and Jia [10]. Moreover, there exists a positive constant \( C \) such that
\[ \|A^n|_{V(v)}\|_p / C \leq \|A^n_{v}\|_p \leq C \|A^n|_{V(v)}\|_p, \quad n \in \mathbb{N}. \]

In our application of joint spectral radius, we consider the space \( V = (\ell_0(Z))^{r \times 1} \).

The collection \( A \) is the set of linear operators \( \{A_\varepsilon : \varepsilon \in E\} \), where \( E \) is a complete set of representatives of the distinct cosets of the quotient group \( Z^s/MZ^s \) containing \( 0 \) and \( A_\varepsilon \) is defined by (2.5). In these circumstances, for any \( v \in (\ell_0(Z^s))^{r \times 1} \), the minimal \( A \)-invariant subspace \( V(v) \) is always finite dimensional. In fact, if \( \Omega := \{\alpha \in Z^s : a(\alpha) \neq 0\} \) and \( G := \{\alpha \in Z^s : v(\alpha) \neq 0\} \), then \( V(v) \subset \ell(K) \), where \( \ell(K) \) denotes the linear subspace of \( (\ell_0(Z^s))^{r \times 1} \) consisting of all sequences supported on \( K = (\sum_{n=1}^{\infty} M^{-n} (MG \cup (\Omega - E) \cup \{0\})) \cap Z^s \). See the proof of Lemma 2.3 in [10]. Therefore, Lemma 2.1 tells us that
\[ \lim_{n \to \infty} \|a_n \ast v\|_p^{1/n} = \lim_{n \to \infty} \|A^n v\|_p^{1/n} = \rho_p(A|_{V(v)}). \]
Combining all the above discussions, we have the following main result on norms of combinations of shifts with subdivision sequence coefficients.

**Theorem 1.** Let \( a \) be a finitely supported sequence of \( r \times r \) complex matrices and \( f \in (L_p(\mathbb{R}^s))^r \) be compactly supported. Suppose that \( 1 \leq p \leq \infty \), \( g \in G(f) \) and (2.2) holds. If \( M \) is a dilation matrix with \( m = |\det(M)| \) and \( E \) is a complete set of representatives of \( \mathbb{Z}^s / M\mathbb{Z}^s \) containing 0, then

\[
\lim_{n \to \infty} \left\| \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha) f(x - \alpha) \right\|_{L_p^1}^{1/n} = \max_{1 \leq j \leq d} \rho_p \left( \{ A_\epsilon | V(\epsilon) \in E \} \right),
\]

where \( e_j \) denotes the \( j \)th column of the \( d \times d \) identity matrix, and \( b \) is the sequence in \((\ell_0(\mathbb{Z}^s))^r \times 1\) given by \( b(\alpha) e_j \) for \( \alpha \in \mathbb{Z}^s \). Moreover, if \( \rho > 0 \) and

\[
\rho^n \left\| \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha) f(x - \alpha) \right\|_p \to 0 \quad (n \to \infty),
\]

then

\[
\max_{1 \leq j \leq d} \rho_p \left( \{ A_\epsilon | V(\epsilon) \in E \} \right) < 1/\rho.
\]

### 3. Convergence of Cascade Algorithms in Sobolev Spaces

The central equation in wavelet analysis is the **refinement equation**

\[
\phi(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \phi(Mx - \alpha), \quad x \in \mathbb{R}^s.
\]

The sequence \( a \in (\ell_0(\mathbb{Z}^s))^r \times r \) is called the **refinement mask**.

To solve the refinement equation, a useful tool is the so-called cascade algorithm. Starting from an initial vector \( \phi_0 \) of compactly supported functions, we define a sequence \( \{\phi_n\}_{n \in \mathbb{N}} \) by iterating (3.1)

\[
\phi_n(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \phi_{n-1}(Mx - \alpha), \quad n \in \mathbb{N}.
\]

If the sequence \( \{\phi_n\} \) converges in some function space, then the limit \( \phi \) is a solution of (3.1) in that function space. In wavelet analysis, we are often interested in \((L_p(\mathbb{R}^s))^r\) and the Sobolev spaces \((W^k_p(\mathbb{R}^s))^r\). The Sobolev space \( W^k_p(\mathbb{R}^s) \) consists of all functions \( f \in L_p(\mathbb{R}^s) \) such that \( D^\mu f \in L_p(\mathbb{R}^s) \) for all multiindices \( \mu = (\mu_1, \ldots, \mu_s) \in (\mathbb{N} \cup \{0\})^s \) with \( |\mu| := |\mu_1| + \cdots + |\mu_s| \leq k \), and is equipped with the norm

\[
\| f \|_{W^k_p(\mathbb{R}^s)} := \sum_{|\mu| \leq k} \| D^\mu f \|_p.
\]

In the univariate case \( s = 1 \), the convergence of the cascade algorithms was characterized for \( r = 1 \) by Dyn et al. [8] in \( C(\mathbb{R}) \) and by Jia [13] in \( L_p(\mathbb{R}) \), and for \( r > 1 \) by Jia et al. [17] in \((L_p(\mathbb{R}))^r\).
In the multivariate case \( s > 1 \), the cascade algorithm was considered for \( r = 1 \) by Cavaretta et al. [3], and for \( r > 1 \) by Dahmen and Micchelli [5]. The characterizations for the convergence were given by Han and Jia [10] for \( r = 1 \) in \( L_p \), independently by Strang [24] and Lawton et al. [18] for \( r = 1 \) in \( L_2 \), by Shen [23] for \( r > 1 \) in \( (L_2)^r \), and by Jia et al. [15] for \( r = 1 \) in \( W^1_k(R^s) \) when \( M \) is isotropic. In all the above characterizations, the initial \( \phi_0 \) is chosen to be either a stable one or a large family of (vectors of) functions.

The purpose of this section is to use our formula given in Theorem 1 and present a general approach to the convergence of cascade algorithms in \((W^k_p(R^s))^r\) with an arbitrary initial vector of functions \( \phi_0 \in (W^k_p(R^s))^r \) and diagonalizable dilation matrix \( M \). The norm on \((W^k_p(R^s))^r\) is given by

\[
\|f\|_{(W^k_p(R^s))^r} := \left( \sum_{j=1}^r \|f^j\|_{W^k_p(R^s)}^p \right)^{1/p}, \quad f = (f^1, \ldots, f^r)^T \in (W^k_p(R^s))^r.
\]

Observe that the definition of the subdivision sequence (1.1) tells us the following equivalent form for \( \{a_n\}_{n \in \mathbb{N}} \).

**Lemma 3.1.** Let \( a \) be a finitely supported sequence of \( r \times r \) matrices and \( \{a_n\} \) be defined by (1.1). Then

\[
a_{n+1}(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a(\beta)a_n(\alpha - M^n \beta), \quad \alpha \in \mathbb{Z}^s, \ n \in \mathbb{N}.
\]

The proof of Lemma 3.1 can be easily given by induction on \( n \), which is similar to the discussion in [29] and is omitted here.

Lemma 3.1 allows us to express the sequence \( \{\phi_n\} \) in (3.2) as

\[
\phi_n(x) = \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha)\phi_0(M^n x - \alpha), \quad n \in \mathbb{N}.
\]

Hence for \( n \in \mathbb{N} \),

\[
\phi_{n+1}(x) - \phi_n(x) = \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha) \left( \sum_{\beta \in \mathbb{Z}^s} a(\beta)\phi_0(M^n x - \beta) - \phi_0 \right)(M^n x - \alpha).
\]

Write

\[
f(x) := \sum_{\beta \in \mathbb{Z}^s} a(\beta)\phi_0(M x - \beta) - \phi_0(x).
\]

Then

\[
\phi_{n+1}(x) - \phi_n(x) = \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha) f(M^n x - \alpha). \quad (3.3)
\]

We are in a position to state our main result on the convergence of cascade algorithms in Sobolev spaces. Let us mention that even for the special case \( p = 2 \), our result is new, since we only assume that the dilation matrix \( M \) is diagonalizable, which is more general than isotropic matrices.
THEOREM 2. Let $a := (a(\alpha))_{\alpha \in \mathbb{Z}^s}$ be a finitely supported sequence of $r \times r$ matrices and $M$ be a diagonalizable dilation matrix such that

$$\Lambda M \Lambda^{-1} = \text{diag}(\sigma_1, \ldots, \sigma_s),$$

where $\Lambda = (\lambda_{jl})_{1 \leq j, l \leq s}$ is an invertible matrix. Suppose that $k \in \mathbb{N} \cup \{0\}$, $1 \leq p \leq \infty$, and $\phi_0 \in (W_p^k(\mathbb{R}^s))^r$ is compactly supported. Define the sequence $\{\phi_n\}$ by (3.2). Then the sequence $\{\phi_n\}$ converges in $(W_p^k(\mathbb{R}^s))^r$ if and only if for each multiindex $\mu = (\mu_1, \ldots, \mu_s)$ with $\mu = 0$ or $|\mu| = k$,

$$\rho_p \left( \left\{ A_\varepsilon \mid V(b_\mu \varepsilon_j) : \varepsilon \in E \right\} \right) < \frac{m^{1/p}}{|\sigma_1|^{|\mu_1|} \cdots |\sigma_s|^{|\mu_s|}}, \quad j = 1, \ldots, d_\mu, \quad (3.4)$$

where for the shift-invariant space $S(F_\mu)$ generated by

$$F_\mu := \left\{ \prod_{l=1}^s \left( \sum_{l=1}^s \lambda_{jl} D_l \right)^{\mu_j} \right\} \left( \sum_{\beta \in \mathbb{Z}^s} a(\beta) \phi_0(M \cdot - \beta) - \phi_0 \right)$$

we take a generator $g_\mu = (g_{1\mu}, \ldots, g_{d_\mu \mu})^T \in G(F_\mu)$ and $b_\mu \in (\ell_0(\mathbb{Z}^s))^r \times d_\mu$ is the coefficient sequence in (2.2) given by

$$F_\mu = \sum_{\alpha \in \mathbb{Z}^s} b_\mu(\alpha) g_\mu(\cdot - \alpha). \quad (3.5)$$

Proof. Define $q_j$ to be the linear polynomial

$$q_j(x) := \sum_{l=1}^s \lambda_{jl} x_l, \quad x = (x_1, \ldots, x_s) \in \mathbb{R}^s.$$

This means that the coefficients of $q_j$ are given by the $j$th row of $\Lambda$. For the multiindex $\mu = (\mu_1, \ldots, \mu_s)$ let

$$q_\mu(x) = q_1(x)^{\mu_1} \cdots q_s(x)^{\mu_s}.$$

Since $\Lambda$ is invertible, the set $\{q_\mu : |\mu| = k\}$ forms a basis of the space of all homogeneous polynomials of exact degree $k$. Then an equivalent norm $\| \cdot \|$ on $(W_p^k(\mathbb{R}^s))^r$ can be defined as

$$\|g\| := \|g\|_p + \sum_{|\mu| = k} \|q_\mu(D)g\|_p, \quad g \in (W_p^k(\mathbb{R}^s))^r.$$ 

Let $g$ be a differentiable function on $\mathbb{R}^s$. Then

$$\begin{bmatrix} D_1 \\ \vdots \\ D_s \end{bmatrix} (g(M^n \cdot))(x) = M^n \begin{bmatrix} D_1 \\ \vdots \\ D_s \end{bmatrix} g(M^n x), \quad x \in \mathbb{R}^s.$$ 

It follows that

$$\Lambda \begin{bmatrix} D_1 \\ \vdots \\ D_s \end{bmatrix} (g(M^n \cdot))(x) = \text{diag}(\sigma_1^n, \ldots, \sigma_s^n) \Lambda \begin{bmatrix} D_1 \\ \vdots \\ D_s \end{bmatrix} g(M^n x).$$
Hence for \( j = 1, \ldots, s \),
\[
q_j(D)(g(M^n\cdot))(x) = \sigma_j^n q_j(D)g(M^n x),
\]
and for \( \mu \in (\mathbb{N} \cup \{ 0 \})^s \),
\[
q_\mu(D)(g(M^n\cdot))(x) = (\sigma_1^{\mu_1} \cdots \sigma_s^{\mu_s})^n q_\mu(D)g(M^n x).
\]
This in connection with (3.3) tells us that
\[
q_\mu(D)(\phi_{n+1} - \phi_n)(x) = (\sigma_1^{\mu_1} \cdots \sigma_s^{\mu_s})^n \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha)q_\mu(D)f(M^n x - \alpha).
\] (3.6)

Suppose that \( \{ \phi_n \} \) converges in \( (W^k_p(\mathbb{R}^s))^r \). Then \( \| \phi_{n+1} - \phi_n \|_{(W^k_p(\mathbb{R}^s))^r} \to 0 \) as \( n \to \infty \) and for any multiindex \( \mu \) with either \( \mu = 0 \) or \( |\mu| = k \),
\[
\| q_\mu(D)(\phi_{n+1} - \phi_n) \|_p \to 0 \quad (n \to \infty).
\]
This in connection with (3.6) implies that for \( \mu = 0 \) or \( |\mu| = k \),
\[
(|\sigma_1^{|\mu_1|} \cdots |\sigma_s^{|\mu_s|}| m^{-n/p}) \left| \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha)q_\mu(D)f(x - \alpha) \right|_p \to 0 \quad (n \to \infty).
\]
Observing that \( g_\mu \in G(F_\mu) = G(q_\mu(D)f) \), Theorem 1 tells us that for \( \mu = 0 \) or \( |\mu| = k \), \( j = 1, \ldots, d_\mu \),
\[
\rho_p(\{ A_\varepsilon | V(b_\varepsilon e_j) : \varepsilon \in E \}) < \frac{m^{1/p}}{|\sigma_1^{|\mu_1|} \cdots |\sigma_s^{|\mu_s|}|}.
\]
This proves the necessity.

To see the sufficiency, suppose that (3.4) holds. By Theorem 1, there exist some \( \rho \in (0, 1) \) and a positive constant \( C_1 \) such that for \( \mu = 0 \) or \( |\mu| = k \),
\[
\left| \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha)q_\mu(D)f(x - \alpha) \right|_p \leq C_1 \left( \frac{\rho m^{1/p}}{|\sigma_1^{|\mu_1|} \cdots |\sigma_s^{|\mu_s|}|} \right)^n, \quad n \in \mathbb{N}.
\]
Hence
\[
\| q_\mu(D)(\phi_{n+1} - \phi_n) \|_p \leq C_1 \rho^n, \quad n \in \mathbb{N}.
\]
This shows that
\[
\| \phi_{n+1} - \phi_n \|_{(W^k_p(\mathbb{R}^s))^r} \leq C_1 (1 + s^k) \rho^n, \quad n \in \mathbb{N}.
\]
Thus, \( \{ \phi_n \} \) is a Cauchy sequence in \( (W^k_p(\mathbb{R}^s))^r \). Therefore, the cascade algorithm \( \{ \phi_n \} \) converges in this space. This proves Theorem 2. \( \blacksquare \)
4. CASCADE ALGORITHMS FOR INHOMOGENEOUS REFINEMENT EQUATIONS

The **inhomogeneous refinement equation** was introduced in the univariate case by Strang and Nguyen [25] as

\[
\phi(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \phi(Mx - \alpha) + F(x), \quad x \in \mathbb{R}^s.
\] (4.1)

The cascade algorithm associated with the inhomogeneous refinement equation (4.1) is defined by

\[
\phi_n(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \phi_{n-1}(Mx - \alpha) + F(x), \quad n \in \mathbb{N}.
\] (4.2)

If the sequence \(\{\phi_n\}\) converges in some function space, then the limit is a solution of (4.1) in that space. The \(L_p\)-convergence of cascade algorithms associated with inhomogeneous refinement equations was first characterized by Strang and Zhou [26] in the univariate scalar case \((s = 1, r = 1)\). Jia et al. [16] gave a characterization for the \(L_2\)-convergence. Li [19] studied the convergence in Sobolev spaces \((W^k_2(\mathbb{R}^s))^r\) with \(M = 2I\). The main difficulty in the \(L_p\)-convergence in the multivariate case is the lack of perfect generators. In this paper we apply our result on norms concerning subdivision sequences to overcome this difficulty and give a characterization for the \(L_p\)-convergence of cascade algorithms associated with inhomogeneous refinement equations as follows.

**THEOREM 3.** Let \(a \in (\ell_0(\mathbb{Z}^s))^r \times r\), \(M\) be a dilation matrix and \(F\) be a vector of compactly supported functions in \(L_p(\mathbb{R}^s)\) with \(1 \leq p \leq \infty\). Suppose that \(\phi_0 \in (L_p(\mathbb{R}^s))^r\) is compactly supported. Define the sequence \(\{\phi_n\}\) by (4.2). Then \(\{\phi_n\}\) converges in \((L_p(\mathbb{R}^s))^r\) if and only if

\[
\rho_p \left( \{ A_\alpha | V(\{b_{\text{ej}}\}) : \varepsilon \in E \} \right) < m^{1/p}, \quad j = 1, \ldots, d,
\]

where \(b\) is the sequence of \(r \times d\) matrices given in (2.2) for

\[
f(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \phi_0(Mx - \alpha) - \phi_0(x) + F(x)
\] (4.3)

and \(g \in G(f)\).

**Proof.** By iterating (4.2), we have

\[
\phi_n(x) = \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha) \phi_0(M^n x - \alpha) + \sum_{j=1}^{n-1} \sum_{\alpha \in \mathbb{Z}^s} a_j(\alpha) F(M^j x - \alpha) + F(x).
\]

It follows that

\[
\phi_{n+1}(x) - \phi_n(x) = \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha) f(M^n x - \alpha),
\]

where \(f\) is given by (4.3).

If \(\{\phi_n\}\) converges in \((L_p(\mathbb{R}^s))^r\), then

\[
\| \phi_{n+1} - \phi_n \|_p = m^{-n/p} \left\| \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha) f(x - \alpha) \right\|_p \to 0 \quad (n \to \infty).
\]
Since \( g \in G(f) \), Theorem 1 with \( \rho = m^{-1/p} \) tells us that for \( j = 1, \ldots, d \),

\[
\rho_p \left( \{A_k | V(\mu_j) : \mu \in E \} \right) < m^{1/p}.
\]

This proves the necessity.

Conversely, if \( \rho_p \left( \{A_k | V(\mu_j) : \mu \in E \} \right) < m^{1/p} \) for \( j = 1, \ldots, d \), then (2.7) tells us that there exist constant \( \rho \in (0, 1) \) and \( C_2 > 0 \) such that

\[
\|a_n * b\|_p \leq C_2 (\rho m^{1/p})^n, \quad n \in \mathbb{N}.
\]

By (2.4) this shows that

\[
\left\| \sum_{a \in \mathbb{Z}^d} a_n(\alpha) f(x - \alpha) \right\|_p \leq CC_2 (\rho m^{1/p})^n, \quad n \in \mathbb{N}.
\]

It follows that

\[
\|\phi_{n+1} - \phi_n\|_p = m^{-n/p} \left\| \sum_{a \in \mathbb{Z}^d} a_n(\alpha) f(x - \alpha) \right\|_p \leq CC_2 \rho^n, \quad n \in \mathbb{N}.
\]

Therefore, \( \{\phi_n\} \) is a Cauchy sequence and converges in \( (L_p(\mathbb{R}^d))^\prime \).

The proof of Theorem 3 is complete.

5. INHOMOGENEOUS MULTIRESOLUTION ANALYSIS

The (homogeneous) refinement equation plays an essential role in constructing multiresolution analyzes and wavelets, see [6, 25]. It is natural for us to investigate inhomogeneous multiresolution analyzes based on the study of inhomogeneous refinement equations. Here we consider only the case when \( r = 1, s = 1 \), and \( M = 2 \).

Suppose that \( \phi \in L_2(\mathbb{R}) \) is a solution of (4.1). We hope to construct a closed subspace \( V_0 \) of \( L_2(\mathbb{R}) \) such that the following two conditions hold:

(i) \( \{\phi(x - k) : k \in \mathbb{Z}\} \subset V_0 \);
(ii) \( V_j \subset V_{j+i}, j \in \mathbb{Z} \), where \( V_j := \{f(2^j x) : f \in V_0\} \).

This in connection with (4.1) implies that \( F(2^{-j}x - k) \in V_0 \) for \( j \in \mathbb{N} \) and \( k \in \mathbb{Z} \). Thus, it is natural for us to define \( V_0 \) as the closed span of \( \{\phi(x - k) : k \in \mathbb{Z}\} \cup \{g(2^{-j}x - k) : j \geq 0, k \in \mathbb{Z}\} \), where \( g \in L_2(\mathbb{R}) \) has orthonormal shifts and for some sequence \( b \),

\[
F(x/2) = \sum_{k \in \mathbb{Z}} b(k) g(x/2 - k).
\]

Hence \( b \in \ell_2(\mathbb{Z}) \). In particular, this holds when \( g \) is a perfect generator of \( S(F(\cdot/2)) \).

Define the symbol of a sequence \( c \in \ell_2(\mathbb{Z}) \) as

\[
\tilde{c}(\xi) = \sum_{k \in \mathbb{Z}} c(k) e^{-ik\xi}/2, \quad \xi \in \mathbb{R}.
\]
Theorem 4. Suppose that $g \in L_2(\mathbb{R})$ has orthonormal shifts and for a sequence $b \in \ell_2(\mathbb{Z})$,

$$F(x/2) = \sum_{k \in \mathbb{Z}} b(k) g(x - k).$$

Let $a \in \ell_2(\mathbb{Z})$. If $\phi \in L_2(\mathbb{R})$ is a solution of (4.1) such that $\{\phi(x - k), g(x - k)\}$ forms an orthonormal system in $L_2(\mathbb{R})$, then

$$\begin{bmatrix} \hat{a}(\xi) & \hat{b}(\xi) \end{bmatrix} \begin{bmatrix} \hat{a}(\xi + \pi) \\ \hat{b}(\xi + \pi) \end{bmatrix} = 1, \quad \text{a.e.} \; \xi \in \mathbb{R}. \tag{5.1}$$

Proof. Taking the Fourier transform in (4.1), we have

$$\sum_{k \in \mathbb{Z}} |\hat{\phi}(\xi + 2k\pi)|^2 = \sum_{k \in \mathbb{Z}} |\hat{\phi}(\xi/2 + k\pi)\hat{\phi}(\xi/2 + k\pi) + \hat{b}(\xi/2 + k\pi)\hat{g}(\xi/2 + k\pi)|^2$$

$$= \sum_{j=0,1} \left| \hat{a}(\xi/2 + j\pi) \right|^2 \langle \hat{\phi}, \hat{\phi} \rangle (\xi/2 + j\pi)$$

$$+ \sum_{j=0,1} \left| \hat{b}(\xi/2 + j\pi) \right|^2 \langle \hat{g}, \hat{g} \rangle (\xi/2 + j\pi)$$

$$+ \sum_{j=0,1} \hat{a}(\xi/2 + j\pi)\hat{b}(\xi/2 + j\pi)\langle \hat{\phi}, \hat{g} \rangle (\xi/2 + j\pi)$$

$$+ \sum_{j=0,1} \hat{b}(\xi/2 + j\pi)\hat{a}(\xi/2 + j\pi)\langle \hat{g}, \hat{\phi} \rangle (\xi/2 + j\pi).$$

Here for $g_1, g_2 \in L_2(\mathbb{R})$,

$$[g_1, g_2](x) := \sum_{k \in \mathbb{Z}} g_1(x + 2k\pi)g_2(x + 2k\pi).$$

We know that almost everywhere

$$[\hat{g}_1, \hat{g}_2](\xi) = \sum_{k \in \mathbb{Z}} \langle g_1(\cdot + k), g_2 \rangle e^{-ik\xi} \hat{g}(\xi).$$

It follows from the orthonormality of $\{\phi(x - k), g(x - k)\}$ that

$$1 = |\hat{a}(\xi/2)|^2 + |\hat{a}(\xi/2 + \pi)|^2 + |\hat{b}(\xi/2)|^2 + |\hat{b}(\xi/2 + \pi)|^2, \quad \text{a.e.} \; \xi \in \mathbb{R}. \tag{5.2}$$

Observe that (5.2) is the same expression as (5.1). Hence Theorem 4 holds.

The necessary condition (5.1) or (5.2) for the orthonormality is a generalization of the orthonormality condition for multiple refinable functions and multiple wavelets.

If $g(x/2) \in \text{span}\{\phi(x - k), g(x - k)\}$, then our framework is simply the multiresolution analysis for multiple wavelets.

Now we apply the convergence of cascade algorithms associated with inhomogeneous refinement equations discussed in Section 4 and give some conditions for the framework of the inhomogeneous multiresolution analysis proposed in this paper.
THEOREM 5. Suppose that $g \in L_2(\mathbb{R})$ has orthonormal shifts and for a sequence $b \in \ell_2(\mathbb{Z})$,

$$F(x/2) = \sum_{k \in \mathbb{Z}} b(k)g(x-k).$$

Assume that a sequence $a \in \ell_2(\mathbb{Z})$ together with $b$ satisfies (5.1). Let \{$\phi_0(x-k)$\}_{k \in \mathbb{Z}} \cup \{2^{-j/2}g(2^{-j}x-k) : j \geq 0, k \in \mathbb{Z}\}$ be an orthonormal system in $L_2(\mathbb{R})$. Define the sequence \{$\phi_n$\} by (4.2). If \{$\phi_n$\} converges to $\phi$ in $L_2(\mathbb{R})$, then \{$\phi(x-k)$\}_{k \in \mathbb{Z}} \cup \{2^{-j/2}g(2^{-j}x-k) : j \geq 0, k \in \mathbb{Z}\}$ forms an orthonormal system in $L_2(\mathbb{R})$. Define

$$V_j := \text{span}\{\phi(2^jx-k), 2^{n/2}g(2^n x-k) : n \leq j, k \in \mathbb{Z}\},$$

then $V_j \subset V_{j+1}$ for $j \in \mathbb{Z}$.

Proof. Let $n \in \mathbb{N}$, $j \geq 0$ and $k, l \in \mathbb{Z}$. Then by (4.2)

$$\langle \phi_n(x-k), 2^{-j/2}g(2^{-j}x-l) \rangle = \sum_{\alpha \in \mathbb{Z}} a(\alpha)(\phi_{n-1}(2x-2k-\alpha), 2^{-j/2}g(2^{-j}x-l))$$

$$+ \sum_{\alpha \in \mathbb{Z}} b(\alpha)(g(2x-2k-\alpha), 2^{-j/2}g(2^{-j}x-l))$$

$$= \sum_{\alpha \in \mathbb{Z}} a(\alpha)/\sqrt{2}(\phi_{n-1}(x-2k-\alpha), 2^{-(j+1)/2}g(2^{-j-1}x-l)).$$

Hence an induction procedure shows that

$$\langle \phi_n(x-k), 2^{-j/2}g(2^{-j}x-l) \rangle = 0, \quad n \in \mathbb{N}, j \geq 0, k, l \in \mathbb{Z}.$$ 

The convergence of \{$\phi_n$\} to $\phi$ tells that

$$\langle \phi(x-k), 2^{-j/2}g(2^{-j}x-l) \rangle = 0, \quad j \geq 0, k, l \in \mathbb{Z}.$$ 

It also follows that

$$[\hat{\phi}_n, \hat{\phi}_n] = 0, \quad n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$. Then the same computation as in the proof of Theorem 4 shows that

$$[\hat{\phi}_n, \hat{\phi}_n](\xi) = |\tilde{a}(\xi/2)|^2[\hat{\phi}_{n-1}, \hat{\phi}_{n-1}](\xi/2) + |\tilde{b}(\xi/2 + \pi)|^2[\hat{\phi}_{n-1}, \hat{\phi}_{n-1}](\xi/2 + \pi)$$

$$+ |\tilde{b}(\xi/2)|^2[\hat{\phi}, \hat{\phi}](\xi/2) + |\tilde{b}(\xi/2 + \pi)|^2[\hat{\phi}, \hat{\phi}](\xi/2 + \pi).$$

This in connection with (5.1) (which is the same as (5.2)) and an induction procedure implies that almost everywhere

$$[\hat{\phi}_n, \hat{\phi}_n](\xi) = \sum_{k \in \mathbb{Z}} \langle \phi_n(\cdot+k), \phi_n \rangle e^{-ik\xi} = 1.$$ 

Hence the shifts \{$\phi(x-k)$\} are orthonormal.
The second statement is trivial, since \( \phi \) satisfies (4.1).

The proof of Theorem 5 is complete. \( \blacksquare \)

It would be interesting to derive conditions (in terms of \( a \) and \( F \)) for the existence of \( \phi_0 \) such that the cascade algorithm converges and the orthogonality in Theorem 5 holds.

6. EXAMPLES

In this section we provide two examples on cascade algorithms associated with inhomogeneous refinement equations to illustrate our general theory. For these examples, sufficient conditions for the \( L^p \)-convergence were given in [26], while necessary and sufficient conditions for the \( L^2 \)-convergence were given in [16].

**Example 1.** Consider the inhomogeneous refinement equation

\[
\phi(x) = t\phi(2x) + F(x), \quad x \in \mathbb{R},
\]

(6.1)

where \( t \) is a nonzero complex number. Let \( 1 \leq p \leq \infty \), \( F \) and \( \phi_0 \) be compactly supported functions in \( L^p(\mathbb{R}) \). If \( |t| < 2^{1/p} \), then the cascade algorithm associated with \( a, F \), and any \( \phi_0 \) converges in \( L^p(\mathbb{R}) \). If \( |t| \geq 2^{1/p} \), then the cascade algorithm associated with \( a, F \), and \( \phi_0 \) converges in \( L^p(\mathbb{R}) \) if and only if \( \phi_0 \) is a solution of the inhomogeneous refinement equation (6.1).

To see the conclusion, let \( f(x) = t\phi_0(2x) + F(x) - \phi_0(x) \). Choose a perfect generator for \( S(f) \). Then \( r = d = 1 \). We observe that for any nonzero sequence \( v \in \ell_0(\mathbb{Z}) \),

\[
\rho_p(\{A_0|V(v)\}, A_1|V(v)\}) = |t|.
\]

Therefore, our conclusion follows from Theorem 3.

**Example 2.** Consider the inhomogeneous refinement equation

\[
\phi(x) = a(0)\phi(2x) + a(1)\phi(2x - 1) + F(x), \quad x \in \mathbb{R},
\]

(6.2)

where \( a(0) \) and \( a(1) \) are nonzero complex numbers. Let \( 1 \leq p \leq \infty \), \( F \) and \( \phi_0 \) be functions in \( L^p(\mathbb{R}) \) supported in \([0, 2]\). Let

\[
f(x) = a(0)\phi_0(2x) + a(1)\phi_0(2x - 1) + F(x) - \phi_0(x).
\]

If \( |a(0)|^p + |a(1)|^p < 2 \), then the cascade algorithm associated with \( a, F \), and any \( \phi_0 \) converges in \( L^p(\mathbb{R}) \). If \( |a(0)|^p + |a(1)|^p \geq 2, |a(0)| < 2^{1/p} \) but \( \phi_0 \) is not a solution of (6.2), then the cascade algorithm associated with \( a, F \), and \( \phi_0 \) converges in \( L^p(\mathbb{R}) \) if and only if \( a(0) = a(1) \) and \( f(x) + f(x + 1) = 0 \) for \( x \in [0, 1) \). If \( |a(0)| \geq 2^{1/p} \), then the cascade algorithm associated with \( a, F \), and \( \phi_0 \) converges in \( L^p(\mathbb{R}) \) if and only if \( \phi_0 \) is a solution of (6.2).

To see the conclusion, we may choose \( r = d = 1 \), \( b \) is supported on \([0, 1]\), and the operators \( A_0, A_1 \) in (2.5) have the corresponding matrices

\[
A_0 = \begin{bmatrix}
a(0) & 0 \\
0 & a(1)
\end{bmatrix}, \quad A_1 = \begin{bmatrix}
a(1) & a(0) \\
0 & 0
\end{bmatrix}.
\]

Observe that \( \rho_p(\{A_0, A_1\}) = (|a(0)|^p + |a(1)|^p)^{1/p} \).
If \( b = (b(0), b(1))^T \neq 0 \), then

\[
\begin{align*}
A_0 b &= \begin{bmatrix} a(0)b(0) \\ a(1)b(1) \end{bmatrix}, \\
A_1 b &= \begin{bmatrix} a(1)b(0) + a(0)b(1) \\ 0 \end{bmatrix}.
\end{align*}
\]

Hence \( \dim V(be_1) = 1 \) if and only if \( b(0)b(1)(a(0) - a(1)) = 0 \) and \( (a(1)b(0) + a(0)b(1))b(1) = 0 \). This happens if and only if either \( b(1) = 0 \) or \( a(0) = a(1) \), \( b(1) = -b(0) \neq 0 \). In the first case, \( \rho_p([A_0|_{V(be_1)}, A_1|_{V(be_1)}) = (|a(0)|^p + |a(1)|^p)^{1/p} \). In the second case, \( \text{supp } g \subset [0, 1] \), \( V(be_1) = \text{span}(1, -1)^T \) and \( \rho_p([A_0|_{V(be_1)}, A_1|_{V(be_1)}) = |a(0)| \). Then our conclusion follows from Theorem 3.

A solution \( \phi \) of the inhomogeneous refinement equation (4.1) (homogeneous refinement equation (3.1)) is called singular in some function space if the cascade algorithm associated with \( a \), \( F \) (the zero function), and \( \phi_0 \) converges to \( \phi \) in that function space only if \( \phi_0 \) is the solution \( \phi \).

### 7. SMOOTHNESS ANALYSIS

In this section we apply our estimates for norms of combinations of shifts with subdivision sequence coefficients to smoothness analysis of refinable functions. To illustrate the method, we only consider the scalar case \( r = 1 \) and the special dilation matrix \( M = 2I \).

Let \( \phi \) be a solution of (3.1) in \( L_p(\mathbb{R}^s) \). The smoothness of \( \phi \) in \( L_p \) can be measured by the critical exponent

\[
v_p(\phi) := \sup\{v > 0 : \phi \in \text{Lip}^*(v, L_p)\}.
\]

Here \( \text{Lip}^*(v, L_p) \) is the generalized Lipschitz space consisting of all functions \( f \) in \( L_p(\mathbb{R}^s) \) such that

\[
\|\nabla^k f\|_p \leq C|h|^v, \quad \forall h \in \mathbb{R}^s,
\]

where \( k \) is a positive integer greater than \( v \) and

\[
\nabla^k f(x) := \sum_{l=0}^{k} \binom{k}{l} (-1)^l f(x - lh), \quad x, h \in \mathbb{R}^s.
\]

It is well known that \( f \in \text{Lip}^*(v, L_p) \) if and only if for \( j = 1, \ldots, s \) and \( n \in \mathbb{N} \),

\[
\|\nabla^k_{2^{-n}e_j} f\|_p \leq C2^{-nv}.
\]

Here \( e_j \) is the \( j \)th column of the \( s \times s \) identity matrix.

Under the assumption of stability, the smoothness of a refinable function \( \phi \) can be easily analyzed by our norm estimates in terms of the refinement mask. The smoothness analysis of univariate refinable functions has been well understood (see, e.g., [7]). The \( L_2 \) smoothness of multivariate refinable functions with isotropic dilation matrices has been characterized by Jia [14] and Cohen et al. [4].
THEOREM 6. Let \( 1 \leq p \leq \infty \), \( a \in \ell_0(\mathbb{Z}^s) \) and \( \phi \in L_p(\mathbb{R}^s) \) satisfy the refinement equation

\[
\phi(x) = \sum_{a \in \mathbb{Z}^s} a(\alpha) \phi(2x - a).
\]

Suppose that the shifts of \( \phi \) are stable. Let \( k \in \mathbb{N} \) and \( b_j \ (j = 1, \ldots, s) \) be the sequence supported on \( \{0, e_j, \ldots, ke_j\} \) and given by

\[
b_j(le_j) = \binom{k}{l} (-1)^l, \quad l = 0, 1, \ldots, k.
\]

Then

\[
\nu_p(\phi) = \min_{1 \leq j \leq s} \left\{ \frac{1}{p} - \log_2 \rho_p \left( \{ A_\varepsilon | V(b_j) : \varepsilon \in E \} \right) \right\},
\]

provided that

\[
\min_{1 \leq j \leq s} \left\{ \frac{1}{p} - \log_2 \rho_p \left( \{ A_\varepsilon | V(b_j) : \varepsilon \in E \} \right) \right\} < k.
\]

Proof. Iterating the refinement equation tells us that

\[
\phi(x) = \sum_{a \in \mathbb{Z}^s} a_n(\alpha) \phi(2^n x - \alpha), \quad n \in \mathbb{N}.
\]

Then for \( j = 1, \ldots, s, x \in \mathbb{R}^s \),

\[
\nabla^{k}_{2^{-n}e_j} \phi(x) = \sum_{a \in \mathbb{Z}^s} a_n(\alpha) \nabla^{k}_{e_j} \phi(2^n x - \alpha).
\]

Hence

\[
\| \nabla^{k}_{2^{-n}e_j} \phi \|_p = 2^{-n/p} \left\| \sum_{a \in \mathbb{Z}^s} a_n(\alpha) f(x - \alpha) \right\|_p.
\]

Here the function \( f \) is given by

\[
f(x) = \nabla^k_{e_j} \phi(x) = \sum_{l=0}^{k} \binom{k}{l} (-1)^l \phi(x - le_j).
\]

Since \( \phi \) is stable, \( \phi \in G(f) \), \( d = 1 \), and (2.2) holds with \( b = b_j \).

If \( \phi \in \text{Lip}^*(\nu, L_p) \) and \( \nu < k \), then for \( j = 1, \ldots, s \) and \( n \in \mathbb{N} \),

\[
\| \nabla^{k}_{2^{-n}e_j} \phi \|_p = 2^{-n/p} \left\| \sum_{a \in \mathbb{Z}^s} a_n(\alpha) f(x - \alpha) \right\|_p \leq C 2^{-n\nu}.
\]

It follows from Theorem 1 that

\[
\rho_p \left( \{ A_\varepsilon | V(b_j) : \varepsilon \in E \} \right) \leq 2^{1/p - \nu}, \quad j = 1, \ldots, s.
\]

Hence

\[
\nu \leq \frac{1}{p} - \log_2 \rho_p \left( \{ A_\varepsilon | V(b_j) : \varepsilon \in E \} \right), \quad j = 1, \ldots, s.
\]
This is true for any \( v < \nu_p(\phi) \). Therefore,

\[
\nu_p(\phi) \leq \min_{1 \leq j \leq s} \left\{ \frac{1}{p} - \log_2 \rho_p \left( \{ A_\varepsilon | V(b_j) : \varepsilon \in E \} \right) \right\}.
\]

Conversely, suppose \((7.2)\) holds. If \( v > 0 \) satisfies

\[
v < \min_{1 \leq j \leq s} \left\{ \frac{1}{p} - \log_2 \rho_p \left( \{ A_\varepsilon | V(b_j) : \varepsilon \in E \} \right) \right\} < k,
\]

then

\[
\rho_p \left( \{ A_\varepsilon | V(b_j) : \varepsilon \in E \} \right) < 2^{1/p - v}, \quad j = 1, \ldots, s.
\]

This in connection with Theorem 1 tells us that for some positive constant \( C \),

\[
\left\| \sum_{\alpha \in \mathbb{Z}^d} a_n(\alpha) f(x - \alpha) \right\|_p \leq C 2^{(1/p - v)n}, \quad \forall n \in \mathbb{N}, \; j = 1, \ldots, s.
\]

Here

\[
f(x) = \nabla^k_{e_j} \phi(x) = \sum_{\alpha \in \mathbb{Z}^d} b_j(\alpha) \phi(x - \alpha).
\]

By the iterations of the refinement equation, this implies that

\[
\left\| \nabla^k_{2^{-n}e_j} \phi \right\|_p = 2^{-n/p} \left\| \sum_{\alpha \in \mathbb{Z}^d} a_n(\alpha) f(x - \alpha) \right\|_p \leq C 2^{-n^v}, \quad j = 1, \ldots, s.
\]

Since \( v < k \), we know that \( \phi \in \text{Lip}^*(v, L_p) \). Thus, we conclude that

\[
\nu_p(\phi) \geq \min_{1 \leq j \leq s} \left\{ \frac{1}{p} - \log_2 \rho_p \left( \{ A_\varepsilon | V(b_j) : \varepsilon \in E \} \right) \right\}.
\]

This completes the proof of (7.1). ■

REFERENCES