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Discrete Mathematics 278 (2004) 11–22

DISCRETE MATHEMATICS

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Roman domination in graphs

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Received 4 July 2000; received in revised form 6 May 2002; accepted 30 June 2003

Abstract

A *Roman dominating function* on a graph $G=(V,E)$ is a function $f: V \to \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u)=0$ is adjacent to at least one vertex v for which $f(v)=2$. The *weight* of a Roman dominating function is the value $f(V)=\sum_{u\in V} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number* of G. In this paper, we study the graph theoretic properties of this variant of the domination number of a graph.

Keywords: Graph theory; Domination; Facilities location

1. Introduction

Let $G=(V,E)$ be a *graph* of *order* $|V|=n$. For any vertex $v\in V$, the *open neighbourhood of v* is the set $N(v)=\{u\in V | uv\in E\}$ and the *closed neighbourhood* is the set $N[v]=N(v)\cup\{v\}$. For a set $S\subseteq V$, the open neighbourhood is $N(S)=\bigcup_{v\in S} N(v)$ and the closed neighbourhood is $N[S]=N(S)\cup S$.

Let $v\in S\subseteq V$. Vertex u is called a *private neighbour of v with respect to S* (denoted by u is an S-pn of v) if $u\in N[v]-N[S-\{v\}]$. An S-pn of v is *external* if it is a vertex of $V-S$. The set $pn(v,S)=N[v]-N[S-\{v\}]$ of all S-pn's of v is called the *private neighbourhood set of v with respect to S*. The set S is said to be *irredundant* if for every $v\in S$, $pn(v,S)\neq\emptyset$.

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2003.06.004

A set $S \subseteq V$ is a *dominating set* if $N[S] = V$, or equivalently, every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G, and a dominating set S of minimum cardinality is called a γ-*set* of G. We note, for later reference, that every minimal dominating set is a maximal irredundant set, but not conversely. For example, two adjacent vertices on the cycle C_5 of length five form a maximal irredundant set which is not a dominating set.

A set S of vertices is called *independent* if no two vertices in S are adjacent. The *independent domination number* $i(G)$ is the minimum cardinality of a set S of vertices which is both independent and dominating, or equivalently, which is a maximal independent set.

A set S of vertices is called a 2-*packing* if for every pair of vertices $u, v \in S$, $N[u] \cap N[v] = \emptyset$. The 2-*packing number* $P_2(G)$ of a graph G is the maximum cardinality of a 2-packing in G.

Finally, a set S of vertices is called a *vertex cover* if for every edge $uv \in E$, either $u \in S$ or $v \in S$.

In this paper, we study a variant of the domination number which is suggested by the recent article in *Scientific American* by Ian Stewart, entitled "Defend the Roman Empire!" [10]. A *Roman dominating function* on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$.

Stated in other words, a Roman dominating function is a colouring of the vertices of a graph with the colours $\{0, 1, 2\}$ such that every vertex coloured 0 is adjacent to at least one vertex coloured 2. The definition of a Roman dominating function is given implicitly in [10] and [9]. The idea is that colours 1 and 2 represent either one or two Roman legions stationed at a given location (vertex v). A nearby location (an adjacent vertex u) is considered to be *unsecured* if no legions are stationed there (i.e. $f(u) = 0$). An unsecured location (u) can be secured by sending a legion to u from an adjacent location (v). But Emperor Constantine the Great, in the fourth century A.D., decreed that a legion cannot be sent from a location v if doing so leaves that location unsecured (i.e. if $f(v) = 1$). Thus, two legions must be stationed at a location ($f(v) = 2$) before one of the legions can be sent to an adjacent location.

The recent book *Fundamentals of Domination in Graphs* [6] lists, in an appendix, many varieties of dominating sets that have been studied. It appears that none of those listed are the same as Roman dominating sets. Thus, Roman domination appears to be a new variety of both historical and mathematical interest.

2. Properties of Roman dominating sets

For a graph $G = (V, E)$, let $f : V \to \{0, 1, 2\}$, and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V \mid f(v) = i\}$ and $|V_i| = n_i$, for $i = 0, 1, 2$. Note that there exists a 1–1 correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus, we will write $f = (V_0, V_1, V_2)$.

A function $f=(V_0,V_1,V_2)$ is a *Roman dominating function* (*RDF*) if $V_2 \succ V_0$, where $\succ$ means that the set V_2 dominates the set V_0, i.e. $V_0 \subseteq N[V_2]$. The weight of f is $f(V)=\sum_{v\in V} f(v)=2n_2+n_1$.

The *Roman domination number*, denoted $\gamma_R(G)$, equals the minimum weight of an RDF of G, and we say that a function $f=(V_0,V_1,V_2)$ is a γ_R*-function* if it is an RDF and $f(V)=\gamma_R(G)$.

Proposition 1. *For any graph G,*

$$\gamma(G) \leqslant \gamma_R(G) \leqslant 2\gamma(G).$$

Proof. Let $f=(V_0,V_1,V_2)$ be a γ_R-function, and let S be a γ-set of G. Then, $V_1 \cup V_2$ is a dominating set of G and $(\emptyset,\emptyset,S)$ is a Roman dominating function. Hence, $\gamma(G) \leqslant |V_1|+|V_2| \leqslant |V_1|+2|V_2| = \gamma_R(G)$. But $\gamma_R(G) \leqslant 2|S| = 2\gamma(G)$. □

Proposition 2. *For any graph G of order n, $\gamma(G)=\gamma_R(G)$ if and only if $G=\overline{K_n}$.*

Proof. It is obvious that if $G=\overline{K_n}$ then $\gamma(G)=\gamma_R(G)$.

Let $f=(V_0,V_1,V_2)$ be a γ_R-function. The equality $\gamma(G)=\gamma_R(G)$ implies that we have equality in $\gamma(G) \leqslant |V_1|+|V_2| = |V_1|+2|V_2| = \gamma_R(G)$. Hence, $|V_2|=0$, which implies that $V_0=\emptyset$. Therefore, $\gamma_R(G)=|V_1|=|V|=n$. This implies that $\gamma(G)=n$, which, in turn, implies that $G=\overline{K_n}$. □

Proposition 3. *Let $f=(V_0,V_1,V_2)$ be* any *γ_R-function. Then*

(a) *$G[V_1]$, the subgraph induced by V_1 has maximum degree* 1.
(b) *No edge of G joins V_1 and V_2.*
(c) *Each vertex of V_0 is adjacent to at most two vertices of V_1.*
(d) *V_2 is a γ-set of $G[V_0 \cup V_2]$.*
(e) *Let $H=G[V_0\cup V_2]$. Then each vertex $v \in V_2$ has at least two H-pn's (i.e. private neighbours relative to V_2 in the graph H).*
(f) *If v is isolated in $G[V_2]$ and has precisely one external H-pn, say $w\in V_0$, then $N(w)\cap V_1=\emptyset$.*
(g) *Let k_1 equal the number of non-isolated vertices in $G[V_2]$, let $C=\{v\in V_0\colon |N(v)\cap V_2| \geqslant 2\}$, and let $|C|=c$. Then $n_0 \geqslant n_2+k_1+c$.*

Proof. We omit the proofs of (a)–(e); they are clear.

(f) Suppose the contrary, that is, $N(w)\cap V_1 \neq \emptyset$. Form a new function by changing the function values of v and each $y\in N(w)\cap V_1$ to 0, and the value $f(w)$ to 2. This is an RDF with smaller weight than f, which is a contradiction.
(g) Let k_0 equal the number of isolated vertices in $G[V_2]$, so that $k_0+k_1=n_2$. By (e), $n_0 \geqslant k_0+2k_1+c = n_2+k_1+c$, as required. □

Proposition 4. *Let $f=(V_0,V_1,V_2)$ be a γ_{R}-function of an isolate-free graph G, such that n_1 is a minimum. Then*

(a) *V_1 is independent, and $V_0 \cup V_2$ is a vertex cover.*
(b) *$V_0 \succ V_1$.*
(c) *Each vertex of V_0 is adjacent to at most one vertex of V_1, i.e. V_1 is a 2-packing.*
(d) *Let $v \in G[V_2]$ have exactly two external H-pn's w_1 and w_2 in V_0. Then there do not exist vertices $y_1, y_2 \in V_1$ such that (y_1,w_1,v,w_2,y_2) is the vertex sequence of a path P_5.*
(e) *$n_0 \geqslant 3n/7$, and this bound is sharp even for trees.*

Proof. Again, we omit the proofs of (a)–(c); they are clear.

(d) Suppose the contrary. Form a new function by changing the function values of (y_1,w_1,v,w_2,y_2) from $(1,0,2,0,1)$ to $(0,2,0,0,2)$. The new function is a γ_{R}-function with fewer 1's, i.e. it has a smaller value of n_1, which is a contradiction.
(e) Define c as in Proposition 3(g). Let a_i, $i=1,2,\ldots,\Delta(G)$, be the number of vertices of V_2 which have exactly i H-pn's in V_0. By Proposition 3(e), (f) and Proposition 4(c), (d), we have

$$n_1 \leqslant \left(a_2 + 3a_3 + \sum_{j=4}^{\Delta} ja_j\right) + c, \tag{1}$$

$$n_0 = \left(a_1 + 2a_2 + 3a_3 + \sum_{j=4}^{\Delta} ja_j\right) + c, \tag{2}$$

$$n_2 = a_1 + a_2 + a_3 + \sum_{j=4}^{\Delta} a_j. \tag{3}$$

Therefore,

$$\begin{aligned} n &= n_0 + n_1 + n_2 \\ &\leqslant n_0 + \left(a_2 + 3a_3 + \sum_{j=4}^{\Delta} ja_j\right) + c, \\ &\quad + \left(a_1 + a_2 + a_3 + \sum_{j=4}^{\Delta} a_j\right), \quad \text{[by (2) and (3)]} \\ &= \left(n_0 + a_1 + 2a_2 + 4a_3 + \sum_{j=4}^{\Delta} (j+1)a_j\right) + c \end{aligned}$$

$$\leqslant \left(n_0 + a_1 + 2a_2 + \sum_{j=4}^{\Delta}(j+1)a_j\right) + c$$

$$+\frac{4}{3}\left(n_0 - a_1 - 2a_2 - \sum_{j=4}^{\Delta} ja_j - c\right), \quad \text{[eliminating } a_3 \text{ by (1)]}$$

$$= \frac{7n_0}{3} - \frac{a_1 + 2a_2}{3} - \sum_{j=4}^{\Delta} a_j\left(\frac{j}{3} - 1\right) - \frac{c}{3}$$

$$\leqslant \frac{7n_0}{3}.$$

Hence, $n_0 \geqslant 3n/7$ as required.

The tree T with seven vertices $V = \{u, u_1, u_2, u_3, v_1, v_2, v_3\}$, where u is adjacent to u_1, u_2 and u_3, and u_i is adjacent to v_i, $i = 1, 2, 3$, has $\gamma_R(T) = 5$, a γ_R-function defined by $f = (V_0, V_1, V_2) = (\{u_1, u_2, u_3\}, \{v_1, v_2, v_3\}, \{u\})$, and achieves the bound $n_0 = 3n/7 = 3$. □

Corollary 1. *For any non-trivial connected graph G,*

$$\gamma_R(G) = \min\{2\gamma(G - S) + |S| : S \text{ is a 2-packing}\}.$$

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_R-function of a graph G. From Proposition 4(a) and (c) we can assume that V_1 is a 2-packing. It follows from Proposition 3(d) that V_2 is a γ-set of the graph $G - S$ obtained from G by deleting all vertices in V_1. Thus, $\gamma_R(G) \geqslant \min\{2\gamma(G - S) + |S|\text{: } S \text{ is a 2-packing}\}$.

Conversely, let V_1 be a 2-packing for which $2\gamma(G - S) + |S|$ is a minimum, and let V_2 be a γ-set of $G - V_1$. Then $(V - V_1 - V_2, V_1, V_2)$ is an RDF, and $\gamma_R(G) \leqslant 2|V_2| + |V_1| = \min\{2\gamma(G - S) + |S|\text{: } S \text{ is a 2-packing}\}$. □

The following lower bound for the Roman domination number of any graph is proved in [3].

Proposition 5. *For any graph G of order n and maximum degree Δ,*

$$\frac{2n}{\Delta + 1} \leqslant \gamma_R(G).$$

We conclude this section with an upper bound on $\gamma_R(G)$ using a probabilistic method similar to that used by Alon and Spencer in [1].

Proposition 6. *For a graph G on n vertices,*

$$\gamma_R(G) \leqslant n\,\frac{2 + \ln((1 + \delta(G))/2)}{1 + \delta(G)}.$$

Proof. Given a graph G, select a set of vertices A, where each vertex is selected independently with probability p (with p to be defined later). The expected size of A is np. Let $B = V - N[A]$, the vertices not dominated by A. Clearly $f = (V - (A \cup B), B, A)$ is an RDF for G.

We now compute the expected size of B. The probability that v is in B is equal to the probability that v is not in A and that no vertex in A is the neighbour of v. This probability is $(1-p)^{1+\deg(v)}$. Since $e^{-x} \geqslant 1 - x$ for any $x \geqslant 0$, and $\deg(v) \geqslant \delta(G)$, we can conclude that $\Pr(v \in B) \leqslant e^{-p(1+\delta(G))}$. Thus, the expected size of B is at most $ne^{-p(1+\delta(G))}$, and the expected weight of f, denoted $E[f(V)]$, is at most $2np + ne^{-p(1+\delta(G))}$. The upper bound for $E[f(V)]$ is minimized when $p = \ln((1+\delta(G))/2)/(1+\delta(G))$ (this is easily shown using calculus), and substituting this value in for p gives:

$$E[f(V)] \leqslant n \frac{2 + \ln((1+\delta(G))/2)}{1+\delta(G)}.$$

Since the expected weight of $f(V)$ is at most $n(2 + \ln((1+\delta(G))/2))/(1+\delta(G))$, there must be some RDF with at most this weight. It turns out that this bound is sharp, being achieved when G is the disjoint union of $n/2$ copies of K_2. □

3. Specific values of Roman domination numbers

In this section, we illustrate the Roman domination number by presenting the value of $\gamma_R(G)$ for several classes of graphs. Some proofs are straightforward and are omitted.

The following two classes of graphs achieve the lower bound of Proposition 5.

Proposition 7. *For the classes of paths P_n and cycles C_n,*

$$\gamma_R(P_n) = \gamma_R(C_n) = \left\lceil \frac{2n}{3} \right\rceil.$$

For the class of complete multipartite graphs $K_{m_1,\ldots,m_n}$ there are three cases to consider.

Proposition 8. *Let $G = K_{m_1,\ldots,m_n}$ be the complete n-partite graph with $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_n$.*

(a) *If $m_1 \geqslant 3$ then $\gamma_R(G) = 4$.*
(b) *If $m_1 = 2$ then $\gamma_R(G) = 3$.*
(c) *If $m_1 = 1$ then $\gamma_R(G) = 2$.*

Proposition 9. *If G is a graph of order n which contains a vertex of degree $n-1$, then $\gamma(G) = 1$ and $\gamma_R(G) = 2$.*

For arbitrary graphs G and H, we define the *Cartesian product* of G and H to be the graph $G \square H$ with vertices $\{(u,v) | u \in G, v \in H\}$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \square H$ if and only if one of the following is true: $u_1 = u_2$ and v_1

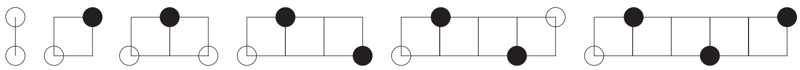

Fig. 1. The constructions for $G_{2,n}$, $1 \leqslant n \leqslant 6$. Filled-in circles denote vertices in V_2, empty circles denote vertices in V_1.

is adjacent to v_2 in H; or $v_1 = v_2$ and u_1 is adjacent to u_2 in G. If $G = P_m$ and $H = P_n$, then the Cartesian product $G \square H$ is called the $m \times n$ *grid graph* and is denoted $G_{m,n}$.

Proposition 10. *For the* $2 \times n$ *grid graph* $G_{2,n}$, $\gamma_R(G_{2,n}) = n + 1$.

Proof. We only show, by construction (cf. Fig. 1), that $\gamma_R(G_{2,n}) \leqslant n+1$. The reverse inequality is straightforward.

Let the vertices of $G_{2,n}$ be denoted $v_{1,1}, \ldots, v_{1,n}, v_{2,1}, \ldots, v_{2,n}$ and define the RDF g as follows: for each i such that $2+4i \leqslant n$, let $g(v_{2,2+4i}) = 2$, and for each j such that $4j \leqslant n$, let $g(v_{1,4j}) = 2$. Let $g(v_{1,1}) = 1$, and if $n \equiv 1 \pmod 4$, let $g(v_{2,n}) = 1$, and if $n \equiv 3 \pmod 4$, let $g(v_{1,n}) = 1$. For all of the remaining vertices u, let $g(u) = 0$. It is easily seen that g is an RDF and that $g(V) = n+1$. □

One final class of graphs is of some interest. For even n, we let $(n/2)K_2$ denote the graph consisting of $n/2$ copies of the complete graph K_2 on two vertices.

Proposition 11. *If* G *is any isolate-free graph of order* n, *then* $\gamma_R(G) = n$ *if and only if* n *is even and* $G = (n/2)K_2$.

Proof. If $G = (n/2)K_2$, then each edge contributes at least two to $\gamma_R(G)$, and hence $n \leqslant \gamma_R(G) \leqslant n$.

Assume therefore that $\gamma_R(G) = n$. If G has two incident edges uv and vw, then $f = (V_0, V_1, V_2)$, where $V_0 = \{u, w\}$, $V_1 = V - \{u, v, w\}$ and $V_2 = \{v\}$, defines a Roman dominating function. Hence, $\gamma_R(G) \leqslant |V_1| + 2|V_2| = n - 1$, which is a contradiction. Hence, no two edges of G are incident, and since G is isolate-free, the conclusion follows. □

4. Graphs with $\gamma_R(G) \leqslant \gamma(G) + 2$

From Proposition 1, we know that:

$$\gamma(G) \leqslant \gamma_R(G) \leqslant 2\gamma(G).$$

But from Proposition 2 we know that this lower bound is achieved only when $G = \overline{K_n}$. Thus, if G is a connected graph then $\gamma_R(G) \geqslant \gamma(G) + 1$. The connected graphs G with γ_R-functions of weight $\gamma(G) + 1$ and $\gamma(G) + 2$ have a very specific structure, which will be shown in the following propositions.

Proposition 12. *If G is a connected graph of order n, then $\gamma_R(G) = \gamma(G) + 1$ if and only if there is a vertex $v \in V$ of degree $n - \gamma(G)$.*

Proof. $\Leftarrow$: Assume G has a vertex v of degree $n - \gamma(G)$. If $V_2 = \{v\}$, $V_1 = V - N[v]$, and $V_0 = V - V_1 - V_2$, then $V_1 \cup V_2$ is a γ-set of G and $f = (V_0, V_1, V_2)$ is an RDF with $f(V) = \gamma(G) + 1$. Since $\gamma_R(G) \geqslant \gamma(G) + 1$ for connected graphs, f is a γ_R-function for G.

$\Rightarrow$: In order for a Roman dominating function $f = (V_0, V_1, V_2)$ to have weight $\gamma(G) + 1$, either (1) $|V_1| = \gamma(G) + 1$ and $|V_2| = 0$ or (2) $|V_1| = \gamma(G) - 1$ and $|V_2| = 1$. Any other arrangement of weight $\gamma(G) + 1$ would have $|V_1| + |V_2| < \gamma(G)$.

In case (1), since $|V_2| = 0$, then $V_1 = V$. By a theorem of Ore [6, p. 41], $\gamma(G) \leqslant n/2$ for a connected graph G on n vertices. Thus, $n = \gamma(G) + 1 \leqslant n/2 + 1$, which implies that $n \leqslant 2$. It is easily verified that $\gamma_R(P_2) = 2 = \gamma(P_2) + 1$, and P_2 has a vertex of degree 1.

In case (2), let $f = (V_0, V_1, V_2)$ be a γ_R-function for G of weight $\gamma(G) + 1$, with $|V_1| = \gamma(G) - 1$ and $|V_2| = 1$. Let $V_2 = \{v\}$. Since no edge of G joins V_1 and $\{v\}$, and $\{v\} \succ V_0$, $\deg(v) = |V_0| = n - |V_1| - |V_2| = n - \gamma(G)$. □

We next characterize the class of trees T for which $\gamma_R(T) = \gamma(G) + 1$. For a positive integer t, a *wounded spider* is a star $K_{1,t}$ with at most $t - 1$ of its edges subdivided. Similarly, for an integer $t \geqslant 2$, a *healthy spider* is a star $K_{1,t}$ with all of its edges subdivided. In a wounded spider, a vertex of degree t will be called the *head vertex*, and the vertices that are distance two from the head vertex will be the *foot vertices*. The head and foot vertices are well defined except when the wounded spider is the path on two or four vertices. For P_2, we will consider both vertices to be head vertices, and in the case of P_4, we will consider both endvertices as foot vertices and both interior vertices as head vertices.

Proposition 13. *If T is a tree on two or more vertices, then $\gamma_R(T) = \gamma(T) + 1$ if and only if T is a wounded spider.*

Proof. $\Leftarrow$: Let T be a wounded spider and let v be the head vertex. Let $S = \{w : d(v, w) = 2\}$ be the set of foot vertices. Clearly, $\{v\} \cup S$ forms a γ-set for T. Also, if $V_0 = V - S - \{v\}$, $V_1 = S$, and $V_2 = \{v\}$, then $f = (V_0, V_1, V_2)$ is an RDF with $f(V) = \gamma(T) + 1$. Therefore, f is a γ_R-function.

$\Rightarrow$: Let $f = (V_0, V_1, V_2)$ be a γ_R-function for T of weight $\gamma(T) + 1$. As in the proof of Proposition 12, either $T = P_2$, or $|V_1| = \gamma(G) - 1$, and $|V_2| = 1$. Let $V_2 = \{v\}$. Then $|N(v)| = |V_0| = n - \gamma(T)$. Since $|V_1|$ is minimized in f, by Proposition 4(c), each vertex of V_0 is adjacent to at most one vertex of V_1. Conversely, since T is connected, V_1 is independent, and V_1 and V_2 have no edges between them, every member of V_1 must be joined to a member of V_0. Furthermore, not every vertex in V_0 can be adjacent to a member of V_1, that is, T cannot be a healthy spider. If this was the case, then V_0 forms a γ-set for T and $\deg(v) = |V_0| < |V_0| + 1 = |V_1| + |V_2| = |V_0| + |V_1| + |V_2| - |V_0| = n - \gamma(T)$, which is a contradiction. Hence, T is a wounded spider. □

We next characterize the class of graphs for which $\gamma_R(G) = \gamma(G) + 2$.

Proposition 14. *If G is a connected graph of order n, then $\gamma_R(G)=\gamma(G)+2$ if and only if*:

(a) *G does not have a vertex of degree $n-\gamma(G)$.*
(b) *either G has a vertex of degree $n-\gamma(G)-1$ or G has two vertices v and w such that $|N[v]\cup N[w]|=n-\gamma(G)+2$.*

Proof. $\Leftarrow$: By (a), we know that $\gamma_R(G)>\gamma(G)+1$. If G has a vertex v of degree $n-\gamma(G)-1$, and we define $V_0=N(v)$, $V_1=V-N[v]$, and $V_2=\{v\}$, then $f=(V_0,V_1,V_2)$ is an RDF with $f(V)=\gamma(G)+2$, and hence is a γ_R-function.

If there are two vertices v and w such that $|N[v]\cup N[w]|=n-\gamma(G)+2$, and we define $V_0=N[v]\cup N[w]-\{v,w\}$, $V_1=V-(N[v]\cup N[w])$, and $V_2=\{v,w\}$, then $f=(V_0,V_1,V_2)$ is an RDF with $f(V)=\gamma(G)+2$, and hence is a γ_R-function.

$\Rightarrow$: In order for an RDF $f=(V_0,V_1,V_2)$ to have weight $\gamma(G)+2$, either (1) $|V_1|=\gamma(G)+2$ and $|V_2|=0$, (2) $|V_1|=\gamma(G)$ and $|V_2|=1$, or (3) $|V_1|=\gamma(G)-2$ and $|V_2|=2$. In order for such an f to be a γ_R-function, there can be no other Roman dominating function of weight $\gamma(G)+1$, which implies that G has no vertex of degree $n-\gamma(G)$.

In case (1), if $|V_2|=0$, then $V_1=V$. Again using Ore's theorem [6, p. 41], $n=\gamma(G)+2\leqslant n/2+2$, which implies that $n\leqslant 4$. A simple analysis of the connected graphs on four or fewer vertices shows that $\gamma_R(G)=\gamma(G)+1$ for all such graphs.

In case (2), let $f=(V_0,V_1,V_2)$ be a γ_R-function for G of weight $\gamma(G)+2$ with $|V_1|=\gamma(G)$ and $|V_2|=1$. Let $V_2=\{v\}$. Since no edge of G joins V_1 and v, and $v\succ V_0$, it follows that $\deg(v)=|V_0|=n-|V_1|-|V_2|=n-\gamma(G)-1$.

In case (3), let $f=(V_0,V_1,V_2)$ be a γ_R-function for G of weight $\gamma(G)+2$ with $|V_1|=\gamma(G)-2$ and $|V_2|=2$. Let $V_2=\{v,w\}$. Since no edge joins V_1 to v or w and $\{v,w\}\succ V_0$, it follows that $|N[v]\cup N[w]|=n-|V_1|=n-(\gamma(G)-2)=n-\gamma(G)+2$. $\square$

Corollary 2. *If G is a connected graph and $\gamma_R(G)=\gamma(G)+2$, then $2\leqslant \mathrm{rad}(G)\leqslant 4$ and $3\leqslant \mathrm{diam}(G)\leqslant 8$.*

It is possible to use Proposition 11 to obtain a characterization of trees for which $\gamma_R(T)=\gamma(T)+2$. Once again, spiders play a major role. This classification is rather technical and we do not give the details.

Proposition 15. *If T is a tree of order $n\geqslant 2$, then $\gamma_R(T)=\gamma(T)+2$ if and only if either* (i) *T is a healthy spider or* (ii) *T is a pair of wounded spiders T_1 and T_2, with a single edge joining $v\in V(T_1)$ and $w\in V(T_2)$, subject to the following conditions*:

(1) *if either tree is a P_2, then neither vertex in P_2 are joined to the head vertex of the other tree.*
(2) *v and w are not both foot vertices.*

5. Graphs for which $\gamma_R(G) = 2\gamma(G)$

From Proposition 1 we know that for any graph G, $\gamma_R(G) \leqslant 2\gamma(G)$. We will say that a graph G is a *Roman graph* if $\gamma_R(G) = 2\gamma(G)$. In this section, we seek to find a characterization of Roman graphs.

Proposition 9 gives us our first class of Roman graphs, i.e. graphs of the form $G = K_1 + H$, where $\gamma(G) = 1$ and $\gamma_R(G) = 2$. Equivalently, any graph G of order n having a vertex of degree $n-1$ is a Roman graph.

Proposition 7 identifies all Roman paths and cycles, i.e. P_{3k}, C_{3k}, P_{3k+2}, and C_{3k+2}.

Proposition 8 identifies which complete bipartite graphs are Roman, i.e. $K_{m,n}$ where $\min\{m,n\} \neq 2$, in which case either $\gamma(G)=1$ and $\gamma_R(G)=2$, or $\gamma(G)=2$ and $\gamma_R(G)=4$.

Two simple characterizations of Roman graphs are as follows.

Proposition 16. *A graph G is Roman if and only if it has a γ_R-function $f=(V_0,V_1,V_2)$ with $n_1 = |V_1| = 0$.*

Proof. Let G be a Roman graph and let $f=(V_0,V_1,V_2)$ be a γ_R-function of G. From Proposition 3(d) we know that $V_2 \succ V_0$ and $V_1 \cup V_2 \succ V$, and hence

$$\gamma(G) \leqslant |V_1 \cup V_2| = |V_1| + |V_2| \leqslant |V_1| + 2|V_2| = \gamma_R(G).$$

But since G is Roman, we know that

$$2\gamma(G) = 2|V_1| + 2|V_2| = \gamma_R(G) = |V_1| + 2|V_2|.$$

Hence, $n_1 = |V_1| = 0$.

Conversely, let $f=(V_0,V_1,V_2)$ be a γ_R-function of G with $n_1 = |V_1| = 0$. Therefore, $\gamma_R(G) = 2|V_2|$, and since by definition $V_1 \cup V_2 \succ V$, it follows that V_2 is a dominating set of G. But by Proposition 3(d), we know that V_2 is a γ-set of $G[V_0 \cup V_2]$, i.e. $|V_2| = \gamma(G)$ and $\gamma_R(G) = 2\gamma(G)$, i.e. G is a Roman graph. □

Proposition 17. *A graph G is Roman if and only if $\gamma(G) \leqslant \gamma(G-S) + |S|/2$, for every 2-packing $S \subseteq V$.*

Proof. From Corollary 1, we know that $\gamma_R(G)$ is the minimum value of $2\gamma(G-S)+|S|$, over all 2-packings $S \subset V$. Thus, if $\gamma_R(G) = 2\gamma(G)$, then $2\gamma(G-S) + |S| \geqslant \gamma(G)$, or $\gamma(G) \leqslant \gamma(G-S) + |S|/2$, for every 2-packing S.

Conversely, if $\gamma(G) \leqslant \gamma(G-S) + |S|/2$, for every 2-packing S, then $2\gamma(G) \leqslant \gamma(G-S) + |S|$, for every 2-packing S. This implies that $2\gamma(G) \leqslant \gamma_R(G)$. But from Proposition 1 we know that $\gamma_R(G) \leqslant \gamma(G)$. Therefore, $\gamma_R(G) = 2\gamma(G)$. □

We conclude this section by noting that a constructive characterization of Roman trees has recently been given by Henning [7].

6. Open problems

Among the many questions raised by this research, the following are of particular interest to the authors.

1. Can you find other classes of Roman graphs?

2. Can Propositions 12 and 14 be generalized to produce a characterization of graphs for which $\gamma_R(G) = \gamma(G) + k$?

3. Can you determine the Roman domination number of the grid graph $G_{m,n}$, for any positive integers m and n? Various values of Roman domination in grid graphs have been determined by Dreyer [4] in his Ph.D. Thesis, and bounds are obtained in [3].

4. What are the algorithmic, complexity and approximation properties of Roman domination? For example, the authors have constructed a linear algorithm for computing the Roman domination number of any tree. Furthermore, McRae [8] has constructed proofs which show that the decision problem RDF, corresponding to the value of a Roman dominating function, is NP-complete, even when restricted to (i) chordal, (ii) bipartite, (iii) split, or (iv) planar graphs. It was also suggested by one of the referees that the inequalities, $\gamma(G) \leqslant \gamma_R(G) \leqslant 2\gamma(G)$, imply immediately that there is a $2 \log n$ approximation algorithm for the Roman domination number, while a $c \log n$ approximation algorithm does not exist for any $c < 1$ unless $P = \mathrm{NP}$. The algorithmic complexity of Roman domination will be the subject of a subsequent paper [2].

5. Can you construct a polynomial algorithm for computing the value $\gamma_R(G)$ for any interval graph G?

6. What can you say about the minimum and maximum values of n_0, n_1 and n_2 for a γ_R-function $f = (V_0, V_1, V_2)$ of a graph G? For example, Proposition 4(e) says that you can always guarantee there is a γ_R-function with $n_0 \geqslant 3n/7$.

7. What are the properties of independent Roman dominating functions? A Roman domination function $f = (V_0, V_1, V_2)$ is called *independent* if the set $V_1 \cup V_2$ is an independent set. McRae [8] has also shown that the decision problem IRDF corresponding to independent Roman dominating functions is NP-complete, even when restricted to bipartite graphs. It is interesting, however, to note that Farber [5] has shown that *independent dominating set* is polynomial when restricted to chordal graphs. This raises the interesting question of whether IRDF is polynomial for chordal graphs.

Acknowledgements

This paper was completed while Dr. Cockayne was visiting the Department of Mathematics, Applied Mathematics and Astronomy of the University of South Africa during January, 2000. He also gratefully acknowledges research support from the Canadian Natural Sciences and Engineering Research Council (NSERC). The authors gratefully acknowledge the three referees of this paper for many excellent suggestions for improving and condensing this paper.

References

[1] N. Alon, J.H. Spencer, The Probabilistic Method, Wiley, New York, 1992.

[2] E.J. Cockayne, P.A. Dreyer, Jr., S.M. Hedetniemi, S.T. Hedetniemi, A.A. McRae, The algorithmic complexity of Roman domination, to be submitted.
[3] E.J. Cockayne, P.J.P. Grobler, W. Gründlingh, J. Munganga, J.H. van Vuuren, Protection of a graph, submitted for publication.
[4] P.A. Dreyer, Jr., Applications and variations of domination in graphs, Ph.D. Thesis, Rutgers University, October 2000.
[5] M. Farber, Independent domination in chordal graphs, Oper. Res. Lett. 1 (1982) 134–138.
[6] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.
[7] M.A. Henning, A characterization of Roman trees, Discuss. Math. Graph Theory 22 (2) (2002) 325–334.
[8] A.A. McRae, Private communication, March 2000.
[9] C.S. ReVelle, K.E. Rosing, Defendens imperium romanum: a classical problem in military strategy, Amer. Math. Monthly 107 (7) (2000) 585–594.
[10] I. Stewart, Defend the Roman Empire!, Sci. Amer. 281 (6) (1999) 136–139.