New oscillation criteria for a class of second-order nonlinear forced differential equations

Fangcui Jiang a,b,∗, Fanwei Meng b

a Center for Systems and Control, Department of Mechanics and Engineering Science, Peking University, Beijing 100871, PR China
b Department of Mathematical Science, Qufu Normal University, Qufu 273165, Shandong, PR China

Received 7 August 2006
Available online 2 March 2007
Submitted by J.S.W. Wong

Abstract

By using classical variational principle and averaging technique, new oscillation criteria are established for second-order nonlinear forced differential equations

\[ r(t)\psi(x(t))\phi(x'(t))' + q(t)f(x(t)) = e(t), \quad t \geq t_0, \]

where the functions \( r, q, e \in C([t_0, \infty), R) \), and \( \phi, \psi, f \in C(R, R) \) improve and extend some recent results.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Nonlinear; Oscillation; Forcing term; Averaging technique

1. Introduction

In this paper we are concerned with the oscillation behavior for the forced second-order nonlinear differential equation

\[ r(t)\psi(x(t))\phi(x'(t))' + q(t)f(x(t)) = e(t), \quad t \geq t_0, \]  

(1.1)

where the functions \( r, q, e \in C([t_0, \infty), R) \), and \( \phi, \psi, f \in C(R, R) \).

Throughout this paper we shall tacitly assume that

* Corresponding author.
E-mail address: jiangfangcui2008@163.com (F. Jiang).
(A1) $r(t) > 0, t \geq t_0$, and $\psi(x) > 0, xf(x) > 0$ for all $x \neq 0$;

(A2) $\phi$ be continuously differentiable and satisfying

$$|\phi(x)|^{(\alpha+1)/\alpha} \leq \gamma_1 x\phi(x),$$

for some constants $\alpha > 0$, $\gamma_1 > 0$ and for all $x \in \mathbb{R}$.

By a solution of (1.1), we mean a function $x(t) \in C^1[T_x, \infty), T_x \geq t_0$, which has the property $r(t)\psi(x(t))\phi(x'(t)) \in C^1[T_x, \infty)$ and satisfies Eq. (1.1). We restrict our attention only to the nontrivial solutions of Eq. (1.1), i.e., to the solutions $x(t)$ such that $\sup\{|x(t)|: t \geq T\} > 0$ for all $T \geq T_x$. A nontrivial solution of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

A close look at the form of Eq. (1.1) with the assumption (A 2) and our further hypotheses about $f$ which will be mentioned in Sections 2 and 3, we will find that Eq. (1.1) can be considered as a natural generalization of the following differential equations

$$(r(t)x'(t))' + q(t)x(t) = e(t), \quad t \geq t_0,$$

$$(r(t)|x'(t)|^{\sigma-1}x'(t))' + q(t)|x(t)|^{\tau-1}x(t) = e(t), \quad \sigma > 0, \quad \tau \geq \sigma, \quad t \geq t_0.$$ (1.4)

During the last two decades, the oscillation for Eqs. (1.2)–(1.4) has been discussed by numerous authors (see [1–15] and references therein). Some results of them are established by making use of a technique introduced by Kartsatos [16] where it is assumed that there exists a second derivative function “$h(t)$” such that $h''(t) = e(t)$ in order to reduce (1.2) or (1.3) to a second order homogeneous equation. However, these results require the information of “$q$” on the entire half-line $[t_0, \infty)$.

In 1993, according to a comparison theorem of Sturm’s type due to Leighton [17], El-Sayed [2] gave an interval oscillation criterion for Eq. (1.3) which depends only on the behavior of “$q$” in certain subintervals of $[t_0, \infty)$. In 1999, El-Sayed’s result was simplified by Wong [3], the result of which follows from “oscillatory intervals” of “$e(t)$” and Leighton’s variational principles for the associated homogeneous equation of (1.3). From the result of Jaros and Kusano [18], we can see that variational principles for oscillation of half-linear differential equations can also be established. Motivated by this idea, Li and Cheng [7] has extended Wong’s result to the half-linear nonhomogeneous differential equation (1.4) with $\sigma = \tau$. Very recently, the result of Li and Cheng [7] has been extended by Çakmak and Tiryaki [10] to the more general forced second-order nonlinear differential equations of the form

$$r(t)\psi(x(t))|x'(t)|^{\sigma-1}x'(t)' + q(t)f(x(t)) = e(t), \quad t \geq t_0,$$ (1.5)

via an integral operator.

In the present paper, we will give some interval oscillation criteria for Eq. (1.1). Equation (1.1) through some new averaging functions $H(t, s) \in C(D, \mathbb{R})$, which satisfy:

(i) $H(t, t) = 0, H(t, s) > 0$ for $t > s$;

(ii) $H$ has partial derivatives $\frac{\partial H}{\partial t}$ and $\frac{\partial H}{\partial s}$ on $D$ such that

$$\frac{\partial H}{\partial t} = h_1(t, s)\sqrt{H(t, s)}, \quad \frac{\partial H}{\partial s} = -h_2(t, s)\sqrt{H(t, s)},$$
where $D = \{ (t, s) : t_0 \leq s \leq t < \infty \}$, $h_1, h_2 \in L_{\text{loc}}(D, R^+)$. And our results extend and improve some earlier results.

2. $f(x)$ be monotone increasing

In this section, we shall deal with the oscillation for Eq. (1.1) under the assumptions (A1)–(A2) and the following assumption:

(C1) $f'(x)$ exists, $xf(x) > 0$ for $x \neq 0$ and

$$\frac{f'(x)}{(\psi(x)|f(x)|^{\alpha-1})^{1/\alpha}} \geq \gamma_2 > 0,$$

for some nonnegative constant $\gamma_2$ and for all $x \in R/\{0\}$.

**Theorem 2.1.** Suppose (A1)–(A2), (C1) be fulfilled and for any $T \geq t_0$, there exist $T \leq a_1 < b_1 \leq a_2 < b_2$ such that

$$e(t) \begin{cases} 
\leq 0, & t \in [a_1, b_1], \\
\geq 0, & t \in [a_2, b_2]. 
\end{cases} \quad (2.1)$$

If there exist some $c_i \in (a_i, b_i)$, $i = 1, 2$, $H(t, s)$ satisfying (i)–(ii) and a positive function $\rho \in C^1([t_0, \infty), R)$ such that

$$1 \int_{a_i}^{c_i} \left[ H_{\alpha+1}(s, a_i)q(s)\rho(s) - \frac{(\alpha \gamma)^{\alpha}}{(\alpha + 1)^{\alpha+1}} r(s)\rho(s)H_{1,\alpha+1}(s, a_i) \right] ds$$

$$+ \int_{c_i}^{b_i} \left[ H_{\alpha+1}(b_i, s)q(s)\rho(s) - \frac{(\alpha \gamma)^{\alpha}}{(\alpha + 1)^{\alpha+1}} r(s)\rho(s)H_{2,\alpha+1}(b_i, s) \right] ds$$

$$> 0, \quad (2.2)$$

for $i = 1, 2$, where $\gamma := \frac{\gamma_1}{\gamma_2}$

$$H_1(t, s) = \left| (\alpha + 1)h_1(t, s)\sqrt{H(t, s)} + H(t, s)\frac{\rho'(t)}{\rho(t)} \right|,$$

$$H_2(t, s) = \left| (\alpha + 1)h_2(t, s)\sqrt{H(t, s)} - H(t, s)\frac{\rho'(s)}{\rho(s)} \right|,$$

then Eq. (1.1) is oscillatory.

**Proof.** Otherwise, suppose that $x(t)$ be a nonoscillatory solution of Eq. (1.1), say $x(t) \neq 0$ on $[T_0, \infty)$ for some sufficiently large $T_0 \geq t_0$. Define

$$w(t) = \rho(t) \frac{r(t)\psi(x(t))\phi(x'(t))}{f(x(t))}, \quad t \geq T_0. \quad (2.3)$$

Then differentiating (2.3) and making use of Eq. (1.1), assumptions (A1)–(A2) and (C1), we have
\[ w'(t) = -q(t)\rho(t) + \frac{e(t)}{f(x(t))}\rho(t) - \frac{\rho(t)r(t)\psi(x(t))}{f^2(x(t))}[x'(t)\phi'(x(t))]f'(x(t)) \]
\[ + \frac{\rho'(t)}{\rho(t)}w(t) \]
\[ \leq -q(t)\rho(t) + \frac{e(t)}{f(x(t))}\rho(t) - \frac{1}{\gamma(r(t)\rho(t))^{1/\alpha}} \rho'(t) w(t). \]  

(2.4)

By the assumptions, we can choose \( a_i, b_i \geq T_0 \) for \( i = 1, 2 \) such that \( e(t) \leq 0 \) on the interval \( I_1 = [a_1, b_1] \) with \( a_1 < b_1 \) and \( x(t) > 0 \), or \( e(t) \geq 0 \) on the interval \( I_2 = [a_2, b_2] \) and \( x(t) < 0 \).

On the intervals \( I_1 \) and \( I_2 \), (2.4) imply that \( w(t) \) satisfies

\[ w'(t) \leq -q(t)\rho(t) - \frac{1}{\gamma(r(t)\rho(t))^{1/\alpha}} \rho'(t) w(t). \]  

(2.5)

On the one hand, multiplying \( H^{\alpha+1}(t, s) \) through (2.5) and integrating it (with \( t \) replaced by \( s \)) over \([c_i, t)\) for \( t \in [c_i, b_i) \), \( i = 1, 2 \), by using hypotheses (i)–(ii), we have for \( s \in [c_i, t) \)

\[ \int_{c_i}^{t} H^{\alpha+1}(t, s)q(s)\rho(s) \, ds \]
\[ \leq - \int_{c_i}^{t} H^{\alpha+1}(t, s)w'(s) \, ds - \int_{c_i}^{t} H^{\alpha+1}(t, s) \left[ \frac{\rho'(s)}{\rho(s)} w(s) - \frac{1}{\gamma(r(s)\rho(s))^{1/\alpha}} |w(s)|^{(\alpha+1)/\alpha} \right] \, ds \]
\[ = H^{\alpha+1}(t, c_i)w(c_i) - \int_{c_i}^{t} (\alpha + 1)H^{\alpha}(t, s)h_2(t, s)\sqrt{H(t, s)}w(s) \, ds \]
\[ + \int_{c_i}^{t} H^{\alpha+1}(t, s) \left[ \frac{\rho'(s)}{\rho(s)} w(s) - \frac{1}{\gamma(r(s)\rho(s))^{1/\alpha}} |w(s)|^{(\alpha+1)/\alpha} \right] \, ds \]
\[ \leq H^{\alpha+1}(t, c_i)w(c_i) + \int_{c_i}^{t} \left[ H^{\alpha}(t, s)H_2(t, s) |w(s)| - \frac{H^{\alpha+1}(t, s)}{\gamma(r(s)\rho(s))^{1/\alpha}} |w(s)|^{(\alpha+1)/\alpha} \right] \, ds. \]  

(2.6)

For a given \( t \) and \( s \), set

\[ F(v) := H^\alpha H_2 v - \frac{H^{\alpha+1}}{\gamma(r^\rho)^{1/\alpha}} v^{(\alpha+1)/\alpha}, \quad v > 0. \]

Because of \( F'(v) = H^\alpha H_2 - \frac{(\alpha+1)H^{\alpha+1}}{\alpha\gamma(r^\rho)^{1/\alpha}} v^{1/\alpha} \), and \( F(v) \) obtains its maximum at

\[ v = r^\rho \left( \frac{\alpha \gamma H_2}{(\alpha + 1)H} \right)^{\alpha}, \]

and

\[ F(v) \leq F_{\max} = \frac{(\alpha \gamma)^{\alpha}}{(\alpha + 1)^{\alpha+1}} r^\rho H^{\alpha+1}. \]  

(2.7)
Then we get, by using (2.7),

\[
\begin{align*}
\int_{c_i}^{t} H^{α+1}(t, s) q(s) ρ(s) \, ds \\
\leq H^{α+1}(t, c_i) w(c_i) + \frac{(αγ)^α}{(α + 1)^{α+1}} \int_{c_i}^{t} r(s) ρ(s) H_2^{α+1}(t, s) \, ds.
\end{align*}
\]  
(2.8)

Letting \( t \to b_i^- \) in (2.6), we obtain

\[
\begin{align*}
\int_{c_i}^{b_i} H^{α+1}(b_i, s) q(s) ρ(s) \, ds \\
\leq H^{α+1}(b_i, c_i) w(c_i) + \frac{(αγ)^α}{(α + 1)^{α+1}} \int_{c_i}^{b_i} r(s) ρ(s) H_2^{α+1}(b_i, s) \, ds.
\end{align*}
\]  
(2.9)

On the other hand, if we multiply \( H^{α+1}(s, t) \) through (2.5) and integrate it (with \( t \) replaced by \( s \)) over \((t, c_i]\) for \( t ∈ (a_i, c_i], \) \( i = 1, 2, \) instead, by using hypotheses (i)–(ii), we yield for \( s ∈ (t, c_i]\)

\[
\begin{align*}
\int_{t}^{c_i} H^{α+1}(s, t) q(s) ρ(s) \, ds \\
\leq -\int_{t}^{c_i} H^{α+1}(s, t) w′(s) \, ds + \int_{t}^{c_i} H^{α+1}(s, t) \left[ \frac{ρ′(s)}{ρ(s)} w(s) - \frac{|w(s)|^{(α+1)/α}}{γ(r(s)ρ(s))^{1/α}} \right] ds \\
= -H^{α+1}(c_i, t) w(c_i) + \int_{t}^{c_i} (α + 1) H^α(s, t) h_1(s, t) \sqrt{H(s, t)} w(s) \, ds \\
+ \int_{t}^{c_i} H^{α+1}(s, t) \left[ \frac{ρ′(s)}{ρ(s)} w(s) - \frac{|w(s)|^{(α+1)/α}}{γ(r(s)ρ(s))^{1/α}} \right] ds \\
\leq -H^{α+1}(c_i, t) w(c_i) \\
+ \int_{t}^{c_i} \left[ H^α(s, t) H_1(s, t) |w(s)| - \frac{H^{α+1}(s, t)}{γ(r(s)ρ(s))^{1/α}} |w(s)|^{(α+1)/α} \right] ds \\
\leq -H^{α+1}(c_i, t) w(c_i) + \frac{(αγ)^α}{(α + 1)^{α+1}} \int_{t}^{c_i} r(s) ρ(s) H_1^{α+1}(s, t) \, ds.
\end{align*}
\]  
(2.10)

(We get the final “≤” in (2.10) by following the proof of (2.8).) Letting \( t \to a_i^+ \) in (2.10), it follows that
\[
\int_{a_i}^{c_i} H^{\alpha+1}(s, a_i)q(s)\rho(s)\,ds \\
\leq -H^{\alpha+1}(c_i, a_i)w(c_i) + \frac{(\alpha\gamma)^\alpha}{(\alpha + 1)^{\alpha + 1}} \int_{a_i}^{c_i} r(s)\rho(s)H_1^{\alpha+1}(s, a_i)\,ds. \tag{2.11}
\]

Finally, dividing (2.9) and (2.11) by \(H^{\alpha+1}(b_i, c_i)\) and \(H^{\alpha+1}(c_i, a_i)\) respectively, and then adding them, we have the following inequality

\[
\frac{1}{H^{\alpha+1}(c_i, a_i)} \int_{a_i}^{c_i} H^{\alpha+1}(s, a_i)q(s)\rho(s)\,ds + \frac{1}{H^{\alpha+1}(b_i, c_i)} \int_{c_i}^{b_i} H^{\alpha+1}(b_i, s)q(s)\rho(s)\,ds \\
\leq \frac{1}{H^{\alpha+1}(c_i, a_i)} \frac{(\alpha\gamma)^\alpha}{(\alpha + 1)^{\alpha + 1}} \int_{a_i}^{c_i} r(s)\rho(s)H_1^{\alpha+1}(s, a_i)\,ds \\
+ \frac{1}{H^{\alpha+1}(b_i, c_i)} \frac{(\alpha\gamma)^\alpha}{(\alpha + 1)^{\alpha + 1}} \int_{c_i}^{b_i} r(s)\rho(s)H_2^{\alpha+1}(b_i, s)\,ds, \tag{2.12}
\]

which contradict to the condition (2.2). Thus the proof of Theorem 2.1 is completed. □

According to Theorem 2.1, it is very easy for us to obtain the following immediate consequence.

**Corollary 2.2.** Suppose that the hypotheses in Theorem 2.1 hold and function \(H(t, s)\) satisfying (i)–(ii). If there exist some \(c_i \in (a_i, b_i), i = 1, 2,\) and some positive function \(\rho \in C^1([t_0, \infty), \mathbb{R})\) such that

\[
\int_{a_i}^{c_i} \left[ H^{\alpha+1}(s, a_i)q(s)\rho(s) - \frac{(\alpha\gamma)^\alpha}{(\alpha + 1)^{\alpha + 1}} r(s)\rho(s)H_1^{\alpha+1}(s, a_i) \right] \,ds > 0, \tag{2.13}
\]

\[
\int_{c_i}^{b_i} \left[ H^{\alpha+1}(b_i, s)q(s)\rho(s) - \frac{(\alpha\gamma)^\alpha}{(\alpha + 1)^{\alpha + 1}} r(s)\rho(s)H_2^{\alpha+1}(b_i, s) \right] \,ds > 0, \tag{2.14}
\]

for \(i = 1, 2,\) where \(\gamma, H_1, H_2\) are similar to ones in Theorem 2.1, then Eq. (1.1) is oscillatory.

Specially, if letting \(H := H(t - s) \in C(D, \mathbb{R})\) satisfy (i)–(ii), we have that \(h_1(t - s) = h_2(t - s)\) and denote them by \(h(t - s).\) Then, from Theorem 2.1 with \(\rho(t) \equiv 1,\) we obtain:

**Corollary 2.3.** Suppose that for any \(T \geq t_0,\) there exists \(T \leq a_1 < 2c_1 - a_1 \leq a_2 < 2c_2 - a_2\) such that

\[
e(t) \begin{cases} 
\leq 0, & t \in [a_1, 2c_1 - a_1], \\
\geq 0, & t \in [a_2, 2c_2 - a_2].
\end{cases} \tag{2.15}
\]

If there exists \(H := H(t - s)\) satisfying (i)–(ii) in Theorem 2.1 such that
\[ \int_{a_i}^{c_i} H^{\alpha+1}(s-a_i) \left[ q(s) + q(2c_i - s) \right] ds > \frac{(\alpha \gamma)^{\alpha}}{(\alpha + 1)^{\alpha+1}} \int_{a_i}^{c_i} \left[ r(s) + r(2c_i - s) \right] \left[ h(s-a_i) \sqrt{H(s-a_i)} \right]^{\alpha+1} ds \] (2.16)

for \( i = 1, 2 \) and \( \gamma := \frac{\gamma_1}{\gamma_2} \). Then Eq. (1.1) is oscillatory.

The proof of the above corollary is similar to Theorem 2.2 in [13] or Theorem 2.4 in [19], so we omit the details.

**Remark 2.4.** If the hypothesis in Theorem 2.1 is replaced by the following condition

\[
e(t) \begin{cases} 
\geq 0, & t \in [a_1, b_1], \\
\leq 0, & t \in [a_2, b_2],
\end{cases}
\]

we will find that the conclusion of the above theorem is valid as well.

**Remark 2.5.** In [19], Li gave some interval oscillation criteria for Eq. (1.4) with \( e(t) \equiv 0, \tau = \sigma \), one of the results are described as follows:

**Theorem C.** Suppose that for any \( T \geq t_0 \), if there exist some function \( H \) satisfying (i)–(ii) in Theorem 2.1, and \( a, b, c \in R \) such that \( T \leq a < c < b \) and

\[
\frac{1}{H(c, a)} \int_{a}^{c} H(s, a)q(s) + \frac{1}{H(b, c)} \int_{c}^{b} H(b, s)q(s)
\]

\[
> \frac{1}{(\sigma + 1)^{\sigma+1}} \left[ \frac{1}{H(c, a)} \int_{a}^{c} r(s)h_1^{\sigma+1}(s, a) \frac{H^{(\sigma-1)/2}(s, a)}{H^{(\sigma-1)/2}(s, a)} ds 
\right.
\]

\[
+ \frac{1}{H(b, c)} \int_{c}^{b} H(b, s) \left. r(s)h_2^{\sigma+1}(b, s) \frac{H^{(\sigma-1)/2}(b, s)}{H^{(\sigma-1)/2}(b, s)} ds \right].
\] (2.17)

Then Eq. (1.4) with \( e(t) = 0, \tau = \sigma \) is oscillatory.

Comparing with the condition (17) in [19] of the above Theorem C, our results are new and avoid the terms \( \frac{1}{H^{(\sigma-1)/2}(s, a)} \) and \( \frac{1}{H^{(\sigma-1)/2}(b, s)} \), which perhaps lead to improper integral arising. Thus, to some extent, the results in this paper extend, improve and unify some earlier results, and are of a high degree of generality.

**3. \( f(x) \) be not monotone increasing**

In this section, we shall mainly consider the oscillation for Eq. (1.1) under assumptions \( (A_1)–(A_2) \) and the following additional conditions:

\( (C_2) \ 0 < \psi(x) \leq C, \) for some positive constants \( C > 0 \) and for \( x \neq 0 \).
(C3) \( f(x) \) satisfies
\[
\frac{f(x)}{x} \geq K|x|^{\beta-1},
\]
for \( x \neq 0 \), where \( K > 0 \) and \( \beta \geq \alpha \) be constant.

**Lemma 3.1** (Hölder inequality). If \( A \) and \( B \) are nonnegative, then
\[
\frac{1}{p} A^p + \frac{1}{q} B^q \geq AB,
\]
for \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Theorem 3.2.** Suppose \((A_1)–(A_2)\) and \((C_2)–(C_3)\) be valid, and for any \( T \geq t_0 \), there exist \( T \leq a_1 < b_1 \leq a_2 < b_2 \) such that (2.1) holds and \( q(t) \geq 0 \neq 0 \) for \( t \in [a_1, b_1] \cup [a_2, b_2] \). If there exist some \( c_i \in (a_i, b_i) \) for \( i = 1, 2 \), \( H(t, s) \) satisfying (i)–(ii) and a positive function \( \rho \in C^1([t_0, \infty), R) \) such that
\[
\frac{1}{H^{\alpha+1}(c_i, a_i)} \int_{a_i}^{c_i} \left[ H^{\alpha+1}(s, a_i)Q(s)\rho(s) - \frac{C_i}{(\alpha + 1)\alpha + 1}r(s)\rho(s)H_1^{\alpha+1}(s, a_i) \right] ds
\]
\[
+ \frac{1}{H^{\alpha+1}(b_i, c_i)} \int_{c_i}^{b_i} \left[ H^{\alpha+1}(b_i, s)Q(s)\rho(s) - \frac{C_i}{(\alpha + 1)\alpha + 1}r(s)\rho(s)H_2^{\alpha+1}(b_i, s) \right] ds > 0,
\]
for \( i = 1, 2 \), where \( H_1, H_2 \) are defined as in Theorem 2.1 and
\[
Q(t) = \left[ K q(t) \right]^{\alpha/\beta} |e(t)|^{(\beta-\alpha)/\beta}.
\]
Then Eq. (1.1) is oscillatory.

**Proof.** Otherwise, let \( x(t) \) be a nonoscillatory solution of Eq. (1.1), say \( x(t) \neq 0 \) on \([T_0, \infty)\) for some sufficiently large \( T_0 \geq t_0 \). Define
\[
v(t) = \rho(t) \frac{r(t)\psi(x(t))\phi(x'(t))}{|x(t)|^{\alpha-1}x(t)}, \quad t \geq T_0.
\]
(3.2)

Then differentiating (3.2) and making use of Eq. (1.1) and assumptions \((A_1)–(A_2), (C_2)–(C_3)\), we obtain
\[
v'(t) = \frac{e(t)\rho(t)}{|x(t)|^{\alpha-1}x(t)} - \frac{q(t)\rho(t)f(x(t))}{|x(t)|^{\alpha-1}x(t)} - \frac{\alpha\rho(t)r(t)\psi(x(t))\phi(x'(t))x'(t)|x(t)|^{\alpha-1}}{|x(t)|^{2\alpha}} + \frac{\rho'(t)}{\rho(t)} v(t)
\]
\[
\leq \frac{e(t)\rho(t)}{|x(t)|^{\alpha-1}x(t)} - K q(t) \rho(t) |x(t)|^{\beta-\alpha} \frac{\alpha r(t)\rho(t)\psi(x(t))|\phi(x'(t))|^{(\alpha+1)/\alpha}}{\gamma_1(|x(t)|^{\alpha})^{(\alpha+1)/\alpha}} + \frac{\rho'(t)}{\rho(t)} v(t)
\]
\[ \leq -\rho(t) \left( -\frac{e(t)}{|x(t)|^{\alpha-1}x(t)} + K q(t) |x(t)|^{\beta-\alpha} \right) - \frac{\alpha|v(t)|^{(\alpha+1)/\alpha}}{\gamma_1(Cr(t)\rho(t))^{1/\alpha}} + \frac{\rho'(t)}{\rho(t)} v(t). \]  

(3.3)

By the hypotheses, we can choose \( a_i, b_i \geq T_0 \) for \( i = 1, 2 \) such that \( e(t) \leq 0 \) on the interval \( I_1 = [a_1, b_1] \) with \( a_1 < b_1 \) and \( x(t) > 0 \); or \( e(t) \geq 0 \) on the interval \( I_2 = [a_2, b_2] \) with \( a_2 < b_2 \) and \( x(t) < 0 \). Thus, on interval \( I_1 = [a_1, b_1] \), from Hölder’s inequality, we have

\[ -e(t) |x(t)|^{\alpha-1}x(t) + K q(t) |x(t)|^{\beta-\alpha} = |e(t)| x^{\alpha}(t) + K q(t) x^{\beta-\alpha}(t) \geq \frac{\beta-\alpha}{\beta} \left[ \frac{|e(t)|^{(\beta-\alpha)/\beta}}{x^{(\beta-\alpha)/\beta}(t)} \right]^{\beta/(\beta-\alpha)} + \frac{\alpha}{\beta} \left[ (K q(t))^{\alpha/\beta} x^{(\beta-\alpha)/\beta}(t) \right]^{\beta/\alpha} \geq Q(t), \]  

(3.4)

on interval \( I_2 = [a_2, b_2] \), we arrive at as well

\[ -e(t) |x(t)|^{\alpha-1}x(t) + K q(t) |x(t)|^{\beta-\alpha} \geq \left[ K q(t) \right]^{\alpha/\beta} e(t)^{(\beta-\alpha)/\beta} = Q(t). \]  

(3.5)

From (3.4) and (3.5), (3.3) implies that on the intervals \( I_1 \) and \( I_2 \), \( v(t) \) satisfies

\[ v'(t) \leq -\rho(t) Q(t) - \frac{\alpha|v(t)|^{(\alpha+1)/\alpha}}{\gamma_1(Cr(t)\rho(t))^{1/\alpha}} + \frac{\rho'(t)}{\rho(t)} v(t). \]  

(3.6)

The rest of the proof is similar to that of Theorem 2.1 and hence omitted. \( \Box \)

The following results are analogous to Corollaries 2.2–2.3 with the assumption (C1) replaced by (C2) and (C3).

**Corollary 3.3.** Suppose that the hypotheses in Theorem 3.2 hold and function \( H(t, s) \) satisfying (i)–(ii). If there exist some \( c_i \in (a_i, b_i) \), \( i = 1, 2 \), and some positive function \( \rho \in C^1([t_0, \infty), \mathbb{R}) \) such that

\[ \int_{c_i}^{b_i} \left[ H^{\alpha+1}(s, a_i) Q(s) \rho(s) - \frac{C \gamma_1^{\alpha}}{(\alpha + 1)^{\alpha+1}} r(s) \rho(s) H^{\alpha+1}_1(a_i, s) \right] ds > 0, \]  

(3.7)

\[ \int_{c_i}^{b_i} \left[ H^{\alpha+1}(b_i, s) Q(s) \rho(s) - \frac{C \gamma_1^{\alpha}}{(\alpha + 1)^{\alpha+1}} r(s) \rho(s) H^{\alpha+1}_2(b_i, s) \right] ds > 0, \]  

(3.8)

for \( i = 1, 2 \), where \( H_1, H_2, Q \) are similar to ones in Theorem 3.2. Then Eq. (1.1) is oscillatory.

Just as Corollary 2.3, if we take \( H := H(t-s) \) satisfying (i)–(ii) and \( h_1(t-s) = h_2(t-s) \) denoted by \( h(t-s) \), from Theorem 3.2 with \( \rho(t) \equiv 1 \), we will get:
Corollary 3.4. Suppose that for any $T \geq t_0$, there exist $T \leq a_1 < 2c_1 - a_1 \leq a_2 < 2c_2 - a_2$ such that (2.15) in Corollary 2.3 holds. If there exist $H := H(t-s)$ satisfying (i)–(ii) in Theorem 2.1 such that
\[
\int_{a_i}^{c_i} H^{a+1}(s-a_i) \left[ Q(s) + Q(2c_i - s) \right] ds > \frac{C\gamma_i^a}{(\alpha + 1)^{\alpha+1}} \int_{a_i}^{c_i} \left[ r(s) + r(2c_i - s) \right] \left( h(s-a_i) \sqrt{H(s-a_i)} \right)^{\alpha+1} ds \tag{3.9}
\]
for $i = 1, 2$. Then Eq. (1.1) is oscillatory.

Remark 3.5. The Remark 2.4 in Section 2 is also valid in this section.

References