On a special class of smooth codimension two subvarieties in $\mathbb{P}^n$, $n \geq 5$

C. Folegatti

Dipartimento di Matematica, 35, via Machiavelli, 44100 Ferrara, Italy

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Abstract

We consider smooth codimension two subcanonical subvarieties in $\mathbb{P}^n$ with $n \geq 5$, lying on a hypersurface of degree $s$ having a linear subspace of multiplicity $s - 2$. We prove that such varieties are complete intersections. We also give a little improvement to some earlier results on the nonexistence of rank two vector bundles on $\mathbb{P}^4$ with small Chern classes.

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1. Introduction

We work on an algebraically closed field of characteristic zero.

By Lefschetz’s theorem, a smooth codimension two subvariety $X \subset \mathbb{P}^n$, $n \geq 4$, which is not a complete intersection, lying on a hypersurface $\Sigma$, verifies $dim(X \cap Sing(\Sigma)) \geq n - 4$.

In this paper, we deal with a situation in which the singular locus of $\Sigma$ is as large as can be, but, at the same time, the simplest possible: we assume $\Sigma$ is an hypersurface of degree $m$ with an $(m - 2)$-uple linear subspace of codimension two.

More generally, we are concerned with smooth codimension two subvarieties $X \subset \mathbb{P}^n$, $n \geq 5$.

In the first part, we consider smooth subcanonical threefolds $X \subset \mathbb{P}^5$ and we prove that if $deg(X) \leq 25$, then $X$ is a complete intersection (Prop. 2.2). In the second section, we study a particular class of codimension two subvarieties and we prove the following result.

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E-mail address: chiara@dm.unife.it (C. Folegatti).
Theorem 1.1. Let $X \subset \mathbb{P}^n$, $n \geq 5$, be a smooth codimension two subvariety (if $n=5$ assume $Pic(X) = \mathbb{Z}H$) lying on a hypersurface $\Sigma$ of degree $m$, which is singular, with multiplicity $m-2$, along a linear subspace $K$ of dimension $n-2$. Then $X$ is a complete intersection.

This gives further evidence to Hartshorne conjecture in codimension two.

It is enough to prove the theorem for $n=5$, the result for higher dimensions will follow by hyperplane sections. For $n=5$ it is necessary to suppose $Pic(X) = \mathbb{Z}H$, whereas for $n \geq 6$, thanks to Barth’s theorem, this hypothesis is always verified.

The proof for $n=5$ goes as follows. Using the result of the first part we may assume $d < 25$, then we prove, under the special assumptions of the theorem, that either $deg(X)$ is less than 25 or we use the result of Lemma 3.3 to conclude that $S$ is a complete intersection.

By the way we give a little improvement of earlier results on the non existence of rank two vector bundles on $\mathbb{P}^5$ with small Chern classes, see Lemma 2.8.

2. Smooth subcanonical threefolds in $\mathbb{P}^5$

Let $X$ be a smooth subcanonical threefold in $\mathbb{P}^5$, of degree $d$, with $\omega_X \cong \mathcal{O}_X(e)$. Let $S = X \cap H$ be the general hyperplane section of $X$, $S$ is a smooth subcanonical surface in $\mathbb{P}^4$, indeed by adjunction it is easy to see that $\omega_S \cong \mathcal{O}_S(e + 1)$. Again we set $C$ the general hyperplane section of $S$, $C$ is a smooth subcanonical curve in $\mathbb{P}^3$, with $\omega_C \cong \mathcal{O}_C(e + 2)$.

We can compute the sectional genus $\pi(S)$, indeed since $\omega_C \cong \mathcal{O}_C(e + 2)$ it follows that $\pi = g(C) = 1 + [d(e + 2)]/2$.

Lemma 2.1. With the notations above, $q(S) = 0$ and all hyperplane sections $C$ of $S$ are linearly normal in $\mathbb{P}^3$.

Proof. By Barth’s theorem we know that if $X \subset \mathbb{P}^5$ is a smooth threefold, then $h^1(\mathcal{O}_X) = 0$. Let us consider the exact sequence: $0 \rightarrow \mathcal{O}_X(-1) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_S \rightarrow 0$. By taking cohomology and observing that $h^2(\mathcal{O}_X(-1)) = h^1(\omega_X(1)) = 0$ by Kodaira, we get the result. \[\square\]

If we look at the surface $S$, we can observe that most of its invariants are known. Hence it seems natural to consider the double points formula in order to get some more information.

Since $q(S) = 0$, $\pi - 1 = [d(e + 2)]/2$ and $K^2 = d(e + 1)^2$, the formula becomes:
$$d(d - 2e^2 - 9e - 17) = -12(1 + p_g(S)),$$
where the quantity $1 + p_g(S)$ is strictly positive. We have the following condition:
$$d(d - 2e^2 - 9e - 17) \equiv 0 \pmod{12}.$$  \(1\)

Proposition 2.2. Let $X \subset \mathbb{P}^5$ be a smooth subcanonical threefold of degree $d$, then if $d \leq 25$, $X$ is a complete intersection.

Proof. We recall that for a smooth subcanonical threefold in $\mathbb{P}^5$ with $\omega_X \cong \mathcal{O}_X(e)$ we have $e \geq 3$, unless $X$ is a complete intersection (see [1]). Let $G(d, 3) = 1 + [d(d - 3) - 2r(3 - r)]/6$ be the maximal genus of a curve of $\mathbb{P}^3$ of degree $d = 3k + r, 0 \leq r \leq 2$, not lying on a surface of degree two. If we compare the value of $\pi$ computed before with this (using $e \geq 3$), we
see that if $d \leq 17$, then $h^0(\mathcal{I}_C(2)) \neq 0$. Since Severi’s and Zak’s theorems on linear normality $h^1(\mathcal{I}_X(1)) = h^1(\mathcal{O}_X(1)) = 0$, it follows that $h^0(\mathcal{I}_X(2)) \neq 0$ and this implies that $X$ is a complete intersection (see [5, Theorem 1.1]).

If $d = 18$, then $\pi = G(18, 3)$. It follows that $C$ is a.C.M. then by the exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O} \to \mathcal{I}_C(e + 6) \to 0$$

we obtain $h^1(\mathcal{O}(k)) = 0$ for all $k \in \mathbb{Z}$. Hence by Horrocks’ theorem $\mathcal{O}$ is split and then $C$ is a complete intersection. Since this holds for the general $\mathbb{P}^3$ section $C$, the same holds for $S$ and for $X$.

If $d = 19$ then $C$ lies on a quadric surface unless $\pi = 1 + (19(e + 2))/2 \leq G(19, 3)$. This inequality yields $e = 3$ but if we look at formula (1) we see that this is not possible.

If $d = 20$, then $\pi = G(20, 4)$, $C$ is a.C.M. and we argue as in the case $d = 18$ to conclude that $X$ is a complete intersection.

If $d = 21, 22, 23$ and if $h^0(\mathcal{I}_C(4)) \neq 0$, then thanks to the “lifting theorems” in $\mathbb{P}^4$ and $\mathbb{P}^5$ (see [10]) we have $h^0(\mathcal{I}_X(4)) \neq 0$ and again by [5] $X$ is a complete intersection. We then assume $h^0(\mathcal{I}_C(4)) = 0$ and using the fact that $\pi = 1 + [d(e + 2)]/2 \leq G(d, 5)$, we obtain $e = 3$. However this is not possible because of formula (1).

If $d = 24$ we still get $e = 3$, but formula (1) is satisfied. We have the following exact sequence:

$$0 \to \mathcal{O} \to \mathcal{O}(5) \to \mathcal{I}_X(9) \to 0$$

where $\mathcal{O}$ is a rank two vector bundle with $c_1(\mathcal{O}) = -1$ and $c_2(\mathcal{O}) = 4$. If $h^0(\mathcal{I}_X(4)) = 0$, then $h^0(\mathcal{O}) = 0$, which is not possible since by [4] there exists no rank two stable vector bundle with such Chern classes. Hence it would be $h^0(\mathcal{I}_X(4)) \neq 0$ and this implies (see [5, Theorem 1.1]) that $X$ is a complete intersection but this is also impossible since the system given by the equations $a + b = -1$ and $ab = 4$ does not have solution in $\mathbb{Z}$.

If $d = 25$, supposing $h^0(\mathcal{I}_C(4)) = 0$ we obtain $e = 4$. In that case we have exactly $\pi = G(25, 5) = 76$ and this means that if $h^0(\mathcal{I}_C(4)) = 0$, then $C$ is a.C.M. It follows that $C$, and then $X$, is a complete intersection. □

**Remark 2.3.** If we perform the same calculations of the proof of 2.2 for $d = 26$, we have that $e = 3$.

Now if we consider subcanonical threefolds in $\mathbb{P}^5$ with $e = 3$, by Kodaira we have that $h^0(\mathcal{O}_X(4)) = \chi(\mathcal{O}_X(4))$. By Riemann–Roch formula for threefolds (see [11]) we compute $\chi(\mathcal{O}_X(4)) = [5d(50 - d)]/24$. Since $h^0(\mathcal{O}_{\mathbb{P}^3}(4)) = 126$, it is easy to see that for $d \geq 30$ it must be $h^0(\mathcal{O}_X(4)) \neq 0$, hence $X$ is a complete intersection.

On the other hand, for $26 \leq d \leq 30$ the unique value of $d$ satisfying (1) is $d = 26$. Thus we have shown that, among smooth threefolds in $\mathbb{P}^5$ with $e = 3$, the only possibility for $X$ not to be a c.i. is if $d = 26$.

We conclude this section with some result about rank two vector bundles. Let us start with a lemma concerning subcanonical double structures.

**Lemma 2.4.** Let $Y \subset \mathbb{P}^n$, $n \geq 4$, be a complete intersection of codimension two. Let $Z$ be a l.c.i. subcanonical double structure on $Y$. Then if $emdim(Y) \leq n - 1$, $Z$ is a complete intersection.

**Proof.** By [8] we have that any doubling of a l.c.i. $Y$ with $emdim(Y) \leq dim(Y) + 1$ is obtained by the Ferrand construction. Hence there is a surjection $\mathcal{N}_Y^L \to \mathcal{L} \to 0$, where
is a locally free sheaf of rank one on $Y$. Taking into account that $\omega_{Z|Y} \cong \omega_Y \otimes \mathcal{L}^y$ (see [2]) and recalling that $Z$ is subcanonical and $Y$ is a c.i., we obtain that $\mathcal{L}' \cong \mathcal{E}_Y(l)$ for a certain $l \in \mathbb{Z}$.

On the other hand, since $Y$ is a complete intersection, say $Y = F_a \cap F_b$, we have $\mathcal{N}_Y \cong \mathcal{E}_Y(a) \oplus \mathcal{E}_Y(b)$, then the sequence above becomes: $\mathcal{E}_Y(a) \oplus \mathcal{E}_Y(-b) \to \mathcal{E}_Y(l) \to 0$. The map $f$ is given by two polynomials of degree, respectively, $a + l$ and $b + l$. If $F$ and $G$ are both not constant, it follows, since $n \geq 4$, that $B := (F)_0 \cap (G)_0 \cap Y \neq \emptyset$. For each $x \in B$ the induced map $f_x$ on the stalks is not surjective: absurd. Thus necessarily $F$ or $G$ is a non zero constant, i.e. either $l = -a$ or $-b$. If $l = -a$ (resp. $l = -b$) we are doubling $Y$ on $F_b$, $Z = F_a^2 \cap F_b$ (resp. we are doubling $Y$ on $F_a$, $Z = F_a \cap F_b^2$). In any case, $Z$ is a complete intersection. □

**Lemma 2.5.** Let $Z \subset \mathbb{P}^4$ be a l.c.i. quartic surface with $\omega_Z \cong \mathcal{O}_Z(-a)$. If $a \geq 3$, then $Z$ is a complete intersection.

**Proof.** Let $C$ be the hyperplane section of $Z$ and let $C_{red} = \tilde{C}_1 \cup \ldots \cup \tilde{C}_s$ be the decomposition of $C_{red}$ in irreducible components, hence $C = C_1 \cup \ldots \cup C_s$, where $C_i$ is a multiple structure on $\tilde{C}_i$ for all $i$. We have $\omega_C \cong \mathcal{O}_C(-a + 1)$, on the other hand, $\omega_{C|C_i} \cong \omega_{C_i}(A)$, where $A$ is the scheme theoretic intersection of $C_i$ and $\bigcup_{i \neq j} C_j$. It follows that $\omega_{C_i} \cong \mathcal{O}_{C_i}(-a + 1 - A)$ and since $\text{deg}(A) \geq 0$, this implies that $p_a(C_i) < 0$, then $C_i$ is a multiple structure on $\tilde{C}_i$ of multiplicity $> 1$.

It turns out that each irreducible component of $Z_{red}$ appears with multiplicity $> 1$, thus since $\text{deg}(Z) = 4$ it follows that $Z$ is a double structure on a quadric surface or a 4-uple structure on a plane. This last case can be readily solved. Indeed $C$ would be a 4-uple structure on a line and thanks to [2] (Remark 4.4) we know that a thick and l.c.i. 4-uple structure on a line is a global complete intersection. Hence we can assume $Z$ quasi-primitive, i.e. we can assume $Z$ does not contain the first infinitesimal neighbourhood of $Z_{red}$. Anyway by [9] (see main theorem and Section B) and since $Z_{red}$ is a plane we also have that $Z$ is a c.i.

We then suppose that $Z$ is a double structure on a quadric surface of rank $\geq 2$, which is a complete intersection (1, 2). By Lemma 2.4 it follows that $Z$ is a c.i. □

**Definition 2.6.** Let $\mathcal{E}$ be a rank two normalized vector bundle (i.e. $c_1(\mathcal{E}) = -1, 0$), we set $r := \min \{n \mid h^0(\mathcal{E}(n)) \neq 0\}$. If $r > 0$, $\mathcal{E}$ is stable. If $r \leq 0$ we call $r$ degree of instability of $\mathcal{E}$.

**Remark 2.7.** The next lemma represents a slight improvement of previous results about the existence of rank two vector bundles in $\mathbb{P}^4$ and $\mathbb{P}^5$.

Indeed Decker proved that any stable rank two vector bundle on $\mathbb{P}^4$ with $c_1 = -1$ and $c_2 = 4$ is isomorphic to the Horrocks–Mumford bundle and that in $\mathbb{P}^5$ there is no stable rank two vector bundle with these Chern classes (see [4]). We show that neither are there such vector bundles with $r = 0$. As for bundles with $c_1 = 0$ and $c_2 = 3$, there are similar results by Barth–Elencwajg (see [3]) and Ballico–Chiantini (see [1]) stating that $r < 0$. We prove that in fact $r < -1$. 
Lemma 2.8. There does not exist any rank two vector bundle $E$ on $\mathbb{P}^4$ such that $r = 0$, $c_1(E) = -1$, $c_2(E) = 4$ or, respectively, $r = -1$, $c_1(E) = 0$, $c_2(E) = 3$.

Proof. We observe first of all that in both cases there are no integers $a$, $b$ satisfying the equations $a + b = c_1$, $ab = c_2$, hence the vector bundle $E$ cannot be split.

Assume $E$ has $r = 0$, $c_1(E) = -1$, $c_2(E) = 4$, then $h^0(E) \neq 0$. There is a section of $E$ vanishing on a codimension two scheme $Z: 0 \to \mathcal{O} \to E \to \mathcal{F}_Z(-1) \to 0$. We have $\deg(Z) = c_2(E) = 4$ and $Z$ subcanonical with $\omega_Z \cong \mathcal{O}_Z(-6)$.

If $r = -1$, $c_1(E) = 0$, $c_2(E) = 3$, then $h^0(E(-1)) \neq 0$ and we get a section of $E(-1)$ vanishing in codimension two along a quartic surface $Z$, with $\omega_Z \cong \mathcal{O}_Z(-7)$.

It is enough to apply 2.5 to conclude that such vector bundles cannot exist. □

3. Codimension two subvarieties in $\mathbb{P}^n$, $n \geq 5$

Let $X \subset \mathbb{P}^n$, $n \geq 5$ be a smooth codimension two subvariety, lying on a hypersurface $\Sigma$ of degree $m \geq 5$ with a $(m - 2)$-uple linear subspace $K$ of codimension two, i.e. $K \cong \mathbb{P}^{n-2}$. If $n = 5$ we assume $Pic(X) = \mathbb{Z} H$, for $n \geq 6$ this is granted by Barth’s theorem. In any case we set $\omega_X \cong \mathcal{O}_X(e)$.

The general $\mathbb{P}^4$ section $S$ of $X$ is a surface lying on a threefold $\Sigma \cap H$ of degree $m$ having a singular plane of multiplicity $(m - 2)$. We will always suppose that $h^0(\mathcal{F}_S(2)) = 0$.

We will prove that $S$ contains a plane curve. First we fix some notations and state some results concerning surfaces containing a plane curve, proofs and more details can be found in [6].

Let $P$ be a plane curve of degree $p$, lying on a smooth surface $S \subset \mathbb{P}^4$. Let $\Pi$ be the plane containing $P$ and let $Z := S \cap \Pi$. We assume that $P$ is the one-dimensional part of $Z$ and we define $\mathcal{R}$ as the residual scheme of $Z$ with respect to $P$, namely $\mathcal{R} := (\mathcal{F}_Z : \mathcal{I}_P)$. The points of the zero-dimensional scheme $\mathcal{R}$ can be isolated as well as embedded in $P$.

Let $\delta$ be the $\infty^1$ linear system cut out on $S$, residually to $P$, by the hyperplanes containing $\Pi$. Severi’s theorem states that unless $S$ is a Veronese surface, then $h^1(\mathcal{F}_S(1)) = 0$ and thus $H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \cong H^0(\mathcal{O}_S(1))$. Moreover if $p \geq 2$, the hyperplanes containing $\Pi$ are exactly those containing $P$. This allows us to conclude that $\delta = [H - P]$ (on $S$). We will denote by $Y_H$ the element of $\delta$ corresponding to the hyperplane $H$ and we call $C_H := P \cup Y_H = S \cap H$.

Let $\mathcal{B}$ be the base locus of $\delta$. We have the following results.

Lemma 3.1. (i) $P$ is reduced, the base locus $\mathcal{B}$ is contained in $\Pi$ and $\dim(\mathcal{B}) \leq 0$. The general $Y_H \in \delta$ is smooth out of $\Pi$ and does not have any component in $\Pi$.

(ii) $\mathcal{B} = \mathcal{R}$ and $\deg(\mathcal{R}) = (H - P)^2 = d - 2p + P^2$.

Proof. See Lemmas 2.1 and 2.4 of [6]. □

In the present situation, $S$ is subcanonical with $\omega_S \cong \mathcal{O}_S(e + n - 4)$. We know $\deg(\mathcal{R}) = d - 2p + P^2$ and we compute $P^2$ by adjunction, knowing $p_a(P)$ since $P$ is a plane curve and recalling that $K_S = (e + n - 4)H$. It turns out that $\deg(\mathcal{R}) = d + P^2 - p(e + n + 1)$. 
Lemma 3.2. If $S \subset \mathbb{P}^4$ is a smooth surface, lying on a degree $m$ hypersurface $\Sigma$ with a $(m-2)$-uple plane, then $S$ contains a (reduced) plane curve, $P$. If $H$ is a general hyperplane through $P$, then $H \cap S = P \cup Y_H$ where $Y_H$ has no irreducible components in $H$ and is smooth out of $H$.

Proof. If $H$ is the plane with multiplicity $(m-2)$ in $\Sigma$ and $H$ is an hyperplane containing $H$, we have $H \cap \Sigma = (m-2)P \cup Q_H$, where $Q_H$ is a quadric surface and $C_H = S \cap H \subset (m-2)P \cup Q_H$. If $\dim(C_H \cap H) = 0$, then $C_H \subset Q_H$, but this is excluded by our assumptions. Indeed by Severi’s theorem $h^0(I_C(2)) \neq 0$ would imply $h^0(I_S(2)) \neq 0$. So $\dim(C_H \cap H) = 1$ and $S$ contains a plane curve. We conclude with Lemma 3.1. \hfill $\square$

If $H$ is an hyperplane through $H$, the corresponding section is $C_H = Y_H \cup P$. Since $Y_H$ does not have any component in $P$, we have $H \subset Q_H$. We denote by $q_H$ the conic $Q_H \cap H$. As $H$ varies, the $q_H$’s form a family of conics in $H$. Let $\mathcal{A}_q$ be the base locus of $\{q_H\}$, we have $\mathcal{A} \subset \mathcal{A}_q$, since $Y_H \cap \mathcal{A} \subset Q_H \cap \mathcal{A} = q_H$.

One can show that $\mathcal{A}_q$ is $(m-1)$-uple in $\Sigma$ (see [6, Lemma 3.3]). To prove this, just consider an equation $\phi$ of $\mathcal{A}$ and note that clearly $\phi \in I^2(H)$. Easy computations show that all $(s-2)$th derivatives of $\phi$ vanish at a point $x \in \mathcal{A}_q$.

The following result concerns in particular subcanonical surfaces.

Lemma 3.3. With notations as above ($S$ subcanonical with $\omega_S \cong \mathcal{O}_S(e)$), we have

(i) $\deg(P) \leq e + 3$.
(ii) If $\mathcal{R} = \emptyset$, then $S$ is a complete intersection.

Proof. (i) We have already computed $\deg(\mathcal{R}) = -p(e+5)+d+p^2$. Recall that $\deg(Y_H) = d - p$ and $\deg(\mathcal{R}) \leq \deg(Y_H)$, this implies $p \leq e + 4$. We will see that the case $p = e + 4$ is not possible. Let $p = e + 4$, then $Y_H \cdot P = p - P^2 = -p(e - 4) = 0$, i.e. $Y_H \cap P = \emptyset$. In other words the curve $C_H = S \cap H = Y_H \cup P$ is not connected, but this is impossible since $h^0(\mathcal{O}_{C_H}) = 1$ (use $0 \rightarrow \mathcal{O}_S(-1) \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_{C_H} \rightarrow 0$ and $h^1(\mathcal{O}_S(-1)) = h^1(\omega_S(1)) = 0$ by Kodaira).

(ii) Since $S$ is subcanonical we can consider the exact sequence $0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \rightarrow \mathcal{F}_S(e + 5) \rightarrow 0$. If we restrict it to $\mathcal{R}$ and divide by an equation of $P$, we get $0 \rightarrow \mathcal{O}_H \rightarrow \mathcal{E}(p) \rightarrow \mathcal{F}_{\mathcal{R}}(e + 5 - 2p) \rightarrow 0$. If $\mathcal{R} = \emptyset$, then $\mathcal{F}_{\mathcal{R}} = \mathcal{O}_H$ and the above sequence splits. It follows that $\mathcal{E}$ splits and $S$ is a complete intersection. \hfill $\square$

Example 3.4. Let $S$ be a smooth section of the Horrocks–Mumford bundle $\mathcal{F}$, $S$ is an abelian variety and has $\omega_S = \mathcal{O}_S$. By Lemma 3.3 we know that if $S$ contains a plane curve $P$, then $p \leq 3$. Moreover, $P$ cannot be a line or a conic, since these curves are rational and this would imply that there exists a non constant morphism $\mathbb{P}^1 \rightarrow S$, factoring through $\text{Jac}^0(\mathbb{P}^1) \cong \{\ast\}$ and this is not possible. Then necessarily $P$ is a plane smooth cubic (hence elliptic).

By the “reducibility lemma” of Poincaré, an abelian surface $S$ contains an elliptic curve if and only if $S$ is isogenous to a product of elliptic curves. It is known that the general section of the Horrocks–Mumford bundle is not isogenous to a product of elliptic curves, but there
exist smooth sections satisfying such property (see [7,11]). Summarizing we can say that among the sections of Horrocks–Mumford bundle we can find smooth surfaces containing a plane curve, but the general one does not contain any.

Now assume $S$ to be one of those smooth surfaces containing a plane cubic, $P$. Let $\Pi$ be the plane spanned by $P$. Recall that we have $0 \rightarrow \mathcal{O} \rightarrow \mathcal{F}(3) \rightarrow \mathcal{F}_S(5) \rightarrow 0$. We restrict the sequence to $\Pi$ and since $s_{\Pi}$ vanishes along $P$, we can divide by an equation of $P$ and obtain a section of $\mathcal{F}_{\Pi}$. We then have $h^0(\mathcal{F}_{\Pi}) \neq 0$, i.e. $\mathcal{F}_{\Pi}$ is not stable, in other words $\Pi$ is an unstable plane for $\mathcal{F}$.

In order to prove Theorem 1.1 we need some other preliminary results.

**Lemma 3.5.** Let $F \subset \mathbb{P}^3$ be a surface of degree $m$, singular along a line $D$ with multiplicity $m - 1$. Then $F$ is the projection of a surface of degree $m$ in $\mathbb{P}^{m+1}$ (minimal degree surface).

**Proof.** The surface $F$ is rational. Let $p : F' \rightarrow F$ be a desingularization of $F$ and let $H$ be a divisor in $p^*\mathcal{O}_F(1)$. We have $0 \rightarrow \mathcal{O}_F \rightarrow \mathcal{O}_F(H) \rightarrow \mathcal{O}_H(H) \rightarrow 0$ and since $F'$ is rational too, then $h^1(\mathcal{O}_F) = 0$. Now $h^0(\mathcal{O}_H(H)) = m + 1$ ($H$ is a rational curve), then $h^0(\mathcal{O}_F(H)) = m + 2$ and we can embed $F'$ in $\mathbb{P}^{m+1}$.

**Remark 3.6.** Minimal degree surfaces in $\mathbb{P}^n$ are classified, in particular they can be: a smooth rational scroll, a cone over a rational normal curve of $\mathbb{P}^{n-1}$ or the Veronese surface if $n = 5$. Except for the Veronese, all these surfaces are ruled in lines.

**Lemma 3.7.** Let $T \subset \mathbb{P}^{m+1}$, $m \geq 3$, be a surface ruled in lines. Let $C \subset T$ be a smooth irreducible curve. If $\text{dim}(\langle C \rangle) = 3$, then $\text{deg}(C) \leq \text{deg}(T) - m + 3$ ($\langle C \rangle$ is the linear space spanned by $C$).

**Proof.** Let us consider $m - 3$ general points on $C$ and let $f_1, \ldots, f_{m-3}$ be the rulings passing through these points. We consider moreover $m - 3$ points $p_1, \ldots, p_{m-3}$ such that $p_i \in f_i$ but $p_i \not\in \langle C \rangle$ and let also $q_1, \ldots, q_4$ be four general points in $\langle C \rangle$. We thus have $m + 1$ points, spanning at most a space of dimension $m$, hence these points are contained in a hyperplane $H$ of $\mathbb{P}^{m+1}$. Now $\langle C \rangle \subset H$ since $q_i \in H \forall i = 1, \ldots, 4$, $f_i \subset H$ since $\text{card}(f_i \cap H) > 1 \forall i = 1, \ldots, m - 3$, so $H \cap T$ contains $C, f_1, \ldots, f_{m-3}$ (which form a degenerate curve in $T$ of degree $m - 3 + \text{deg}(C)$) and this yields: $\text{deg}(T) \geq \text{deg}(C) + m - 3$.

**Lemma 3.8.** Let $X, K \subset \mathbb{P}^n$, $n \geq 4$, $X$ smooth of codimension two, $K \cong \mathbb{P}^{n-2}$ a linear subspace. Let $\text{dim}(X \cap K) = n - 3$. If the general hyperplane section of $X \cap K$ contains a linear subspace of dimension $n - 4$, then $X$ contains a linear subspace of dimension $n - 3$.

**Proof.** We see $X_K = X \cap K$ as a hypersurface in $K \cong \mathbb{P}^{n-2}$. A general hyperplane of $K$ is cut on $K$ by a general hyperplane of $\mathbb{P}^n$. Then the hypersurface $X_K$ of $K$ is such that its general hyperplane section contains a linear subspace of dimension $n - 4$. We claim that $X_K$ contains an hyperplane of $K$. Indeed we may assume $X_K$ reduced. Let $X_K = T_1 \cup \ldots \cup T_r$ be the decomposition of $X_K$ into irreducible components. Now using the fact that the general
We recall that if \( C \) is a curve in \( \mathbb{P}^3 \), we follow the method used in the proof of Theorems 1.1 and 1.2 of [6]. We must distinguish three cases, depending on the behaviour of the base locus \( B \) of \( C \).

If \( \dim(B) = 0 \), at least two of the conics intersect properly and then \( \deg(B) \leq 4 \). It follows that \( r := \deg(B) \leq 4 \) too, since \( X \subset B \). If \( \dim(B) = 1 \), there are two possibilities: the one-dimensional part of \( B \) can be a line or a conic.

If \( \dim(B) = 0 \), we have seen that \( r = \deg(B) \leq 4 \). We observe that actually we can suppose \( r \geq 1 \), indeed by 3.3 if \( X = \emptyset \), then \( S \) (and \( X \)) is a complete intersection.

If \( H \) is a general hyperplane, \( Y_H \cap P \subset q_H \cap P \) and since at least one conic intersects \( P \) properly, we obtain \( Y_H \cdot P \leq 2p \). We have \( Y_H \cdot P = p - P^2 \) and recalling that \( r = d - 2p + P^2 \), it follows \( Y_H \cdot P = d - p - r \). Putting everything together: \( p \geq (d - r)/3 \geq (d - 4)/3 \). On the other hand, we have \( Y_H \cdot P = p(e + 5 - p) \) and clearly this implies \( p \geq e + 3 \). Comparing this with the result stated in 3.3 and setting \( \omega_S \cong \mathcal{O}_S(e + 1) \), we are left with only two possibilities: \( p = e + 3 \) or \( p = e + 4 \). We have already observed that \( d = p(e + 6) - P^2 + r \), then considering the two cases above, we can express \( d \) in terms of \( e \) and \( r \) and we get the following formulas:

\[
\begin{align*}
\text{if } p &= e + 3, \quad & \text{then } d &= 3(e + 3) + r, \\
\text{if } p &= e + 4, \quad & \text{then } d &= 2(e + 4) + r.
\end{align*}
\]

We recall that if \( C \) lies on a quartic surface and \( d \) is large enough, \( X \) lies on a quartic hypersurface too, then \( X \) is a complete intersection. We know that \( \pi - 1 = (d(e + 2))/2 \), then since \( (d - 4)/3 \leq p \leq e + 4 \) we obtain \( \pi - 1 \geq (d(d - 10))/6 \). If we compare this quantity with \( G(d, 5) \), we see that if \( d \geq 33 \), then \( h^0(\mathcal{F}_C(4)) \neq 0 \) and \( X \) is a complete intersection.

Thanks to the result in Proposition 2.2 we know that if \( d \leq 25 \), \( X \) is a complete intersection too, then we only have to check the cases \( 26 \leq d \leq 32 \).

We assume \( h^0(\mathcal{F}_C(4)) = 0 \), then it must be \( \pi = 1 + (d(e + 2))/2 \leq G(d, 5) \). Thanks to this inequality it is easy to see that for \( d \leq 32 \), we always have \( e \leq 5 \). Now if we look at formulas (2) and (3) above, clearly \( e \leq 5 \) implies \( d \leq 28 \).

On the other hand, in order to have \( d \geq 26 \), \( e \) must be at least equal to 4.

If \( d = 26, 27, 28 \), the condition on the genus \( \pi \) yields \( e = 4 \) again. However, if we look at formulas (2) and (3) we see that if \( e = 4 \), \( d \) is at most equal to 25.

If \( B \) contains a line, \( D \), then \( D \) has multiplicity \( m - 1 \) in \( B \), so if \( H \) is an hyperplane containing \( D \) (but not \( P \) ), \( F = Y \cap H \) is a surface of degree \( m \) in \( \mathbb{P}^3 \) having a \( (m - 1) \)-uple line. This kind of surface is a projection of a degree \( m \) surface in \( \mathbb{P}^{m+1} \), by Lemma 3.5. The hyperplane section \( C = S \cap H \) is a curve contained in \( F \). We must distinguish two cases: \( D \subset S \) or \( D \not\subset S \).

If \( D \not\subset S \), we claim that the general \( C \) is smooth. Let \( |L| \) be the linear system cut on \( S \) by the hyperplanes containing \( D \) and let \( B = D \cap S = \{p_1, \ldots, p_r\} \). Since \( B \) is the base locus of \( |L| \), the general element of \( |L| \) is smooth out of \( B \). If all curves in \( |L| \) were singular at a point \( p_i \in B \), it would be \( T_{p_i} S \subset H \), \( \forall H \supset D \). Anyway the intersection of \( H \supset D \) is
only $D$, so this is not possible. It follows that the curves of $|L|$ singular at a $p_i \in B$ form a closed subset of $|L|$. The same holds for all $p \in B$, hence the claim.

Let $F'$ be a surface in $\mathbb{P}^{m+1}$ projecting down to $F$. Since $C$ is not contained in the singular locus of $F$, there exists a curve $C' \subset F'$ such that the projection restricted to $C'$ is an isomorphism over $C$. In particular $\mathcal{O}_{C'}(1) \cong \mathcal{O}_C(1)$ and since $C$ is linearly normal in $\mathbb{P}^3$, this implies that $C'$ is degenerate. Now we can apply Lemma 3.7 to $F'$ and $C'$ (we have already pointed out that $F'$ is ruled in lines unless $F'$ is the Veronese surface) and we get $d = \deg(C') \leq m - m + 3 = 3$. If $F'$ is the Veronese surface we have anyway $d \leq 4$.

If $D \subset S$, then $D$ is a component of the plane curve $P$ (the one-dimensional part of $S \cap \Pi$). Back to the variety $X \subset \Sigma \subset \mathbb{P}^5$ with $K \cong \mathbb{P}^3 \subset \Sigma$ a linear subspace of multiplicity $m - 2$, we have a surface $X_K = X \cap K \subset K \cong \mathbb{P}^3$ such that its general hyperplane section contains a line. This implies by Lemma 3.8 that $X_K$ contains a plane and thus $X$ contains a plane, say $E$. This plane is a Cartier divisor on the smooth threefold $X$. Since we are supposing $Pic(X) = \mathbb{Z}H$, there exists an hypersurface such that $E$ is cut on $X$ by this hypersurface, but this could happen only if $\deg(X) = 1$.

To complete the proof we only have to consider the case in which $\mathcal{B}_q = q$, where $q$ is an irreducible conic (if $q$ is reducible, $\mathcal{B}_q$ contains a line). For every $Y_H \in |H - P|$ we consider the zero-dimensional scheme $A_H = Y_H \cap q$. For every $H$, $A_H$ is a subset of $d - p$ points of $q$.

There are two possibilities: $q \subset S$ or $q \not\subset S$. If $q \not\subset S$, then $A_H$ is fixed (otherwise the points of $A_H$ would cover the conic, as $H$ varies, i.e. $q \subset S$). It must be $A_H = \mathcal{R}$. It is enough to compare the degrees of $A_H$ and $\mathcal{R}$ to see that this implies $p^2 = p$ and then $Y_H \cdot P = P^2 - p = 0$. This is not possible since the corresponding hyperplane section $C_H$ of $S$ would be disconnected.

Hence $q \subset S$ and then $q \subset P$. In other words: $A_H = Y_H \cap P$, thus $Y_H P = d - p$ and $r = 0$. By Lemma 3.3 we conclude that $X$ is a complete intersection. \hfill $\square$

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References