JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 113, 468–472 (1986)

Comparison Theorems for Difference Inequalities

S. R. GRACE AND B. S. LALLI

Deparment of Mathematics, University of Saskatchewan, Saskatoon, Canada S7N OWO

AND

C. C. YEH

Department of Mathematics, Central University, Chung-Li, Taiwan, Republic of China

Submitted by S. M. Meerkov

Let $R=(-\infty, \infty)$, $R^+=[0, \infty)$ and let N denote the set of nonnegative integers. For the sequence $\{u_n\}: N \to R$ and for the function $u: R^+ \to R$, we define

$$\Delta u_n = u_{n+1} - u_n \qquad \text{for} \quad n \in N$$

and

$$\Delta u(t) = u(t+1) - u(t) \qquad \text{for} \quad t \in R^+.$$

Recently, Redheffer and Walter [3] established a comparison theorem for difference inequalities which is similar to that of the first-order differential inequalities. The purpose of this Note is to give some comparison theorems for difference inequalities which are different from the result of Redheffer and Walter [3]. For related results we refer to [1, 2, 4, 5].

As in the classical definition a real-valued function $f(t, u)$ defined on $R^+ \times R$ is called increasing in u if

$$u \leqslant v \qquad \text{implies}\, f(t, u) \leqslant f(t, v)$$

for $t \in R^+$. If $-f(t, u)$ is increasing in u, then f is called decreasing in u.

THEOREM 1. *For the two sequences $\{v_n\}, \{w_n\}: N \to R$, if*

(A) *$f(t, u)$ is increasing in u and*

$$\Delta v_n - f(n, v_n) \leqslant \Delta w_n - f(n, w_n) \qquad \textit{for all } n \in N, \tag{1}$$

0022-247X/86 $3.00

or

(B) *$f(t, u)$ is decreasing in u and*

$$\Delta v_n - f(n, v_{n+1}) \leqslant \Delta w_n - f(n, w_{n+1}) \qquad \text{for all } n \in N, \tag{2}$$

then $v_0 \leqslant w_0$ implies $v_n \leqslant w_n$ for all $n \in N$.

Proof. We consider the case (A). If the assertion were untrue, then there would exist a positive integer m such that

$$v_m \leqslant w_m,$$

but

$$v_{m+1} > w_{m+1}.$$

Thus

$$\Delta v_m > \Delta w_m$$

and

$$f(m, v_m) \leqslant f(m, w_m).$$

Hence

$$\Delta v_m - f(m, v_m) > \Delta w_m - f(m, w_m)$$

which is a contradiction. This contradiction proves the case (A).

A similar argument can be used to prove the case (B). Thus the proof is complete. ∎

Remark. The case (A) of Theorem 1 is a discrete analogue for first-order differential inequalities, for example, see [1].

The following is a continuous analogue of Theorem 1.

THEOREM 2. *Let f: $R^+ \times R \to R$ and v, w: $R^+ \to R$. If*

(A) *$f(t, u)$ is increasing in u and*

$$\Delta v(t) - f(t, v(t)) \leqslant \Delta w(t) - f(t, w(t)) \qquad \text{for } t \in R^+, \tag{3}$$

or

(B) *$f(t, u)$ is decreasing in u and*

$$\Delta v(t) - f(t, v(t+1)) \leqslant \Delta w(t) - f(t, w(t+1)) \qquad \text{for } t \in R^+, \tag{4}$$

then $v(t) \leqslant w(t)$ for $t \in [0, 1)$ implies $v(t) \leqslant w(t)$ for $t \in R^+$.

Next, we treat with a comparison theorem for difference inequalities which is similar to that of the integral inequalities due to Walter [4]. As usual, we define $\sum_{s=0}^{-1} u(s)=0$.

THEOREM 3. *Let $F\colon N\times N\times R\to R$ satisfy*

$$u\leqslant v \text{ implies } F(t,s,u)\leqslant F(t,s,v)$$

for all $(t,s)\in N\times N$. If for three sequences $\{v_n\},\{w_n\}, \{g_n\}\colon N\to R$ the following two inequalities hold:

$$v_n\leqslant g_n+\sum_{s=0}^{n-1} F(n,s,v_s), \qquad w_n\geqslant g_n+\sum_{s=0}^{n-1} F(n,s,w_s), \tag{5}$$

where the equality holds in at most one place for each n, then

$$v_n<w_n \qquad \text{for all } n\in N.$$

Proof. For $n=0$, it follows from (5) that

$$v_0\leqslant g_0$$

and

$$w_0\geqslant g_0,$$

where the equality holds in at most one place; thus

$$v_0<w_0.$$

If the conclusion were false, then there would exist a positive integer m such that $v_m=w_m$ and $v_s<w_s$ for $s=0, 1,..., m-1$. By the increasing of F,

$$F(m,s,v_s)\leqslant F(m,s,w_s) \qquad \text{for } s=0,1,..., m-1.$$

This and (5) imply

$$v_m\leqslant g_m+\sum_{s=0}^{m-1} F(m,s,v_s)\leqslant g_m+\sum_{s=0}^{m-1} F(m,s,w_s)\leqslant w_m,$$

where there is strictly inequality in at least one place; hence

$$v_m<w_m.$$

This contradiction proves our theorem.

It follows from Theorem 3 that we have the following

COROLLARY 4. *Let F, v_n, and w_n be defined as in Theorem 3 for $n \in N$. If*

$$v_n - \sum_{s=0}^{n-1} F(n, s, v_s) < w_n - \sum_{s=0}^{n-1} F(n, s, w_s) \qquad \text{for } n \in N,$$

then $v_n < w_n$ for all $n \in N$.

Finally, we consider the case that "$<$" is replaced by "$\leqslant$" in Corollary 4.

THEOREM 5. *Let F, v_n, w_n, and g_n be defined as in Theorem 3. Suppose the difference equation*

$$w_n = g_n + \sum_{s=0}^{n-1} F(n, s, w_s)$$

has a unique solution $w_n(g_n)$ for each g_n and

$$\max_{0 \leqslant s \leqslant n-1} |h_k(s) - g_s| \to 0 \qquad \text{implies } |w_n(h_k(n)) - w_n(g_n)| \to 0 \tag{6}$$

where $h_k \colon N \to R$ for $k = 1, 2, \ldots$. If

$$v_n - \sum_{s=0}^{n-1} F(n, s, v_s) \leqslant w_n - \sum_{s=0}^{n-1} F(n, s, w_s) \qquad \text{for } n \in N,$$

then $v_n \leqslant w_n$ for all $n \in N$.

Proof. Let

$$g_n = w_n - \sum_{s=0}^{n-1} F(n, s, w_s),$$

and for any $\varepsilon > 0$, let w_n^{ε} be the solution of

$$g_n + \varepsilon = w_n - \sum_{s=0}^{n-1} F(n, s, w_s).$$

Thus

$$v_n - \sum_{s=0}^{n-1} F(n, s, v_n) \leqslant g_n < g_n + \varepsilon = w_n^{\varepsilon} - \sum_{s=0}^{n-1} F(n, s, w_s^{\varepsilon})$$

From Corollary 4, we have

$$v_n < w_n^{\varepsilon} \qquad \text{for } n \in N.$$

It follows from (6) that

$$\varepsilon \to 0 \qquad \text{implies } w_n^{\varepsilon} \to w_n^0 \equiv w_n.$$

Thus $v_n \leqslant w_n$ for all $n \in N$.

The continuous analogue of Theorem 5 is due to Kato [2, p. 122].

As an application of Theorem 5, let $F(n, s, u) = L(n, s)u$ and

$$w_n = H_n + \sum_{s=0}^{n-1} L(n, s) w_s, \tag{7}$$

then we can estimate the solution of (7) under suitable assumptions on L and H_n. Hence we derive the Gronwall inequality of discrete type from Theorem 5. For this technique we refer to [5].

References

1. J. Eisenfeld and V. Lakshmikanthan, Differential inequalities and flow-invariance via linear functionals, *J. Math. Anal. Appl.* **69** (1979), 104–114.
2. J. Kato, "Functional Equations," 1974. (Japanese)
3. R. Redheffer and W. Walter, A comparison theorem for difference inequalities, *J. Differential Equations* **44** (1982), 111–117.
4. W. Walter, "Differential and Integral Inequalities," Springer-Verlag, New York/Berlin, 1970.
5. C. C. Yeh, Discrete inequalities of the Gronwall–Bellman type in n independent variables II, *J. Math. Anal. Appl.*, in press.