On Mutually Nearest and Mutually Furthest Points in Reflexive Banach Spaces

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Let $G$ be a nonempty closed (resp. bounded closed) subset in a reflexive strictly convex Kadec Banach space $X$. Let $\mathcal{K}(X)$ denote the space of all nonempty compact convex subsets of $X$ endowed with the Hausdorff distance. Moreover, let $\mathcal{K}G(X)$ denote the closure of the set $\{A \in \mathcal{K}(X) : A \cap G = \emptyset\}$. A minimization problem $\min(A, G)$ (resp. maximization problem $\max(A, G)$) is said to be well posed if it has a unique solution $(x_0, z_0)$ and every minimizing (resp. maximizing) sequence converges strongly to $(x_0, z_0)$. We prove that the set of all $A \in \mathcal{K}G(X)$ (resp. $A \in \mathcal{K}(X)$) such that the minimization (resp. maximization) problem $\min(A, G)$ (resp. $\max(A, G)$) is well posed contains a dense $G$-subset of $\mathcal{K}G(X)$ (resp. $\mathcal{K}(X)$), extending the results in uniformly convex Banach spaces due to Blasi, Myjak and Papini. © 2000 Academic Press

I. INTRODUCTION

Let $X$ be a real Banach space. We denote by $\mathcal{B}(X)$ the space of all nonempty closed bounded subsets of $X$. For a closed subset $G$ of $X$ and $A \in \mathcal{B}(X)$, we set

$$\lambda_{AG} = \inf \{\|z - x\| : x \in A, z \in G\},$$

and for $G \in \mathcal{B}(X)$, we set

$$\mu_{AG} = \sup \{\|z - x\| : x \in A, z \in G\}.$$

Given a nonempty closed subset $G$ of $X$ (resp. $G \in \mathcal{B}(X)$), according to [9], a pair $(x_0, z_0)$ with $x_0 \in A$, $z_0 \in G$ is called a solution of the minimization (resp. maximization) problem, denoted by $\min(A, G)$ (resp. $\max(A, G)$), if $\|x_0 - z_0\| = \lambda_{AG}$ (resp. $\|x_0 - z_0\| = \mu_{AG}$). Moreover, any sequence $\{(x_n, z_n)\}$

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$x_n \in A, z_n \in G$, such that $\lim_{n \to \infty} \|x_n - z_n\| = \lambda_{AG}$ (resp. $\lim_{n \to \infty} \|x_n - z_n\| = \mu_{AG}$) is called a minimizing (resp. maximizing) sequence. A minimization (resp. maximization) problem is said to be well posed if it has a unique solution $(x_0, z_0)$, and every minimizing (resp. maximizing) sequence converges strongly to $(x_0, z_0)$.

Set

$\mathscr{C}(X) = \{ A \in \mathscr{B}(X) : A \text{ is convex} \},$

and let $\mathscr{C}(X)$ be endowed with the Hausdorff distance $h$ defined as follows:

$h(A, B) = \max\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \}, \quad \forall A, B \in \mathscr{C}(X).$

As is well known, under such metric, $\mathscr{C}(X)$ is complete.

In [9], the authors considered the well posedness of the minimization and maximization problems. If $X$ is a uniformly convex Banach space they proved that the set of all $A \in \mathscr{G}_c(X)$ (resp. $A \in \mathscr{C}(X)$), such that the minimization (resp. maximization) problem $\min(A, G)$ (resp. $\max(A, G)$) is well posed, is a dense $G_\delta$-subset of $\mathscr{G}_c(X)$ (resp. $\mathscr{C}(X)$), where $\mathscr{C}(X)$ is the closure of the set $\{ A \in \mathscr{C}(X) : \lambda_{AG} > 0 \}$.

Furthermore, let

$\mathscr{K}(X) = \{ A \in \mathscr{C}(X) : A \text{ is compact} \}$

and $\mathscr{K}_c(X) = \mathscr{K}(X) \cap \mathscr{G}_c(X)$. Clearly, $X$ can be embedded as a subset of $\mathscr{K}(X)$ in a natural way that, for any $x \in X$, $A_x \in \mathscr{K}(X)$ is defined by $A_x = \{ x \}$.

It is our purpose in the present note to extend the results, with a completely different approach, to a reflexive strictly convex Kadec Banach space. We prove that if $X$ is a reflexive strictly convex Kadec Banach space, then the set of all $A \in \mathscr{K}_c(X)$ (resp. $A \in \mathscr{K}(X)$), such that the minimization problem $\min(A, G)$ (resp. maximization problem $\max(A, G)$) is well posed, contains a dense $G_\delta$-subset of $\mathscr{K}_c(X)$ (resp. $\mathscr{K}(X)$).

It should be noted that the problems considered here are in the spirit of Stechkin [27]. Some further developments of Stechkin’s ideas can be founded in [2–6, 8, 11–17, 20, 24, 26] and in the monograph [10], while some generic results in spaces of convex sets and bounded sets can be founded in [2, 3, 7, 19, 21].

In sequel, let $X^*$ denote the dual of $X$. We use $B(x, r)$ to denote the closed ball with center at $x$ and radius $r$. As usual, if $A \subset X$, by $A$ and $\text{diam} A$ we mean the closure and the diameter of $A$, respectively, while $\overline{\text{co}} A$ stands for the closed convex hull of $A$. 

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Definition 1.1. Let $D$ be an open subset of $X$. A real-valued function $f$ on $D$ is said to be Frechet differentiable at $x \in D$ if there exists an $x^* \in X^*$ such that
$$\lim_{y \to x} \frac{f(y) - f(x) - \langle x^*, y - x \rangle}{\|y - x\|} = 0.$$ $x^*$ is called the Frechet differential at $x$ which is denoted by $Df(x)$.

The following proposition on the Frechet differentiability of Lipschitz functions due to [24] is useful.

Proposition 1.1. Let $f$ be a locally Lipschitz continuous function on an open set $D$ of a Banach space with equivalent Frechet differentiable norm (in particular, $X$ reflexive will do). Then $f$ is Frechet differentiable on a dense subset of $D$.

Definition 1.2. A Banach space $X$ is said to be (sequentially) Kadec provided that for each sequence $\{x_n\} \subset X$ which converges weakly to $x$ with $\lim_{n \to \infty} \|x_n\| = \|x\|$ we have $\lim_{n \to \infty} \|x_n - x\| = 0$.

Definition 1.3. A Banach space $X$ is said to be strongly convex provided it is reflexive, Kadec and strictly convex.

We also need a result concerning the characterization of strongly convex spaces, which is due to Konjagin [15], see also Borwein and Fitzpatrick [5].

Proposition 1.2. A Banach space $X$ is strongly convex if and only if for every closed nonempty subset $G$ of $X$ there is a dense set of points $X \setminus G$ possessing unique nearest points.

2. MINIMIZATION PROBLEMS

Let $x \in X$, $A \in \mathcal{X}(X)$ and $G$ be a closed subset of $X$. We set
$$d_G(x) = \inf_{z \in G} \|x - z\|,$$
$$d_G(A) = \inf_{x \in A} d_G(x) = \lambda_{dG}$$
and
$$P_A(G) = \{x \in A : d_G(x) = d_G(A)\}.$$
Then
\[ |d_G(A) - d_G(B)| \leq h(A, B), \quad \forall A, B \in \mathcal{X}(X). \]

For \( A \in \mathcal{X}(X) \), let \( f_A \) be the functional on \( X \) defined as follows:
\[ f_A(x) = d_G(A + x), \quad \forall x \in X. \]

Then \( f_A \) is 1-Lipschitz and satisfies \( f_A(x) = f_A(x_0) + x(x_0) \).

**Lemma 2.1.** Suppose that \( f_A \) is Fréchet differentiable at \( x = 0 \) with \( Df_A(0) = x^* \). Then \( \|x^*\| = 1 \) and for any \( x_0 \in P_\alpha(G) \), \( \{z_n\} \subset G \) with \( \lim_{n \to \infty} \|x_0 - z_n\| = d_G(x_0) \), we have
\[ d_G(x_0) = \lim_{n \to \infty} \langle x^*, x_0 - z_n \rangle. \]

**Proof.** Let \( x_0, \{z_n\} \) satisfy the assumptions of the lemma. Then for each \( 1 > t > 0 \),
\[ f_A(t(z_n - x_0)) - f_A(0) = d_G(A + t(z_n - x_0)) - d_G(A) \]
\[ \leq \|x_0 + t(z_n - x_0) - z_n\| - d_G(A) \]
\[ = (1 - t) \|x_0 - z_n\| - d_G(A) \]
\[ = -t \|x_0 - z_n\| + \|x_0 - z_n\| - d_G(A). \]

Let \( t_n = 2^{-n} + \|x_0 - z_n\| - d_G(A) \). Then from the Fréchet differentiability of \( f_A(x) \) at \( x = 0 \), we have that
\[ \lim_{n \to \infty} \frac{f_A(t(z_n - x_0)) - f_A(0) - \langle x^*, z_n - x_0 \rangle}{t_n} = 0, \]
so that
\[ \lim_{n \to \infty} \left[ -\|x_0 - z_n\| + \langle x^*, x_0 - z_n \rangle \right] \geq 0 \]
and
\[ d_G(A) = \lim_{n \to \infty} \|x_0 - z_n\| \leq \lim_{n \to \infty} \langle x^*, x_0 - z_n \rangle. \]

Note that \( \|x^*\| \leq 1 \) since \( f_A \) is 1-Lipschitz. It follows that
\[ \lim_{n \to \infty} \|x_0 - z_n\| \geq \lim_{n \to \infty} \|x^*\| \|x_0 - z_n\| \geq \limsup_{n \to \infty} \langle x^*, x_0 - z_n \rangle. \]
Comparison of the last two inequalities shows the desired results, proving the lemma.

**Lemma 2.2.** The set-valued map $P_A(G)$ with respect to $A$ is upper semi-continuous in the sense that for each $A_0 \in \mathcal{K}_G(X)$ and any open set $U$ with $P_A(G) \subseteq U$, there exists $\delta > 0$ such that for any $A \in \mathcal{K}_G(X)$ with $h(A, A_0) < \delta$, $P_A(G) \subseteq U$.

**Proof.** Suppose on the contrary that there exist $\{A_n\} \subseteq \mathcal{K}_G(X)$ and $A \in \mathcal{K}_G(X)$ with $\lim_{n \to \infty} h(A_n, A) = 0$, such that $P_A(G) \not\subseteq U$ for some open subset $U$ with $P_A(G) \subseteq U$ and each $n$. Let $x_n \in P_A(G) \setminus U$ for any $n$. Note that $\bigcup_n A_n$ is relatively compact and $\{x_n\} \subseteq \bigcup_n A_n$. It follows that there exists a subsequence, denoted by itself, such that $\lim_{n \to \infty} \|x_n - a_n\| = 0$ for some $x_0 \in X$. Clearly, $x_0 \notin U$. However, by $\lim_{n \to \infty} h(A_n, A) = 0$, there exists $\{a_n\} \subseteq A$ such that $\lim_{n \to \infty} \|x_n - a_n\| = 0$ so that

$$\limsup_{n \to \infty} \|a_n - x_0\| \leq \lim_{n \to \infty} \|x_n - a_n\| + \lim_{n \to \infty} \|x_n - x_0\| = 0$$

and $x_0 \in A$. Furthermore, for each $n$,

$$\inf_{z \in G} \|z - x_0\| \leq \inf_{z \in G} \|z - x_n\| + \|x_n - x_0\|$$

$$\leq d_G(A) + h(A_n, A) + \|x_n - x_0\|,$$

which shows that $x_0 \in P_A(G)$, contradicting that $x_0 \notin U$. The proof is complete.

Let

$$L_\alpha(G) = \left\{ A \in \mathcal{K}_G(X) : \|x\| \leq \frac{\|x\|}{\|x\|} + \delta, \quad \text{for some } \delta > 0, \|x\| = 1 \right\}$$

and let

$$L(G) = \bigcap_\alpha L_\alpha(G).$$

**Lemma 2.3.** Suppose that $X$ is reflexive. Then $L(G)$ is a dense $G_\delta$-subset of $\mathcal{K}_G(X)$. 
Proof. To show that \( L(G) \) is a \( G \)-subset of \( \mathcal{K}_G(X) \), we only need prove that \( L_n(G) \) is open for each \( n \). Let \( A \in L_n(G) \). Then there exist \( x^* \in X^* \) with \( \|x^*\| = 1 \) and \( \delta > 0 \) such that

\[
\beta = \inf \{ \langle x^*, x - z \rangle : x \in P_A(G), z \in G \cap B(x, d_G(x) + \delta) \} - (1 - 2^{-n}) d_G(A) > 0.
\]

Let \( \lambda > 0 \) be such that \( \lambda < \min \{ (\delta/2), (\beta/2) \} \). It follows from Lemma 2.2 that there exists \( 0 < \varepsilon < \lambda \) such that for any \( F \in \mathcal{K}_G(X) \) with \( h(F, A) < \varepsilon \) and each \( y \in P_F(G) \) there exists \( x \in P_A(G) \) satisfying \( \|y - x\| < \lambda \). For \( \delta^* = \delta - 2\lambda \) we have

\[
H = G \cap B(y, d_G(y) + \delta^*) \subset G \cap B(x, d_G(x) + \delta).
\]

Thus if \( z \in H \),

\[
\langle x^*, x - z \rangle \geq \beta + (1 - 2^{-n}) d_G(A)
\]

and

\[
\langle x^*, y - z \rangle \geq \beta + (1 - 2^{-n}) d_G(F) - \lambda.
\]

Then

\[
\inf \{ \langle x^*, y - z \rangle : z \in H, y \in P_F(G) \} > (1 - 2^{-n}) d_G(F)
\]

and \( F \in L_n(G) \) for all \( F \in \mathcal{K}_G(X) \) with \( h(F, A) < \varepsilon \), which implies that \( L_n(G) \) is open in \( \mathcal{K}_G(X) \).

In order to prove the density of \( L(G) \) in \( \mathcal{K}_G(X) \), from Proposition 1.1, it suffices to prove that if \( f_A(x) \) is Fréchet differentiable at \( x = 0 \) then \( A \in L(G) \).

Suppose on the contrary that for some \( n \) there exist \( \{x_m\} \subset P_A(G) \) and \( \{z_m\} \subset G \cap B(x_m, d_G(x_m) + 2^{-m}) \) such that

\[
\langle x^*, x_m - z_m \rangle \leq (1 - 2^{-n}) d_G(A), \quad \forall m,
\]

where \( x^* = Df_A(0) \). With no loss of generality, we assume that \( \lim_{m \to \infty} \|x_m - x_0\| = 0 \) for some \( x_0 \in P_A(G) \). Observe that \( \lim_{m \to \infty} \|x_m - z_m\| = d_G(A) \). Then \( \lim_{m \to \infty} \|x_0 - z_m\| = d_G(A) \). Thus Lemma 2.1 implies that

\[
\lim_{m \to \infty} \langle x^*, x_0 - z_m \rangle = d_G(A)
\]

so that

\[
\lim_{m \to \infty} \langle x^*, x_m - z_m \rangle = d_G(A)
\]
which contradicts that
\[ \langle x^*, x_m - z_m \rangle \leq (1 - 2^{-m}) \, d_G(A), \quad \forall m. \]

This completes the proof.

**Lemma 2.4.** Suppose \( X \) is a reflexive Kadec Banach space. Let \( A \in L(G) \). Then any minimizing sequence \( \{(x_n, z_n)\} \) with \( x_n \in A, \ z_n \in G \) has a subsequence which converges strongly to a solution of the minimization problem \( \min(A, G) \).

**Proof.** Let \( A \in L(G) \). Then \( A \in L_m(G) \) for any \( m = 1, 2, \ldots \). By the definition of \( L_m(G) \), there exist \( \delta_m > 0, \ x^*_m \in X^*, \ |x^*_m| = 1 \) such that
\[
\inf \{ \langle x^*_m, x - z \rangle : z \in G \cap B(x, d_G(x) + \delta_m), x \in P_A(G) \} > (1 - 2^{-m}) \, d_G(A).
\]

Let \( \{(x_n, z_n)\} \) with \( x_n \in A, \ z_n \in G \) be any minimizing sequence. With no loss of generality, we assume that \( x_n \to x_0 \) strongly and \( z_n \to z_0 \) weakly as \( n \to \infty \) for some \( x_0 \in P_A(G), \ z_0 \in X \), since \( A \) is compact and \( X \) is reflexive. Then we have that
\[
\|x_0 - z_0\| \leq \liminf_{n \to \infty} \|x_0 - z_n\| = d_G(A).
\]

We also assume that \( \delta_n \leq \delta_m \) if \( m < n \) and \( z_n \in G \cap B(x_n, d_G(x_n) + \delta_m) \) for all \( n > m \). Thus,
\[
\langle x^*_m, x_0 - z_n \rangle > (1 - 2^{-m}) \, d_G(A), \quad \forall n > m
\]
and
\[
\langle x^*_m, x_0 - z_0 \rangle > (1 - 2^{-m}) \, d_G(A), \quad \forall m.
\]

Hence we have
\[
\|x_0 - z_0\| \geq \limsup_{m \to \infty} \langle x^*_m, x_0 - z_0 \rangle \geq d_G(A).
\]

This shows that \( \|x_0 - z_0\| = d_G(A) \). Now the fact that \( X \) is Kadec implies that \( \lim_{n \to \infty} \|z_n - z_0\| = 0 \) and \( z_0 \in G \). Clearly, \( (x_0, z_0) \) is a solution of the minimization problem \( \min(A, G) \) and completes the proof.

Let
\[
Q_n(G) = \left\{ A \in \mathcal{A}(X) : \text{diam} \ P_A(G) < \frac{1}{n} \right\}
\]
and let
\[ Q(G) = \bigcap_{n} Q_n(G). \]

**Lemma 2.5.** Suppose that \( X \) is reflexive Kadec Banach space. Then \( Q(G) \) is a dense \( G_\delta \)-subset of \( K_G(X) \).

**Proof.** Given \( n \) and \( A \in Q_n(X) \), we define
\[ c = \frac{1}{n} \text{diam } P_A(G) \]
and
\[ U = \left\{ x \in X : d_{P_A(G)}(x) < \frac{c}{3} \right\}. \]
Then
\[ \text{diam } U < \text{diam } P_A(G) + \frac{2c}{3} < \frac{1}{n}. \]

It follows from Lemma 2.2 that there exists \( \lambda > 0 \) such that \( P_F(G) \subseteq U \) for any \( F \in K(X) \) with \( h(F, A) < \lambda \). This shows \( \text{diam } P_F(G) < (1/n) \) for any \( F \in K(X) \) with \( h(F, A) < \lambda \) so that \( Q_n(G) \) is open and \( Q(G) \) is a \( G_\delta \)-subset of \( K_G(X) \).

Now let us prove that \( Q(G) \) is dense. From Lemma 2.3 and 2.4 it suffices to prove that for any \( A \in L(G) \) and a solution \((x_0, z_0)\) of \( \min(A, G) \), the set \( A_x \) defined by
\[ A_x = \overline{co} \left( A \cup \{ x\} \right) \]
is in \( Q(G) \) for all \( 0 < \alpha < 1 \), where \( x_\alpha = \alpha x_0 + (1-\alpha) z_0 \).

Observe that for each \( 0 < \alpha < 1 \), if \( x \in A_x \), \( x \neq x_\alpha \), then \( x = ta + (1-t) x_\alpha \) for some \( 0 < t \leq 1 \) and \( a \in A \). Set \( a_0 = ta + (1-t) x_0 \). Then \( a_0 \in A \) and
\[
\inf_{z \in G} \| z - x \| \geq \inf_{z \in G} \| z - a_0 \| - \| a_0 - x \|
\]
\[
\geq \| z_0 - x_0 \| - (1-t) \| x_0 - x_\alpha \|
\]
\[
= (1 - (1-t)(1-\alpha)) \| z_0 - x_0 \|
\]
\[
> \alpha \| z_0 - x_0 \| = \| z_0 - x_\alpha \| \geq \lambda_{A,G}.
\]
This shows \( P_{A_x}(G) = x_\alpha \) and proves the lemma.
Now we are ready to give the main theorem of this section.

**Theorem 2.1.** Suppose that $X$ is a strongly convex Banach space. Let $G$ be a closed subset of $X$. Then the set of all $A \in \mathcal{K}_G(X)$ such that the minimization problem $\min(A, G)$ is well posed contains a dense $G_\delta$-subset of $\mathcal{K}_G(X)$.

**Proof.** It suffices to prove that $\min(A, G)$ is well posed if $A \in Q(G) \cap L(G)$, as $Q(G) \cap L(G)$ is a dense $G_\delta$-subset of $\mathcal{K}_G(X)$.

We first show that $\min(A, G)$ has a unique solution. Suppose there is $A \in Q(G) \cap L(G)$ such that $\min(A, G)$ has two solutions $(x_0, z_0), (x_1, z_1)$. Clearly $x_1 = x_0$ because $A \in Q(G)$. On the other hand, since $A \in L(G)$, for each $n$, there exists $x_n^* \in X, \|x_n^*\| = 1$ satisfying

$$\langle x_n^*, x_0 - z_i \rangle > (1 - 2^{-n}) d_G(A), \quad i = 0, 1$$

so that

$$\|x_0 - z_0 + x_1 - z_1\| \geq \limsup_{n \to \infty} \langle x_n^*, x_0 - z_0 + x_0 - z_1 \rangle \geq 2d_G(A).$$

Thus, using the strict convexity of $X$, we have $z_0 = z_1$, proving the uniqueness.

Now let $(x_n, z_n)$ with $x_n \in A, z_n \in G$ be any minimizing sequence. Then from the uniqueness and Lemma 2.4 it follows that $(x_n, z_n)$ converges strongly to the unique solution of the minimization problem $\min(A, G)$. The proof is complete.

**Remark 2.1.** Theorem 2.1 is a multivalued version of a theorem due to Lau [17].

Note that if $\min(A, G)$ has a unique solution $(x_0, z_0)$, then $x_0$ has a unique nearest point in $G$. This, with Proposition 1.2 and Theorem 2.1, make us prove the following theorem.

**Theorem 2.2.** Let $X$ be a Banach space. Then the following statements are equivalent:

(i) $X$ is strongly convex;

(ii) for every closed non-empty subset $G$ of $X$, the set of all $A \in \mathcal{K}_G(X)$ such that the minimization problem $\min(A, G)$ is well posed contains a dense $G_\delta$-subset of $\mathcal{K}_G(X)$;

(iii) for every closed non-empty subset $G$ of $X$, the set of all $A \in \mathcal{K}_G(X)$ such that the minimization problem $\min(A, G)$ is well posed contains a dense subset of $\mathcal{K}_G(X)$. 

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Proof. By Theorem 2.1, it suffices to prove that (iii) implies (i). For any fixed \( x \in X \setminus G \) and any \( \varepsilon > 0, \varepsilon < d_G(x) \), let \( A_\varepsilon \) denote the closed ball at \( x \) with radius \( \varepsilon / 2 \). From (iii) it follows that there exists \( B_\varepsilon \in \mathcal{K}_G(X) \) such that \( h(A_\varepsilon, B_\varepsilon) < (\varepsilon / 2) \) and \( \min(B_\varepsilon, G) \) is well posed so that \( \min(B_\varepsilon, G) \) has a unique solution \( (x', z') \). Thus,

\[
\|x' - x\| \leq h(A_\varepsilon, B_\varepsilon) + \frac{\varepsilon}{2} < \varepsilon
\]

and \( x' \) has a unique nearest point \( z' \) in \( G \). Using Proposition 1.2, we complete the proof.

Remark 2.2. Let \( X \) be a space of finite dimensions. It follows from Remark 3.4 in [9] that Theorem 2.1 and so Theorem 2.2 may not hold if \( \mathcal{K}_G(X) \) is replaced by \( \mathcal{K}(X) \).

3. MAXIMIZATION PROBLEMS

In order to establish the well posedness result of maximization problems we need some lemmas on furthest points.

Let \( E \) be a real Banach space and \( G \) be a bounded closed subset of \( E \). We set

\[
F_G(x) = \sup_{z \in G} \|x - z\|, \quad \forall x \in E.
\]

Thus \( z \in G \) is called a furthest point of \( x \) with respect to \( G \) if \( \|z - x\| = F_G(x) \). The set of all furthest point of \( x \) with respect to \( G \) is denoted by \( R_G(x) \), that is,

\[
R_G(x) = \{z \in G : \|z - x\| = F_G(x)\}.
\]

Lemma 3.1. Suppose that \( F_G(\cdot) \) is Frechet differentiable at \( x \in E \) with \( DF_G(x) = x^* \). Then \( \|x^*\| = 1 \), and for any \( \{z_n\} \subset G \) with \( \lim_{n \to \infty} \|x - z_n\| = F_G(x) \), we have

\[
\lim_{n} \langle x^*, x - z_n \rangle = F_G(x).
\]

Proof. Let \( \{z_n\} \subset G \) such that \( \lim_{n \to \infty} \|x - z_n\| = F_G(x) \). It follows that for \( \forall t < 0, \)

\[
F_G(x + t(z_n - x)) - F_G(x) \geq -t \|x - z_n\| + \|x - z_n\| - F_G(x).
\]
Taking $t_n < 0$, $t_n \to 0$ with $t_n^2 > F_G(x) - \|x - z_n\|$, we have

$$\lim_n \left( \frac{F_G(x + t_n(z_n - x)) - F_G(x)}{t_n} - \langle x^*, z_n - x \rangle \right) = 0.$$  

This implies that

$$\liminf_n (\|x - z_n\| - t_n + \langle x^*, x - z_n \rangle) \geq 0.$$

Now $\|x^*\| \leq 1$ since $F_G(\cdot)$ is 1-Lipschitz. It follows that

$$F_G(x) \leq \liminf_n \langle x^*, x - z_n \rangle \leq \limsup_n \langle x^*, x - z_n \rangle \leq \lim_n \|x^*\| \|x - z_n\| \leq \|x^*\| F_G(x) \leq F_G(x).$$

This shows that $\|x^*\| = 1$ and

$$\lim_n \langle x^*, x - z_n \rangle = F_G(x).$$

The proof is complete.

For $y \in E$, define

$$S = \overline{\text{span}} G, \quad E_y = S \oplus \overline{\text{span}} \{ y \},$$

and let $J(G)$ denote the set of all $y \in E$ such that $F_G(\cdot)$ is Frechet differentiable at $y$ when $F_G(\cdot)$ is restricted on the subspace $E_y$.

**Lemma 3.2.** $J(G)$ is a $G_\delta$-subset of $E$.

**Proof.** For any $y \in E$, let $J_y(G)$ denote the set of all points $x \in E_y$ such that $F_G(\cdot)$ is Frechet differentiable at $x$ when $F_G(\cdot)$ is restricted on the subspace $E_y$. Clearly, $J_y(G) \subset J(G)$ for any $y \in E$. Then $J(G) = \bigcup_{y \in F} J_y(G)$ is a $G_\delta$-subset of $E$ from Proposition 1.25 of [23] or [20] since $F_G(\cdot)$ is convex on $E$.

**Lemma 3.3.** Let $\mathcal{D}$ be a closed convex subset of $E$. Suppose that $S$ is reflexive and $S \subset \mathcal{D}$. Then $\mathcal{D} \cap J(G)$ is a dense $G_\delta$-subset of $\mathcal{D}$.
Proof. From Lemma 3.2, it suffices to prove that $\mathcal{D} \cap J(G)$ is dense in $\mathcal{D}$. Toward this end, for fixed $y \in \mathcal{D}$, set

$$O = \{ xy + x : x \in S, 0 < a < 1 \}.$$ 

Then $O \subset \mathcal{D}$ is open in $E$, and $E$ is reflexive. It follows from the convexity of the function $F_G$ and Proposition 1.1 (see also [23]) that $F_G(\cdot)$ is Frechet differentiable on a dense subset of $E_y$, so that there exists $\{ x_n \} \subset O$ such that $F_G(\cdot)$ is Frechet differentiable at $x_n$ and $x_n \to y$. Observe that $E_{x_n} = E_y$ for any $n$. It follows that $\mathcal{D} \cap J(G)$ is dense in $\mathcal{D}$. The proof is complete.

Now we suppose $\mathcal{H}(X)$ to be endowed with the addition and multiplication as follows:

$$A + B = \{ a + b : a \in A, b \in B \}, \quad \forall A, B \in \mathcal{H}(X),$$

$$\lambda A = \{ \lambda a : a \in A \}, \quad \forall A \in \mathcal{H}(X), \quad \lambda \geq 0.$$ 

Then it follows from the proof of Theorem 2 in [25] that

**Lemma 3.4.** Suppose that $X$ is a reflexive Banach space. Then there exists a Banach space $(E, \| \cdot \|_E)$ such that $\mathcal{H}(X)$ is embedded as a convex cone in such a way that

(i) the embedding is isometric, that is, $\forall A, B \in \mathcal{H}(X)$, $h(A, B) = \| A - B \|_E$;

(ii) addition in $E$ induces addition in $\mathcal{H}(X)$;

(iii) multiplication by nonnegative scalars in $E$ induces the corresponding operation in $\mathcal{H}(X)$;

(iv) linear operation in $E$ induces linear operation in $X$.

Thus, from $X \subset E$, for $G \in \mathcal{B}(X)$, $A \in \mathcal{H}(X) \subset E$, we have

$$R_G(A) = \{ z \in G : \| A - z \|_E = F_G(A) \} = \{ z \in G : \sup_{x \in A} \| x - z \| = \mu_{AG} \}.$$

**Lemma 3.5.** Suppose that $X$ is reflexive Kadec Banach space. Let $E$ be given by Lemma 3.4 and $G \in \mathcal{B}(X)$. Then for $A \in J(G)$ any sequence $\{ z_n \} \subset G$ with $\lim_{n \to \infty} \sup_{x \in A} \| x - z_n \| = \mu_{AG}$ has a subsequence which converges strongly to an element of $R_G(A)$. 

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Proof. Let $A \in J(G)$ and let $\{z_n\} \subseteq G$ such that $\lim_{n \to \infty} \sup_{x \in d} \|x - z_n\| = \mu_{dG}$. Using Lemma 3.1 and Lemma 3.4, there exists $x^*_n \in E^*$ such that $\|x^*_n\| = 1$ and

$$
\lim_{n} \langle x^*_n, A - z_n \rangle = F_G(A).
$$

By the reflexivity of $X$, there exists a subsequence $z_{n_k}$ denoted by itself, which converges weakly to $z \in X$. Thus,

$$
\|A - z\| \geq \lim_{n} \langle x^*_n, A - z \rangle = \lim_{n} \langle x^*_n, A - z_n \rangle = F_G(A).
$$

Note that

$$
\|A - z\| \leq \lim_{n} \|A - z_n\| \leq F_G(A).
$$

Then

$$
\lim_{n} \|A - z_n\| = \|A - z\|.
$$

Since $A$ is compact, we take $a_0 \in A$ and $x^* \in X^*$, $\|x^*\| \leq 1$ such that

$$
\|a_0 - z\| = \sup_{a \in A} \|a - z\| = F_G(A)
$$

and

$$
\langle x^*, a_0 - z \rangle = \|a_0 - z\| = F_G(A).
$$

From the fact that $\{x_n\}$ converges weakly to $z$, we have

$$
\|a_0 - z\| = \langle x^*, a_0 - z \rangle = \lim_{n} \langle x^*, a_0 - z_n \rangle
$$

$$
\leq \liminf_{n} \|a_0 - z_n\| \leq \limsup_{n} \|a_0 - z_n\|
$$

$$
\leq \sup_{a \in A, \, x \in G} \|a - x\| = F_G(A),
$$

so that

$$
\lim_{n} \|a_0 - z_n\| = \|a_0 - z\|.
$$

Then the fact that $X$ is Kadec shows $\lim_{n \to \infty} \|z_n - z\| = 0$ and $z \in G$, proving the lemma.
Let
\[ V_n = \left\{ A \in \mathcal{K}(X) : \text{diam } R_A(G) < \frac{1}{n} \right\} \]
and let
\[ V(G) = \bigcap_n V_n(G), \]
where \( R_A(G) = \{ x \in A : \sup_{z \in G} \|z - x\| = \mu_{A,G} \}. \)

**Lemma 3.6.** Suppose that \( X \) is reflexive Kadec Banach space. Then \( V(G) \) is a dense \( G_\delta \)-subset of \( \mathcal{K}(X) \).

**Proof.** Exactly as in the proof of Lemma 2.5 we can obtain that \( V(G) \) is a \( G_\delta \)-subset of \( \mathcal{K}(X) \). To prove the density, for any \( A \in J(G) \), by Lemma 3.5, we may take \((x_0, z_0)\) to be a solution of \( \max(A, G) \) with \( x_0 \in A \), \( z_0 \in G \), and let \( x_\alpha = \alpha x_0 + (1 - \alpha) z_0 \) for \( \alpha > 1 \). We define \( A_\alpha = \text{co}(A \cup \{ x_\alpha \}) \). Thus, using Lemma 3.3, the proof will be completed if we can prove that \( A_\alpha \in V(G) \) for all \( \alpha > 1 \).

Now for any \( x \in A_\alpha \) if \( x \neq x_\alpha \) then \( x = tx_\alpha + (1 - t) a \) for some \( a \in A \) and \( 0 < t < 1 \). Thus we have
\[
\sup_{z \in G} \|z - x\| \leq t \sup_{z \in G} \|z - x_\alpha\| + (1 - t) \sup_{z \in G} \|z - a\| \\
\leq t \left( \sup_{z \in G} \|z - x_0\| + \|x_0 - x_\alpha\| \right) + (1 - t) \|z_0 - x_0\| \\
= t \left( \|z_0 - x_0\| + (\alpha - 1) \|z_0 - x_0\| \right) + (1 - t) \|z_0 - x_0\| \\
= (\alpha + 1 - t) \|z_0 - x_0\| < \alpha \|z_0 - x_0\| = \|z_0 - x_\alpha\| \leq \mu_{A,G}.
\]
This implies that \( R_A(G) = x_\alpha \) and \( A_\alpha \in V(G) \) for all \( \alpha > 1 \).

The main theorem of this section is stated as follows:

**Theorem 3.1.** Suppose that \( X \) is a strongly convex Banach space and \( G \in \mathcal{B}(X) \). Then the set of all \( A \in \mathcal{K}(X) \) such that the maximization problem \( \max(A, G) \) is well posed contains a dense \( G_\delta \)-subset of \( \mathcal{K}(X) \).

**Proof.** Note that for any \( A \in J(G) \cap \mathcal{K}(X) \), \( R_G(A) = \{ z_0 \} \) is a singleton. In fact, suppose that \( R_G(A) \) contains at least two distinct elements \( x_0, x_1 \in G \). Then by Lemma 3.1 there exists \( x^* \in E^* \) satisfying
\[
\langle x^*, A - x_0 \rangle = \langle x^*, A - x_1 \rangle = F_G(A).
\]
Hence
\[ \|A - x_0 + A - x_1\|_E = 2F_G(A). \]

Take \( a_0 \in A \) such that
\[ \|a_0 - \frac{1}{2}(x_0 + x_1)\| = h(A, \frac{1}{2}(x_0 + x_1)) = \|A - \frac{1}{2}(x_0 + x_1)\|_E. \]

Then
\[ \|a_0 - x_0 + a_0 - x_1\| = \|A - x_0 + A - x_1\|_E \]
and
\[ 2F_G(A) = \|a_0 - x_0 + a_0 - x_1\| \leq \|a_0 - x_0\| + \|a_0 - x_1\| = 2F_G(A). \]

This implies that
\[ \|a_0 - x_0 + a_0 - x_1\| = \|a_0 - x_0\| + \|a_0 - x_1\|. \]

It follows from the strict convexity of \( X \) that \( x_0 = x_1 \), which is a contradiction. So \( R_G(A) \) is a singleton.

Note that for any \( A \in J(G) \cap I(G) \), the maximization problem \( \max(A, G) \) has a unique solution. Now let \((x_n, z_n)\) with \( x_n \in A \), \( z_n \in G \) be any maximizing sequence. Then, using Lemma 3.5 and the compactness of \( A \), we have that \((x_n, z_n)\) converges strongly to the unique solution and complete the proof by Lemma 3.3 and 3.6.

**Remark 3.1.** Theorem 3.1 is a multivalued version of results due to Asplund [1], Panda & Kapoor [22], Zhivkov [28] and Fitzpatrick [13].

**Remark 3.2.** Note that if \( \max(A, G) \) has a unique solution \((x_0, z_0)\) then \( x_0 \) has a unique furthest point in \( G \), which implies that there is a dense set of \( X \) possessing unique furthest points in \( G \) provided that the result of Theorem 3.1 holds. This enables us to construct some counterexamples to which Theorem 3.1 fails if \( X \) is not strongly convex. In fact, in this case, either \( X \) is not both reflexive and strictly convex, or \( X \) is not Kadec. In the first case Example 5.3 in [13] and Remark 4.4 in [9] apply. In the second case, let \( X \) be the renormed space \( l_2 \oplus R \) in [12] by taking
\[ \|(x, r)\| = \max\{\|x\|, |r|\} + \left[ r^2 + \sum_{n} 2^{-2n} x_n^2 \right]^{1/2} \]
for \((x, r) \in X\). Let
\[ G = \{ (e_n, 2 - n^{-1}) : n = 2, 3, \ldots \} \]
and

\[ U = \{(u, r) : \|u\| < \frac{1}{2}, |r| < \frac{1}{4}\}. \]

Then, for \((u, r) \in U\),

\[ F_G(u, r) = 2 - r + \left( (2 - r)^2 + \sum_{n} 2^{-n}u_n^2 \right)^{1/2}. \]

However, for each \((e_n, 2 - n^{-1}) \in G\)

\[ \|(u, r) - (e_n, 2 - n^{-1})\| > F_G(u, r), \]

which shows no points in the set \(U\) has a furthest point in \(G\). Hence Theorem 3.1 fails. Obviously, \(X\) is both reflexive and strictly convex.

REFERENCES


