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Cyclic Cohomology of Crossed Products with $\mathbb{Z}$

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INTRODUCTION

Given a C^*-algebra A, we shall call a dense subalgebra $\mathscr{A} \subset A$ smooth if it is a Fréchet algebra in some nuclear topology stronger than the norm topology induced from A.

The standing assumptions throughout this paper are as follows. A is unital, $\mathscr{A}$ is a smooth subalgebra of A containing the unit, and α is an automorphism of A mapping $\mathscr{A}$ onto $\mathscr{A}$ and such that both $\alpha|\mathscr{A}$ and $\alpha^{-1}|\mathscr{A}$ are continuous with respect to each of the seminorms defining the topology of $\mathscr{A}$.

In this situation Section 1 gives a construction of what will be called the smooth crossed product of $\mathscr{A}$ by α, denoted $\mathscr{A} \hat{\times}_\alpha \mathbb{Z}$, which is a smooth subalgebra of the C^*-crossed product $A \times_\alpha \mathbb{Z}$ and which contains the algebraic crossed product of $\mathscr{A}$ by α.

The rest of this paper is devoted to the study of the cyclic cohomology of the smooth crossed product and the main results, as given in Sections 11 and 12, can be stated as follows.

THEOREM A. *There is a linear map*

$$\# : H^n_\lambda(\mathscr{A}) \rightarrow H^{n+1}_\lambda(\mathscr{A} \hat{\times}_\alpha \mathbb{Z}),$$

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compatible with the boundary map in the Pimsner–Voiculescu six-term eaxt sequence of the K-theory of the C-crossed product and such that the following diagram is exact.*

$$\begin{tikzcd}
HC^{\mathrm{ev}}(\mathscr{A}) \arrow[r, "\#"] & HC^{\mathrm{odd}}(\mathscr{A} \hat{\times}_\alpha \mathbb{Z}) \arrow[r] & HC^{\mathrm{odd}}(\mathscr{A}) \arrow[d, "1-\alpha"] \\
HC^{\mathrm{ev}}(\mathscr{A}) \arrow[u, "1-\alpha"] & HC^{\mathrm{ev}}(\mathscr{A} \hat{\times}_\alpha \mathbb{Z}) \arrow[l] & HC^{\mathrm{odd}}(\mathscr{A}). \arrow[l, "\#"']
\end{tikzcd}$$

The proof decomposes naturally into two distinct parts. Sections 2 to 5 are devoted to the construction of a convenient representation for the Hochschild cohomology of $\mathscr{A} \hat{\times}_\alpha \mathbb{Z}$ and an analysis of the $\mathbb{E}_1$-term of the spectral sequence associated to the exact couple

$$\begin{tikzcd}
H_\lambda(\mathscr{A} \hat{\times}_\alpha \mathbb{Z}) \arrow[rr] & & H_\lambda(\mathscr{A} \hat{\times}_\alpha \mathbb{Z}) \arrow[dl] \\
& H(\mathscr{A} \hat{\times}_\alpha \mathbb{Z}, \mathscr{A} \hat{\times}_\alpha \mathbb{Z}^*) \arrow[ul] &
\end{tikzcd}$$

(see [2]).

The results obtained here are sufficient for a relatively complete description, given in Section 6, of the periodic cyclic cohomology of the smooth crossed product in the case where $\mathscr{A}$ is the algebra of C^∞-functions on some compact C^∞-manifold.

On the other hand, we obtain in Section 5 the following result (Theorem 5.1).

PROPOSITION B. *The periodic cyclic cohomology of $\mathscr{A} \hat{\times}_\alpha \mathbb{Z}$ can be computed using only the homogeneous cochains on $\mathscr{A} \hat{\times}_\alpha \mathbb{Z}$, i.e., such that given $a_0, ..., a_n \in \mathscr{A}$,*

$$\phi(u^{m_0}a_0, ..., u^{m_n}a_n) = \phi(u^{m_0}a_0, ..., u^{m_n}a_n)\, \delta_{m_0 + \cdots + m_n, 0}.$$

All the cohomology groups computed with the help of the homogeneous cochains will be distinguished by the subscript "hom."

Sections 8 to 10 are devoted to a construction of the map # and a direct proof of the following result (Theorem 10.1).

THEOREM C. *The following long cohomology sequence is exact:*

$$\cdots \to H_\lambda^{n-1}(\mathscr{A}) \xrightarrow{1-\alpha} H_\lambda^{n-1}(\mathscr{A}) \xrightarrow{\#} H_\lambda^n(\mathscr{A} \hat{\times}_\alpha \mathbb{Z})_{\mathrm{hom}} \to H_\lambda^n(\mathscr{A}) \xrightarrow{1-\alpha} \cdots.$$

This result is then applied to the construction of the six-term exact sequence for the periodic cyclic cohomology of the smooth crossed product in Section 11.

Finally, Section 12 is devoted to a construction of the pairing between cyclic cohomology of the smooth crossed product $\mathscr{A}\hat{\times}_\alpha\mathbb{Z}$ and the K-groups of the C^*-crossed product $A\hat{\times}_\alpha\mathbb{Z}$ (under a condition which essentially says that $\mathscr{A}$ is sufficiently large to detect all the K-classes of A, see Theorem 12.5) and to the proof of the compatibility of the six-term exact sequence in cyclic cohomology of $\mathscr{A}\hat{\times}_\alpha\mathbb{Z}$ with the six-term exact sequence in the K-theory of $A\hat{\times}_\alpha\mathbb{Z}$.

1. SMOOTH CROSSED PRODUCT

Our starting point is the following data:

$\mathscr{A}$: a unital topological algebra with its topology given by an increasing sequence of seminorms $\|\cdot\|_k$, $k=1, 2, ...$.

α: an automorphism of $\mathscr{A}$.

The following will hold through all that follows.

Assumption. (1) $\mathscr{A}$ is a complete, nuclear vector spaces;

(2) both α and α^{-1} are continuous in each of the seminorms $\|\cdot\|_k$.

We shall use the sequence of functions

$$\rho_k : \mathbb{Z} \to \mathbb{R}^+, \qquad k=1, 2, ...,$$
$$n \mapsto \sup_{i\leqslant k}\left(\sum_{l=-n}^{n}\|\alpha^l\|_i\right)^k.$$

The following is easily established.

LEMMA 1.1. (1) $|n|\,\rho_k(n)\leqslant\rho_{k+1}(n)$;

(2) $\left(\sum_{l=-n}^{n}\|\alpha^l\|_k\right)\rho_k(n)\leqslant\rho_{k+1}(n)$;

(3) $\rho_k(m)\leqslant\rho_k(n)\,\rho_k(m-n)$;

(4) $\rho_k(n)^m\leqslant\rho_{km}(n)$.

Let now $\mathscr{A}_\alpha[u^{-1}, u]$ denote the algebraic crossed product of $\mathscr{A}$ with $\mathbb{Z}$. This means that the elements of $\mathscr{A}_\alpha[u, u^{-1}]$ are given by finite sums

$$\sum_n a_n u^n, \qquad n\in\mathbb{Z}, a_n\in\mathscr{A},$$

and the product structure is determined by the relation

$$uau^{-1}=\alpha(a), \qquad a\in\mathscr{A}.$$

Using the above defined ρ_k's we define now an increasing sequence of seminorms on $\mathscr{A}_\alpha[u, u^{-1}]$ by

$$\left\|\sum_n a_n u^n\right\|_k=\sup_n \rho_k(n)\,\|a_n\|_k.$$

It is easy to check, using Lemma 1, that the algebraic operations in $\mathscr{A}_\alpha[u, u^{-1}]$ are continuous in the topology defined by the sequence of seminorms $\|\cdot\|_k$, $k=1, ...,$ and hence we can make the following definition.

DEFINITION 1.2. The smooth crossed product

$$\bar{\mathscr{A}}=A\,\hat{\times}_\alpha\,\mathbb{Z}$$

is the topological algebra obtained by the completion of $\mathscr{A}_\alpha[u, u^{-1}]$ in the topology defined by the sequence of seminorms $\|\cdot\|_k$.

Note that the coefficient maps

$$c_m\colon \mathscr{A}_\alpha[u, u^{-1}]\to\mathscr{A},$$
$$\sum a_n u^n\to a_m$$

are continuous and hence extend to the completion $\bar{\mathscr{A}}$.

In fact each element x of the smooth crossed product has a unique representation

$$x=\sum_n a_n u^n, \qquad a_n\in\mathscr{A}.$$

Using the pairing between two-sided sequences $\{\phi_n\}_{n\in\mathbb{Z}}$, $\phi_n\in\mathscr{A}^*$, and $x=\sum a_n u^n\in\bar{\mathscr{A}}$ given by

$$\left\langle\sum a_n u^n, \{\phi_n\}\right\rangle=\sum_n \phi_n(a_n)$$

we get

PROPOSITION 1.3. *The topological dual $\bar{\mathscr{A}}^*$ can be identified with the space of two-sided sequences $\{\phi_n\}$, $\phi_n\in\mathscr{A}^*$, satisfying the condition*

$$\exists c, k \textit{ with } \sup_n \frac{\|\phi_n\|_k}{\rho_k(n)}\leqslant c. \qquad (*)$$

Proof. Suppose that $\{\phi_n\}_{n\in\mathbb{Z}}$ defines a continuous linear functional on $\bar{\mathscr{A}}$; then for some c, k we have

$$\left|\sum_n \phi_n(a_n)\right| \leqslant c \left\|\sum_n a_n u^n\right\|_k = c \sup_n \rho_k(n)\, \|a_n\|_k$$

for all $\sum a_n u^n$ in $\bar{\mathscr{A}}$. Choosing monomials au^n we get $(*)$.

Conversely, suppose that $(*)$ holds. Then, by Lemma 1,

$$\begin{aligned}\left|\sum_n \phi_n(a_n)\right| &\leqslant \sum_n \|\phi_n\|_k\, \|a_n\|_k \leqslant c \sum_n \rho_k(n)\, \|a_n\|_k \\ &\leqslant c \sum_n \frac{1}{n^2}\, \rho_{k+2}(n)\, \|a_n\|_{k+2} \\ &\leqslant c \sum_n \frac{1}{n^2} \sup_n \rho_{k+2}(m)\, \|a_m\|_{k+2} \\ &= \left(c \sum_n \frac{1}{n^2}\right) \cdot \left\|\sum_n a_n u^n\right\|_{k+2}.\end{aligned}$$

Another result which will be useful later is the following

PROPOSITION 1.4. *$\bar{\mathscr{A}}$ is nuclear (as a topological vector space).*

Proof. This consists mainly of unravelling the definitions. Given any vector space E and a seminorm $\|\cdot\|_k$ on E, we denote by $\hat{E}^k$ the Banach space given by completing $E/\mathrm{Ker}\, \|\cdot\|_k$ in the induced norm. Then the fact that $\mathscr{A}$ is nuclear can be formulated as follows:

For each i we can find $\lambda_k \in \mathbb{C}$, $\phi_k \in \mathscr{A}^*$, and $b_k \in \hat{\mathscr{A}}^i$, $k \in \mathbb{N}$, with

$$\begin{aligned}&\sum_k |\lambda_k| < \infty, \\ &\sup_k \|b_k\|_i < \infty, \\ &\sup_k \|\phi_k\|_j < \infty \qquad \text{for some } j,\end{aligned}$$

such that the induced map $\mathscr{A} \to \hat{\mathscr{A}}^i$ has the representation [5]

$$a \mapsto \sum_k \lambda_k \phi_k(a)\, b_k.$$

We have to show that the same statement holds for $\bar{\mathscr{A}}$. Fix an index i.

For each $x \in \mathscr{A}$, we have

$$x = \sum_n a_n u^n \mapsto \sum_{n,k} \lambda_k n^{-2} \psi_{n,k}(x)\, y_{n,k} \in \hat{\mathscr{A}}^i,$$

where $y_{n,k} = (1/\rho_i(n))\, b_k u^n$ and

$$\psi_{n,k}(x) = n^2 \rho_i(n)\, \phi_k(a_n).$$

Now we have

$$\sup_{n,k} \| y_{n,k} \|_i = \sup_k \|b_k\| < \infty,$$

$$\sum_{n,k} |\lambda_k n^{-2}| = \sum_k |\lambda_k| \sum_n n^{-2} < \infty,$$

and, with $j' = \max(i, j)$, by Lemma 1,

$$\sup_{n,k} \|\psi_{n,k}\|_{j'+2} = \sup_{n,k} \frac{n^2 \rho_i(n)\, \|\phi_k\|_{j'+2}}{\rho_{j'+2}(n)} < \infty.$$

This shows that the quotient map $\mathscr{A} \to \hat{\mathscr{A}}^i$ has a representation of the desired type.

EXAMPLE. Let $\mathscr{A} = C^\infty(\mathbb{T})$ and suppose that α is the automorphism of $\mathscr{A}$ induced by a rotation. $\mathscr{A}$ is the algebra of functions

$$\sum a_n z^n, \qquad \{a_n\} \in \mathscr{S}(\mathbb{Z}),$$

where $\mathscr{S}(\mathbb{Z})$ denotes the space of rapidly decreasing sequences, topologized by the norms

$$\| \{a_n\} \|_k = \sup_n (1 + n^2)^{k/2} |a_n|.$$

Since $\alpha(z) = e^{i\theta} z$, where θ is the rotation angle, we have $\|\alpha^k\|_i = \|\alpha^{-k}\|_i = 1$ for all k, i, and hence

$$\rho_k(n) = (2\,|n| + 1)^k.$$

The smooth crossed product of $\mathscr{A}$ by α is the algebra of double sums

$$\sum_{k,m} a_{km} z^k u^m, \qquad \{a_{km}\} \in \mathscr{S}(\mathbb{Z}^2),$$

where $uzu^{-1} = \alpha(z)$, and is thus precisely the dense subalgebra of the rotation C^*-algebra considered by A. Connes in [2]. In general, however, one cannot choose the functions ρ_k to be of polynomial growth.

2. PROJECTIVE RESOLUTION OF $\bar{\mathscr{A}}$

Let

$$\mathscr{B} = \mathscr{A} \otimes \mathscr{A}^{\mathrm{op}},$$
$$\bar{\mathscr{B}} = \bar{\mathscr{A}} \otimes \bar{\mathscr{A}}^{\mathrm{op}},$$

where all the tensor products considered are the projective tensor products. Recall the standard projective resolutions

$$(M_n, b) \to \mathscr{A},$$
$$(\bar{M}_n, b) \to \bar{\mathscr{A}},$$

where

$$M_n = \mathscr{B} \otimes \bigotimes^{n} \mathscr{A},$$
$$\bar{M}_n = \bar{\mathscr{B}} \otimes \bigotimes^{n} \bar{\mathscr{A}},$$

with the boundary operator

$$\begin{aligned} b: x_0 \otimes x_1 \otimes \cdots \otimes x_n & \\ \mapsto \sum_{i=0}^{n-1} & (-1)^i x_0 \otimes x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes x_{i+2} \otimes \cdots \otimes x_n \\ & + (-1)^n x_0 \mathring{x}_n \otimes x_1 \otimes \cdots \otimes x_{n-1}. \end{aligned}$$

We use here the notation

$$\mathring{x} \in \mathscr{A}^{\mathrm{op}} \qquad \text{for} \quad x \in \mathscr{A},$$
$$x\mathring{y} = x \otimes \mathring{y} \in \mathscr{B}$$

(resp. $x, y \in \bar{\mathscr{A}}, \ldots$).

Now denote $\bar{\mathscr{B}} \otimes \bigotimes^n \mathscr{A}$ by $\bar{\mathscr{B}}M_n$ and note the inclusions

$$\bar{\mathscr{B}}M_n \subset \bar{M}_n,$$
$$b(\bar{\mathscr{B}}M_n) \subset \bar{\mathscr{B}}M_{n-1}.$$

These say that $(\bar{\mathscr{B}}M_n, b)$ is a subcomplex of $(\bar{M}_n, b)$ and thus we have an exact sequence of complexes

$$0 \longrightarrow (\bar{\mathscr{B}}M_n, b) \xrightarrow{i} (\bar{M}_n, b) \xrightarrow{\pi} (\bar{M}_n/\bar{\mathscr{B}}M_n, b) \to 0. \tag{1}$$

Let us now, in terms of the coefficient maps c_n, introduced in Section 1, define a subspace $Q_n \subset \bar{M}_n$ by

$$Q_n = \bigoplus_{i\neq 0} \mathrm{Ker}(\mathrm{id}\otimes \cdots \otimes \underset{\substack{\uparrow \\ i\text{th position.}}}{c_0} \otimes \cdots \otimes \mathrm{id})$$

Then we have

LEMMA 2.1. *$\mathrm{Im}\, i$ is closed in $\bar{M}_n$ and has Q_n as a closed complement. The short exact sequence* (1) *splits topologically.*

Proof. Since the coefficient maps c_m are continuous, the maps

$$\mathrm{id}\otimes \cdots \otimes c_m \otimes \cdots \otimes \mathrm{id} : \bar{M}_n \to \bar{M}_n$$

are continuous as well. Since we can write

$$\mathrm{Im}\, i = \bigcap_{\substack{m\neq 0\\ i\neq 0}} \mathrm{Ker}(\mathrm{id}\otimes \cdots \otimes \underset{\substack{\uparrow \\ i\text{th position,}}}{c_m} \otimes \cdots \otimes \mathrm{id})$$

the continuity of c_m's implies closedness of both Q_n and $\mathrm{Im}\, i$ in $\bar{M}_n$. The splitting follows from the decomposition

$$x = \sum a_n u^n = a_0 + \sum_{n\neq 0} a_n u^n$$

applied to each of the factors in the tensor product $\bar{M}_n$.

Applying the functor

$$(\ \ )^* = \mathrm{Hom}_{\mathscr{B}}(\ \ , \mathscr{A}^*)$$

to the exact sequence (1) we get the following

PROPOSITION 2.2. *There is a long exact cohomology sequence*

$$\cdots \xrightarrow{\delta} H^n(Q^*) \xrightarrow{\pi} H^n(\mathscr{A}, \mathscr{A}^*) \xrightarrow{i} H^n(\mathscr{B}M_n^*) \xrightarrow{\delta} H^{n+1}(Q^*) \longrightarrow \cdots.$$

Proof. Using Lemma 2.1, we can write the exact sequence (1) as a split exact sequence

$$0 \longrightarrow (\mathscr{B}M_n, b) \xrightarrow{i} (\bar{M}_n, b) \xrightarrow{\pi} (Q_n, b) \longrightarrow 0.$$

Applying the functor ( )* we get a short exact sequence

$$0 \longrightarrow (Q_n^*, b) \xrightarrow{\pi} (\bar{M}_n^*, b) \xrightarrow{i} (\bar{\mathscr{B}} M_n^*, b) \longrightarrow 0, \tag{2}$$

where for simplicity we use the same symbols to denote both the operators b, π, i in (1) and their transposes in (2). The long exact cohomology sequence corresponding to (2) gives the desired result.

Our aim is now to identify the elements of the long exact sequence from Proposition 2.2 and thus obtain information about the Hochschild cohomology of the crossed product.

3. PRELIMINARY COMPUTATIONS

We will start with a convenient representation for the elements of $\bar{\mathscr{B}} M_n^*$. Set

$$M_{n,k} = \mathscr{A} u^k \otimes \bigotimes^n \mathscr{A}.$$

Then $\bigoplus_k M_{n,k}$ is a dense subspace of $\bar{\mathscr{B}} M_n$. Representing the elements of $(\bigoplus_k M_{n,k})^*$ as sequences

$$\{\phi_k\}_{k \in \mathbb{Z}}, \qquad \phi_k \in M_{n,k}^*,$$

we can state the following

LEMMA 3.1. *A sequence $\{\phi_k\}_{k \in \mathbb{Z}}$, $\phi_k \in M_{n,k}^*$ extends to an element of $\bar{\mathscr{B}} M_{n,k}^*$ if and only if, given $x_1, ..., x_n \in \mathscr{A}$, we can find $i, j_1, ..., j_n \in N$ and a constant C so that*

$$\|\phi_k(\cdot, x_1, ..., x_n)\|_i \leqslant C\rho_i(k)\, \|x_1\|_{j_1} \cdot \cdots \cdot \|x_n\|_{j_n}, \qquad k = 1, 2, ... \tag{1}$$

Proof. Since, according to Proposition 1.4, $\bar{\mathscr{A}}$ is nuclear, separate and joint continuity on $\bar{\mathscr{A}} \otimes \bigotimes^n \mathscr{A}$ are equivalent, and it suffices to show that (1) is equivalent to the separate continuity of $\{\phi_k\}_{k \in \mathbb{Z}}$ on $\mathscr{A}_\alpha[u, u^{-1}] \otimes \bigotimes^n \mathscr{A}$. This follows, however, from Proposition 1.3.

Let us make, for later reference, the following

DEFINITION 3.2. A sequence $\{\phi_k\}_{k \in \mathbb{Z}}$, $\phi_k \in M_{n,k}^*$ is called tempered if it satisfies (1). The vector space of all such sequences will be denoted by $\hat{\bigoplus}_k M_{n,k}^*$.

Note next that the decomposition

$$\bar{\mathscr{B}}M_{n,k}^{*} \to \widehat{\bigoplus_{k}} M_{n,k}^{*}$$

is preserved by b. We let

$$b_k = b\,|_{M_{n,k}^{*}},$$

$$H_k^n(\mathscr{A}, \mathscr{A}^*) = H^n(M_{n,k}^{*}, b_k),$$

$$\widehat{\bigoplus_{k}} H_k^n(\mathscr{A}, \mathscr{A}^*) = \text{vector space of sequences } \{\xi_k\}_{k \in \mathbb{Z}},$$

$$\xi_k \in H_k^n(\mathscr{A}, \mathscr{A}^*),\ \{\xi_k\}_{k \in \mathbb{Z}} \text{ representable}$$

$$\text{by a tempered sequence } \{\phi_k\} \in \widehat{\bigoplus_{k}} M_{n,k}^{*},$$

$$H_{\mathrm{res}}^n = \text{the quotient of the vector space of sequences}$$

$$\{\phi_k\} \in \widehat{\bigoplus_{k}} M_{n,k}^{*},\ \phi_k \in \operatorname{Im} b_k, \text{ by } \operatorname{Im} b.$$

LEMMA 3.3. $H^n(\bar{\mathscr{B}}M_n^*, b) = \widehat{\bigoplus}_k H_k^n(\mathscr{A}, \mathscr{A}^*) \oplus H_{\mathrm{res}}^n$.

Proof. Since, according to Lemma 3.1, the map

$$\bar{\mathscr{B}}M_n^* \to \widehat{\bigoplus_{k}} M_{n,k}^{*}$$

$$\phi \mapsto \phi \Big|_{\bigoplus_{k} M_{n,k}}$$

is an isomorphism commuting with the coboundary operator b, the result follows directly from the definitions of the vector spaces appearing in the statement of the lemma.

Let us next tackle the complex (Q_n, b). First define a $\bar{\mathscr{B}}$-module map

$$\begin{aligned} &h: \bar{\mathscr{B}}M_n \to Q_{n+1} \\ &(-1)^n \otimes x_1 \otimes \cdots \otimes x_n \mapsto \sum_{i=1}^{n} (-1)^{i-1} u^{-1} \otimes \alpha(x_1) \\ &\qquad\qquad \otimes \cdots \otimes \alpha(x_{i-1}) \otimes u \otimes x_i \otimes \cdots \otimes x_n. \end{aligned} \tag{2}$$

A straightforward computation inside $\bar{M}_n$ gives

$$\begin{aligned} &bh(1 \otimes x_1 \otimes \cdots \otimes x_n) \\ &\quad = hb(1 \otimes x_1 \otimes \cdots \otimes x_n) \\ &\qquad + (-1)^n (1 \otimes x_1 \otimes \cdots \otimes x_n - \mathring{u}u^{-1} \otimes \alpha(x_1) \otimes \cdots \otimes \alpha(x_n)). \end{aligned} \tag{3}$$

In particular, we can state

LEMMA 3.4. *$h\colon (\bar{\mathscr{B}}M_n, b) \to (Q_{n+1}, b)$ is a morphism of complexes.*

Proof. Just remember that b acts on Q_{n+1} modulo $\bar{\mathscr{B}}M_{n+1}$ via the isomorphism

$$Q_{n+1} \to \bar{M}_{n+1}/\bar{\mathscr{B}}M_{n+1},$$

given by Lemma 2.1, and $bh = hb \bmod \bar{\mathscr{B}}M_n$.

Next note that both $(\bar{\mathscr{B}}M_n, b)$ and (Q_n, b) are acyclic in positive dimensions. In fact, we define for $n \geqslant 0$ an $\mathscr{A}^{\mathrm{op}}$-module map

$$\begin{gathered} \rho\colon \bar{\mathscr{B}}M_n \to \bar{\mathscr{B}}M_{n+1} \\ (u^k x_0 \otimes 1) \otimes x_1 \otimes \cdots \otimes x_n \mapsto (u^k \otimes 1) \otimes x_0 \otimes x_1 \otimes \cdots \otimes x_n. \end{gathered} \tag{4}$$

Then it is easy to check that $\rho b + b\rho = \mathrm{id}$, so that $(\bar{\mathscr{B}}M_n, b)$ is acyclic. As far as (Q_n, b) is concerned, take an $x \in \bar{M}_n$ with $bx \in \bar{\mathscr{B}}M_{n-1}$. Then $bx = (\rho b + b\rho)bx$, i.e.,

$$b(x - \rho bx) = 0.$$

Since $(\bar{M}_n, b)$ is acyclic, we can find $x' \in M_{n+1}$ with $x - \rho bx = bx'$. But then

$$x = bx' + \rho bx \equiv bx' \qquad \bmod \bar{\mathscr{B}}M_n,$$

and hence (Q_n, b) is acyclic as well.

Now define a $\bar{\mathscr{B}}$-module map

$$\begin{gathered} k\colon Q_m \to \bar{\mathscr{B}}M_{m-1}, \qquad m \geqslant 1 \\ (-1)^m \otimes u^{n_1} \otimes \cdots \otimes u^{n_m} x_m \mapsto \sum_{i=1}^{n_1}{}' u^{n_1 + \cdots + n_m} u^i \alpha^{n_1 - i}(x_1) \\ \otimes \cdots \otimes \alpha^{n_1 + \cdots + n_m - 1}(x_m). \end{gathered} \tag{5}$$

We use here the notation

$$\sum_{i=1}^{n_1}{}' = \begin{cases} \displaystyle\sum_{i=1}^{n_1} & \text{for} \quad n_1 > 0 \\ 0 & \text{for} \quad n_1 = 0 \\ \displaystyle -\sum_{i=n_1+1}^{0} & \text{for} \quad n_1 < 0. \end{cases}$$

The following holds:

$$kb = bk. \tag{6}$$

This can be established either by a direct computation or by noticing the identity

$$\begin{aligned} &k(1 \otimes u^{n_1}x_1 \otimes \cdots \otimes u^{n_m}x_m) \\ &\quad = \rho kb(1 \otimes u^{n_1}x_1 \otimes \cdots \otimes u^{n_m}x_m) \\ &\qquad - b\left(\sum_{i=1}^{n_1}{}' u^i \mathring{u}^{n_1+\cdots+n_m-i} \otimes \alpha^{n_1-i}(x_1) \otimes \cdots \otimes \alpha^{n_1+\cdots+n_m-i}(x_m)\right), \end{aligned}$$

and using induction together with the contracting homotopy property of ρ.

The last piece of information that we need is the fact

$$\operatorname{Im} b \,|\, \bar{\mathscr{B}}M_1 = \operatorname{Im} kb \,|\, Q_2,$$

but since $bkb = kb^2 = 0$ and $(\bar{\mathscr{B}}M_n, b)$ is acyclic, it is enough to show the inclusion

$$b(\bar{\mathscr{B}}M_1) \subseteq kb(Q_2).$$

But the left-hand side is generated by elements of the form $x - \mathring{x}$, $x \in \mathscr{A}$, and directly from the definition

$$x - \mathring{x} = kb(u^{-1} \otimes u \otimes x).$$

Collecting all the above results we get

Lemma 3.5. $H^n(Q^*, b) = H^{n-1}(\bar{\mathscr{B}}M^*, b)$.

Proof. We have the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r, "b"] & Q_3 \arrow[r] & Q_2 \arrow[r, "kb"] & \bar{\mathscr{B}} \arrow[d, equal] \\ \cdots \arrow[r, "b"] & \bar{\mathscr{B}}M_2 \arrow[r] \arrow[u, "h"'] & \bar{\mathscr{B}}M_1 \arrow[r, "b"] \arrow[u, "h"'] & \bar{\mathscr{B}} \end{tikzcd}$$

with both rows free acyclic (see, e.g., [1]).

Note that k and h are a priori defined only on topological bases of the respective vector spaces. It is, however, easy to see, using Lemma 1.1, that they extend to continuous maps defined on all of the respective topological tensor products.

4. HOCHSCHILD COHOMOLOGY OF THE CROSSED PRODUCT

THEOREM 4.1. *The Hochschild cohomology of the crossed product $\bar{\mathscr{A}}$ fits into the long exact sequence*

$$\cdots \xrightarrow{\delta} \hat{\bigoplus_k} H_k^n(\mathscr{A}, \mathscr{A}^*) \oplus H_{\text{res}}^n \xrightarrow{\pi} H^{n+1}(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*)$$

$$\xrightarrow{i} \hat{\bigoplus_k} H_k^{n+1}(\mathscr{A}, \mathscr{A}^*) \oplus H_{\text{res}}^{n+1}$$

$$\xrightarrow{\delta} \hat{\bigoplus_k} H_k^{n+1}(\mathscr{A}, \mathscr{A}^*) \oplus H_{\text{res}}^{n+1} \xrightarrow{\pi} \cdots,$$

where δ is defined by $\delta\phi = \phi - \phi \circ \alpha$.

Proof. Recall the long exact cohomology sequence of Proposition 2.2:

$$\cdots \xrightarrow{\delta} H^n(Q^*) \xrightarrow{\pi} H^n(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*) \xrightarrow{i} H^n(\bar{\mathscr{B}}M^*) \xrightarrow{\delta} H^{n+1}(Q^*) \xrightarrow{\pi} \cdots.$$

Using Lemma 3.3 and Lemma 3.5 we can identify the spaces

$$H^n(\bar{\mathscr{B}}M^*) \cong H^{n+1}(Q^*) \cong \hat{\bigoplus_k} H_k^n(\mathscr{A}, \mathscr{A}^*) \oplus H_{\text{res}}^n,$$

and hence obtain the exact sequence of the theorem. To compute δ recall the definition of the connecting homomorphism. It is given by the following procedure:

(1) Start with $\{\phi_k\}_{k \in \mathbb{Z}}$ representing a class, $\xi \in H^n(\bar{\mathscr{B}}M^*)$.

(2) Pull this back to an element of $\bar{M}_n^*$, say

$$\tilde{\phi}(x_0 u^{m_0}, ..., x_n u^{m_n}) = \phi_{m_0}(x_0 u^{m_0}, x_1, ..., x_n)\, \delta_{m_1,0} \cdot \cdots \cdot \delta_{m_n,0}.$$

(3) Now $b\tilde{\phi} = \pi\psi$ for some $\psi \in Q_{n+1}^*$, and

$$\delta\xi = [\psi] \in H^{n+1}(Q^*) \cong H^n(\bar{\mathscr{B}}M^*).$$

But since the isomorphism $H^{n+1}(Q^*) \cong H^n(\bar{\mathscr{B}}M^*)$ is obtained by composing cocycles on Q_{n+1} with h and $(\pi\psi) \circ h = \psi$, we have, by formula (3) of Section 3,

$$\psi = (b\tilde{\phi}) \circ h = b(\tilde{\phi} \circ h) + (-1)^n(\phi - \phi \circ \alpha).$$

With the above choice of lifting $\tilde{\phi}$, $\tilde{\phi} \circ h = 0$ and we have the desired description of δ:

$$(-1)^n\, \delta[\phi] = [\phi - \phi \circ \alpha].$$

5. THE $\mathbb{E}_1$-TERM OF THE SPECTRAL SEQUENCE

Our goal in this section is the computation of the $\mathbb{E}_1$-term of the spectral sequence associated with the exact couple of Theorem 37 in [2]:

$$\begin{tikzcd}
H_\lambda(\bar{\mathscr{A}}) \arrow[rr, "S"] & & H_\lambda(\bar{\mathscr{A}}) \arrow[dl, "I"] \\
& H(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*). \arrow[ul, "B"] &
\end{tikzcd}$$

The $\mathbb{E}_1$-term is given by the homology of the complex

$$(H^n(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*), d_0), \qquad d_0 = IB.$$

Here the boundary map d_0 is given by

$$\begin{aligned} &d_0\phi(x_0, x_1, ..., x_n) \\ &\quad = \text{cyclic antisymmetrization of} \\ &\qquad \phi(1, x_0, ..., x_n) + (-1)^n \phi(x_0, x_1, ..., x_n, 1). \end{aligned} \tag{1}$$

However, the following simple result will allow us to dispense with the second term, as long as the computations are done inside $H^n(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*)$.

LEMMA 5.1. *Given any n-cochain ϕ, let*

$$\phi|_k(x_0, x_1, ..., x_{n-1}) = \phi(x_0, x_1, ..., x_{k-1}, 1, x_k, ..., x_{n-1}),$$
$$\phi\|_k(x_0, x_1, ..., x_{n-2}) = \phi(x_0, x_1, ..., x_{k-1}, 1, 1, x_k, ..., x_{n-2}),$$

and denote by N the cyclic antisymmetrization operator. Then the following identities hold:

$$\sum_{k>0} (b\phi)\|_k = b\left(\sum_{k>0} \phi\|_k\right) + \sum_{k>0} (-1)^{k-1}\phi|_k, \tag{1}$$

$$N((b\phi)\|_n) = bN(\phi\|_{n-1}) + (-1)^{n-1}N(\phi|_n). \tag{2}$$

We omit the proof, which consists of a straightforward computation.

Let us now turn to the detailed study of the action of d_0 in terms of the exact sequence of Section 4.

$$\begin{tikzcd}
\cdots \arrow[r, "\delta"] & \hat{\bigoplus}_k H_k^{n-1}(\mathscr{A}, \mathscr{A}^*) \oplus H_{\mathrm{res}}^{n-1} \arrow[r, "\pi"] & H^n(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*) \arrow[r, "i"] \arrow[d, "d_0"] & \hat{\bigoplus}_k H_k^n(\mathscr{A}, \mathscr{A}^*) \oplus H_{\mathrm{res}}^n \arrow[r, "\delta"] & \cdots \\
\cdots \arrow[r, "\delta"] & \hat{\bigoplus}_k H_k^{n-2}(\mathscr{A}, \mathscr{A}^*) \oplus H_{\mathrm{res}}^{n-2} \arrow[r, "\pi"] & H^{n-1}(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*) \arrow[r, "i"] & \hat{\bigoplus}_k H_k^{n-1}(\mathscr{A}, \mathscr{A}^*) \oplus H_{\mathrm{res}}^{n-1} \arrow[r, "\delta"] & \cdots.
\end{tikzcd}$$

As a reminder, note that i is just the restriction map

$$H^n(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*) = H^n(\bar{M}^*, b) \xrightarrow{i} H^n(\bar{\mathscr{B}}M^*, b),$$

while π is given by the composition of cochains with the map $k\colon Q_n \to \bar{\mathscr{B}}M_{n-1}$, defined by (5), Section 3. We decompose the side terms of the exact sequence as

$$\hat{\bigoplus_{k}} H_k^n(\mathscr{A}, \mathscr{A}^*) \oplus H_{\text{res}}^n = \underbrace{H_0^n(\mathscr{A}, \mathscr{A}^*)}_{\substack{\text{homogeneous}\\ \text{part}}} \oplus \underbrace{\hat{\bigoplus_{k \neq 0}} H_k^n(\mathscr{A}, \mathscr{A}^*) \oplus H_{\text{res}}^n}_{\substack{\text{non-homogeneous}\\ \text{part}}}. \tag{3}$$

Note, finally, the identity

$$H_0^n(\mathscr{A}, \mathscr{A}^*) = H^n(\mathscr{A}, \mathscr{A}^*).$$

Since the computations are rather lengthy, we shall break them into three parts.

(A) d_0, *the Non-homogeneous Part*

Let us look at the map

$$id_0\pi\colon \operatorname{Coker} \delta \to \operatorname{Ker} \delta.$$

Using the notation

$$\phi = \{\phi_k\} \in \hat{\bigoplus_{k}} M_{n,k}^*,$$

we have

LEMMA 5.2. $id_0\pi[\{\phi_k\}_{k \in \mathbb{Z}}] = [\{\sum\nolimits'^{k}_{i=1} \phi_k \circ \alpha^{k-1}\}_{k \in \mathbb{Z}}]$.

Proof. According to the definitions of π and d_0 as well as Lemma 5.1, we have

$$\begin{aligned}&(id_0\pi\phi)_k(x_0u^k, x_1, ..., x_n)\\ &\qquad = \text{cyclic antisymmetrization of } \phi(k(1 \otimes x_0u^k \otimes x_1 \otimes \cdots \otimes x_n)).\end{aligned}$$

But

$$\begin{aligned}&k(1 \otimes x_i \otimes \cdots \otimes x_n \otimes x_0u^k \otimes x_1 \otimes \cdots \otimes x_{i-1}) = 0 \qquad \text{for} \quad i \neq 0\\ &\qquad k(1 \otimes x_0u^k \otimes x_1 \otimes \cdots \otimes x_n)\\ &\qquad\qquad = \sum_{i=1}^{k}{}' \mathring{u}^{k-i}x_0u^i \otimes \alpha^{k-i}(x_1) \otimes \cdots \otimes \alpha^{k-i}(x_n),\end{aligned}$$

and hence

$$\begin{aligned}(id_0\pi\phi)_k(x_0u^k, x_1, ..., x_n) &= \sum_{i=1}^{k}{}' \phi(u^{k-i}x_0u^i, \alpha^{k-i}(x_1), ..., \alpha k - i(x_n)) \\ &= \sum_{i=1}^{k}{}' \phi(\alpha^{k-i}(x_0u^k), \alpha^{k-i}(x_1), ..., \alpha^{k-i}(x_n)) \\ &= \sum_{i=1}^{k}{}' \phi_k \circ \alpha^{k-i}(x_0u^k, x_1, ..., x_n).\end{aligned}$$

From this lemma we can conclude the following results.

Corollary 5.3. (I) $\delta[\{\phi_k\}_{k\in\mathbb{Z}}] = 0 \Rightarrow id_0\pi[\{\phi_k\}_{k\in\mathbb{Z}}] = [\{k\phi_k\}_{k\in\mathbb{Z}}]$,

(II) $d_0\pi[\{\phi_k\}_{k\in\mathbb{Z}}] = 0 = \phi_0 \Rightarrow [\{\phi_k\}_{k\in\mathbb{Z}}] \in \operatorname{Im}\delta$,

(III) $id_0\pi$ *is onto* $\operatorname{Ker}\delta \ominus H_0^n(\mathscr{A}, \mathscr{A}^*)$,

(IV) $(id_0\pi\phi)_0 = 0$ *for any cocycle* ϕ.

Proof. (I) $\delta[\{\phi_k\}] = 0$ means that

$$\phi_k - \phi_k \circ \alpha = b\omega_k, \qquad k \in \mathbb{Z},$$

for some tempered sequence $\{\omega_k\}_{k\in\mathbb{Z}}$; see Section 3, Definition 3.2. Then

$$\begin{aligned}\sum_{i=1}^{k}{}' \phi_k \circ \alpha^{k-i} &= k\phi_k - \sum_{i=1}^{k}{}' \sum_{t=1}^{k-i} (\phi_k - \phi_k \circ \alpha) \circ \alpha^{k-i-t} \\ &= k\phi_k - b\left(\sum_{i=1}^{k}{}' \sum_{t=1}^{k-i} \omega_k \circ \alpha^{k-i-t}\right), \qquad (*)\end{aligned}$$

and the term $\{\sum'^{k}_{i=1} \sum_{t=1}^{k-i} \omega_k \circ \alpha^{k-i-t}\}_{k\in\mathbb{Z}}$ defines a tempered sequence (Lemma 1.1(2)). Using Lemma 5.2 we get

$$[\{k\phi_k\}] = \left[\left\{\sum_{i=1}^{k}{}' \phi_k \circ \alpha^{k-i}\right\}\right] = id_0\pi[\{\phi_k\}].$$

(II) The first line of (∗) can be written as

$$id_0\pi[\{\phi_k\}] = [\{k\phi_k\}] + \delta[\{\psi_k\}],$$

where

$$\psi_k = -\sum_{i=1}^{k}{}' \sum_{i=1}^{k-i} \phi_k \circ \alpha^{k-i-t}.$$

If $\phi_0 = 0$, then also $\psi_0 = 0$ and according to Lemma 1.1(2), $\{\psi_k\}$ is a tempered sequence. Thus we can write

$$k\phi_k + \delta\psi_k = b\omega_k, \qquad \omega_0 = 0,$$

for some tempered sequence $\{\omega_k\}_{k \in \mathbb{Z}}$. But then

$$\{\phi_k\} = b\left\{\frac{1}{k}\omega_k\right\} - \delta\left\{\frac{1}{k}\psi_k\right\},$$

and both $\{(1/k)\omega_k\}$ and $\{(1/k)\psi_k\}$ are tempered (Lemma 1.1(2)).

(III) This follows immediately from (I) and the fact that, according to Lemma 1.1(1), given a tempered cochain $\{\phi_k\}$, $\phi_0 = 0$, also $\{(1/k)\phi_k\}$ is tempered.

(IV) This follows from Lemma 5.2.

(B) d_0, *the Homogeneous Part*

Let us now look at the action of d_0 on the 0th component, i.e., on $H^n(\mathscr{A}, \mathscr{A}^*)$. Note that (1) defines d_0 on $H^n(\mathscr{A}, \mathscr{A}^*)$ as the derivative of the spectral sequence, relating Hochschild and cyclic cohomology of $\mathscr{A}$.

LEMMA 5.4. (I) *Given a cocucle ϕ on $\bar{\mathscr{A}}$, we have $d_0(i\phi)_0 = (id_0\phi)_0$.*

(II) *Given a cocycle ϕ on $\mathscr{A}$, we have $d_0\pi\{\phi \cdot \delta_{k,0}\} = \pi\{d_0\phi \cdot \delta_{k,0}\}$.*

Proof. (I) This follows immediately from the fact that i is the restriction map.

(II) Since, according to Corollary 5.3(IV), $id_0\pi\{\phi \cdot \delta_{k,0}\} = 0$, we can, using the exactness of the long cohomology sequence of Theorem 4.1, write

$$d_0\pi\{\phi \cdot \delta_{k,0}\} = \pi\{\psi \cdot \delta_{k,0}\}$$

for some cocycle ψ on $\mathscr{A}$. Since the composition with $h: \mathscr{B}M_{n-1} \to Q_n$ inverts π, we have

$$\psi = (d_0\pi\phi) \circ h.$$

This is given (using Lemma 5.1(2)) by

$$\psi(x_0, x_1, ..., x_n) = \phi(k(1 \otimes Nh(x_0 \otimes \cdots \otimes x_n))).$$

Look at a typical term

$$k(1 \otimes x_i \otimes \cdots \otimes x_n \otimes x_0 u^{-1}$$
$$\otimes \alpha(x_1) \otimes \cdots \otimes \alpha(x_k) \otimes u \otimes x_{k+1} \otimes \cdots \otimes x_{i-1}).$$

It can be non-zero only if we have either

$$k(1\otimes x_0u^{-1}\otimes\alpha(x_1)\otimes\cdots\otimes\alpha(x_k)\otimes u\otimes x_{k+1}\otimes\cdots\otimes x_n)$$
$$=-x_0\otimes x_1\otimes\cdots\otimes x_k\otimes 1\otimes x_{k+1}\otimes\cdots\otimes x_n$$

or

$$k(1\otimes u\otimes x_{k+1}\otimes\cdots\otimes x_n\otimes x_0u^{-1}\otimes\alpha(x_1)\otimes\cdots\otimes\alpha(x_k))$$
$$=1\otimes x_{k+1}\otimes\cdots\otimes x_n\otimes x_n\otimes x_0\otimes x_1\otimes\cdots\otimes x_k.$$

Combining now the signs coming from cyclic permutation and from the position of u in $h(x_0\otimes x_1\otimes\cdots\otimes x_n)$ we get

$$\begin{aligned}\psi(x_0, x_1, ..., x_n) &= \text{cyclic antisymmetrization of } \phi(1, x_0, ..., x_n)\\ &\quad+\sum_{k>0}(-1)^k\phi|_k(x_0, x_1, ..., x_n).\end{aligned}$$

Note that there is no cyclic antisymmetrization in the second line. Then Lemma 5.1 gives

$$\psi=d_0\phi+\text{coboundary},$$

and hence completes the proof.

(C) *Computation of the* $\mathbb{E}_1$*-Term*

Let us now go back to the decomposition into homogeneous and non-homogeneous parts given by (3). We introduce a similar decomposition in $H^n(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*)$, given by

$$\begin{gathered}H^n(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*)=H^n_{\text{hom}}(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*)\oplus H^n_e(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*),\\ \phi\mapsto\phi_{\text{hom}}+\phi_e,\end{gathered}\tag{4}$$

where

$$\phi_{\text{hom}}(x_0u^{m_0}, ..., x_nu^{m_n})=\phi(x_0u^{m_0}, ..., x_nu^{m_n})\cdot\delta_{m_0+\cdots+m_n,0}.\tag{5}$$

Since the operator d_0 preserves this splitting, we can write

$$\mathbb{E}_1(\bar{\mathscr{A}})=\mathbb{E}_1(\bar{\mathscr{A}})_{\text{hom}}\oplus\mathbb{E}_1(\bar{\mathscr{A}})_e.$$

THEOREM 5.5. $\mathbb{E}_1(\bar{\mathscr{A}})_e=0.$

Proof. Set

$$(\mathrm{Ker}\,\delta)_e^n = \mathrm{Ker}\,\delta\,\Big|_{k\neq 0}\ \hat{\bigoplus}\ H_k^n(\mathscr{A}, \mathscr{A}^*) \oplus H^n_{\mathrm{res}},$$

$$(\mathrm{Coker}\,\delta)_e^n = \mathrm{Coker}\,\delta\,\Big|_{k\neq 0}\ \hat{\bigoplus}\ H_k^n(\mathscr{A}, \mathscr{A}^*) \oplus H^n_{\mathrm{res}}.$$

Let us look at the diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathrm{Coker}\,\delta)_e^n \arrow[r, "\pi"] & H_e^{n+1}(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*) \arrow[r, "i"] \arrow[d, "d_0"] & (\mathrm{Ker}\,\delta)_e^{n+1} \arrow[r] & 0 \\
0 \arrow[r] & (\mathrm{Coker}\,\delta)_e^{n-1} \arrow[r, "\pi"] & H_e^{n}(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*) \arrow[r, "i"] \arrow[d, "d_0"] & (\mathrm{Ker}\,\delta)_e^{n} \arrow[r] & 0 \\
0 \arrow[r] & (\mathrm{Coker}\,\delta)_e^{n-2} \arrow[r, "\pi"] & H_e^{n-1}(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*) \arrow[r, "i"] & (\mathrm{Ker}\,\delta)_e^{n-1} \arrow[r] & 0.
\end{tikzcd}$$

The rows are exact by Theorem 4.1. Suppose we are given $\phi \in H_e^n(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*)$ such that $d_0\phi = 0$. By Corollary 5.3(III) we can find an element $\psi_1 \in (\mathrm{Coker}\,\delta)_e^n$ such that

$$i\phi = i d_0 \pi \psi_1.$$

Since the middle row is exact, we can find an element $\psi_2 \in (\mathrm{Coker})_e^{n-1}$ such that

$$\phi = d_0\pi\psi_1 + \pi\psi_2.$$

But then

$$d_0\pi\psi_2 = d_0(\phi - d_0\pi\psi_1) = 0,$$

and hence, from Corollary 5.3(II), $\psi_2 = 0$. This gives $\phi = d_0\pi\psi_1$, i.e., $\mathbb{E}_1(\bar{\mathscr{A}})_e = 0$.

For future reference, note that the decomposition of cocycles given by (4) works equally well in the cyclic case, i.e., we can write

$$H_\lambda^n(\bar{\mathscr{A}}) = H_\lambda^n(\bar{\mathscr{A}})_{\mathrm{hom}} \oplus H_\lambda^n(\bar{\mathscr{A}})_e. \tag{6}$$

Using this notation we have

COROLLARY 5.6. $H_\lambda^n(\bar{\mathscr{A}})_e \subset \mathrm{Ker}\, S.$

Proof. Since S preserves the decomposition (6), and since for a cyclic cocycle we have

$$\phi \in \operatorname{Im} d_0 \Leftrightarrow \phi \in \operatorname{Im} S + \operatorname{Ker} S,$$

we can conclude from Theorem 5.5 that

$$SH^n_\lambda(\bar{\mathscr{A}})_e \subset S^2 H^{n-2}_\lambda(\bar{\mathscr{A}})_e.$$

Iterating this relation we get

$$SH^n_\lambda(\bar{\mathscr{A}})_e \subset S^k H^{n-2k}_\lambda(\bar{\mathscr{A}})_e, \qquad \text{for each } k = 1, 2, \ldots,$$

and choosing $k > n/2$ we get the desired result.

To describe $\mathbb{E}_1(\bar{\mathscr{A}})_{\text{hom}}$ we need some extra notation. Set

$$H^n_{\text{eq}}(\mathscr{A}) = \text{homology of } (\operatorname{Ker} \delta \mid H(\mathscr{A}, \mathscr{A}^*), d_0),$$

$$H^n_{\text{coeq}}(\mathscr{A}) = \text{homology of } (\operatorname{Coker} \delta \mid H(\mathscr{A}, \mathscr{A}^*), d_0).$$

Then the following holds.

THEOREM 5.7. *The $\mathbb{E}_1$-term of the spectral sequence of the crossed product $\bar{\mathscr{A}}$ fits into a long exact sequence*

$$\cdots \xrightarrow{\Delta} H^{n-1}_{\text{coeq}}(\mathscr{A}) \xrightarrow{\pi} \mathbb{E}^n_1(\bar{\mathscr{A}}) \xrightarrow{i} H^n_{\text{eq}}(\mathscr{A}) \xrightarrow{\Delta} H^{n-2}_{\text{coeq}}(\mathscr{A}) \longrightarrow \cdots.$$

Proof. The homogeneous part of the exact sequence of Theorem 4.1 shows that the rows of the following diagram are exact.

$$\begin{tikzcd}
 & {} \arrow[d, "d_0"] & {} \arrow[d, "d_0"] & {} \arrow[d, "d_0"] & \\
0 \arrow[r] & \operatorname{Coker} \delta \mid H^{n-1}(\mathscr{A}, A \hat{\times}_\alpha \mathbb{Z}^*) \arrow[r] \arrow[d, "d_0"] & H^n(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*)_{\text{hom}} \arrow[r] \arrow[d, "d_0"] & \operatorname{Ker} \delta \mid H^n(\mathscr{A}, \mathscr{A}^*) \arrow[r] \arrow[d, "d_0"] & 0 \\
0 \arrow[r] & \operatorname{Coker} \delta \mid H^{n-1}(\mathscr{A}, \mathscr{A}^*) \arrow[r] \arrow[d, "d_0"] & H^{n-1}(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*)_{\text{hom}} \arrow[r] \arrow[d, "d_0"] & \operatorname{Ker} \delta \mid H^{n-1}(\mathscr{A}, \mathscr{A}^*) \arrow[r] \arrow[d, "d_0"] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

The diagram is commutative by Lemma 5.4, and, according to Theorem 5.5, the homology of the middle column is $\mathbb{E}_1$ $(\bar{\mathscr{A}})$. Applying the long exact homology sequence to the short exact sequence of complexes given by this diagram, we get

$$0 \longrightarrow (\operatorname{Coker} \delta, d_0) \longrightarrow (H^n(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*)_{\text{hom}}, d_0) \longrightarrow (\operatorname{Ker} \delta, d_0) \longrightarrow 0.$$

For future reference we will describe the connecting homomorphism of this theorem. The proof consists of straightfroward diagram chasing and will be omitted.

Lemma 5.8. *The connecting homomorphism of Theorem* 5.7 *is given as follows*:

For a class $[\phi] \in H^n_{\mathrm{eq}}(\mathscr{A})$, *we have*

$$\delta\phi = b\rho,$$

$$d_0\phi = b\gamma$$

for some cochains ρ, γ *on* $\mathscr{A}$. *Then if* $\tilde{\phi}$, $\tilde{\gamma}$ *are the liftings of the cochains to cochains on* $\tilde{\mathscr{A}}$ *as described in Section* 4, (1), *then*

$$\Delta[\phi] = [(d_0(\tilde{\phi} - \pi\rho) - b\tilde{\gamma}) \circ h].$$

6. Example: Diffeomorphism of a Compact C^∞-Manifold

Suppose that $\mathscr{A} = C^\infty(X)$, where X is a compact C^∞-manifold. Then any $\alpha \in \operatorname{Aut} \mathscr{A}$ is induced by a differomorphism of x. Giving $\mathscr{A}$ its C^∞-topology of uniform convergence of derivatives, it is easy to see that the assumption of Section 1 is satisfied. We will apply the preceding results to the crossed product $\tilde{\mathscr{A}} = C^\infty(x) \hat{\times}_\alpha \mathbb{Z}$.

First of all denote by $\mathscr{D}'_n(x)$ the space of the Rham n-currents on X, and note that according to Lemma 4.5 of [2] the map

$$M^*_n \ni \phi \to \text{antisymmetrization of } \phi \in \mathscr{D}'_n(x)$$

induces an isomorphism

$$H^n(\mathscr{A}, \mathscr{A}^*) \to \mathscr{D}'_n(x),$$

and the operator d_0 is just the standard de Rham boundary operator

$$d_0 \colon \mathscr{D}'_n(x) \to \mathscr{D}'_{n-1}(x).$$

The complexes defining $H_{\mathrm{eq}}(\mathscr{A})$ and $H_{\mathrm{coeq}}(\mathscr{A})$ become, respectively,

$$0 \longrightarrow \mathscr{D}'_N(x)^\alpha \xrightarrow{d_0} \cdots \xrightarrow{0} \mathscr{D}'_1(x)^\alpha \xrightarrow{d_0} \mathscr{D}'_0(x)^\alpha \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Coker} \delta \,|\, \mathscr{D}'_N(x) \xrightarrow{d_0} \cdots \xrightarrow{d_0} \operatorname{Coker} \delta \,|\, \mathscr{D}'_0(x) \longrightarrow 0,$$

where $\delta = \mathrm{id} - \alpha$ acts on n-currents, and

$$\mathscr{D}'_n(x)^\alpha = \operatorname{Ker} \delta \,|\, \mathscr{D}'_n(x).$$

Let us now look at the exact sequence of Theorem 5.7:

$$\cdots \xrightarrow{\Delta} H^{n-1}_{\mathrm{coeq}}(\mathscr{A}) \xrightarrow{\pi} \mathbb{E}^n_1(\bar{\mathscr{A}}) \xrightarrow{i} H^n_{\mathrm{eq}}(\mathscr{A}) \xrightarrow{\Delta} \cdots.$$

LEMMA 6.1. $\Delta = 0$.

Proof. Given a cochain ϕ on $\mathscr{A}$ representing a class in $H^n_{\mathrm{eq}}(\mathscr{A})$, we may suppose that ϕ is an α-invariant n-current on X. Set

$$\bar{\phi}(u^{m_0}x_0, ..., u^{m_n}x_n)$$
$$= \begin{cases} \phi(\alpha^{m_0}(x_0), \alpha^{m_0+m_1}(x_1), ..., \alpha^{m_0+\cdots+1m_n}(x_n)), & m_0+\cdots+m_n=0 \\ 0, & m_0+\cdots+m_n \neq 0. \end{cases} \tag{1}$$

Then $\bar{\phi}$ is a cyclic cocycle on $\bar{\mathscr{A}}$, and hence, from Lemma 5.8,

$$\Delta[\phi] = [(d_0\bar{\phi}) \circ h] = 0.$$

LEMMA 6.2. $\mathbb{E}^n_1(\bar{\mathscr{A}}) \cong H^{n-1}_{\mathrm{coeq}}(\mathscr{A}) \oplus H^n_{\mathrm{eq}}(\mathscr{A})$, *where the splitting is given by*

$$\phi = \phi_1 + \phi_2,$$

$$\phi_2 = \textit{antisymmetrization of } \phi \,|\, \mathscr{A},$$

$$\phi_1 = \textit{antisymmetrization of } (\phi - \bar{\phi}_2) \circ h.$$

Proof. By Theorem 5.7 and Lemma 6.1 we have a short exact sequence. This splits in the stated way (which, however, depends on our choice of h).

Before going on we will need certain computational results which hold for any $\mathscr{A}$. With ϕ an n-cochain on $\bar{\mathscr{A}}$, set

$$T\phi(x_0, x_1, ..., x_n) = (-1)^n(x_n, x_0, ..., x_{n-1}),$$

$$R\phi = \frac{1}{n+1}(n+1+nT+(n-1)\,T^2+\cdots+2T^{n-1}+T^n)\phi,$$

$$N\phi = \frac{1}{n+1}(1+T+\cdots+T^n)\phi,$$

$$b'\phi(x_0, x_1, ..., x_{n+1}) = \sum_{i=0}^{n}(-1)^i\phi(x_0, ..., x_ix_{i+1}, ..., x_{n+1}).$$

T, R, and N map $\bar{M}^*_n$ to $\bar{M}^*_n$ and M^*_n to M^*_n, while b' maps $\bar{M}^*_n$ to $\bar{M}^*_{n+1}$ and M^*_n toM^*_{n+1}. Define a map $\bar{\pi}$ from M^*_n to $(\bar{M}^*_n)_{\mathrm{hom}}$:

$$\bar{\pi}\phi(x_0, x_1, ..., x_n) = \phi(k(1 \otimes x_0 \otimes \cdots \otimes x_n)), \qquad x_i \in \bar{\mathscr{A}}.$$

Considering π as a map from M^*_{n-1} to $(\bar{M}^*_n)_{\text{hom}}$, set

$$\begin{aligned} \tilde{\#} : M^*_{n-1} &\to (\bar{M}^*_n)_{\text{hom}} \\ \phi &\mapsto \pi\phi - bR\bar{\pi}\phi - R\bar{\pi}b\phi. \end{aligned} \tag{2}$$

The following holds for any $\mathscr{A}$.

PROPOSITION 6.3. (I) $bN\phi = Nb'\phi$, $(1-T)b\phi = b'(1-T)\phi$,

(II) $(1-T)\phi = b'(\phi|_0) + (b\phi)|_0$, $(1-T)R = 1-N$,

(III) $N\bar{\pi}N = 0$,

(IV) $\tilde{\#}$ *maps cyclic cocycles to cyclic cocycles, cyclic coboundaries to cyclic coboundaries, and commutes with* b.

Proof. (I) is well known and (II) is shown by a straightforward computation, once we recall the definition of $\phi|_0$ given in Lemma 5.1.

(III) First note the identity

$$\sum_{i=1}^{n_1}{}' \phi\circ\alpha^{-i} + \sum_{i=1}^{m}{}' \phi\circ\alpha^{-i-n} = \sum_{i=1}^{n+m}{}' \phi\circ\alpha^{-i}. \tag{$*$}$$

Suppose that ϕ is a cyclic n-cochain on $\mathscr{A}$ and set

$$\begin{aligned} x_i &= u^{m_i}a_i, \qquad a_i \in \mathscr{A},\ i = 1, ..., n, \\ \tilde{a}_i &= \alpha^{m_0+m_1+\cdots+m_i}(a_i), \\ m_0 + m_1 + \cdots + m_n &= 0. \end{aligned}$$

We have

$$\begin{aligned} &T^{-i}\bar{\pi}\phi(x_0, x_1, ..., x_n) \\ &\quad = (-1)^{ni}\pi\phi(1, x_i, x_{i+1}, ..., x_n, x_0, x_1, ..., x_{i-1}) \\ &\quad = (-1)^{ni} \sum_{k=1}^{m_i}{}' \phi\circ\alpha^{-k}(\alpha^{m_i}(a_i), \alpha^{m_i+m_{i+1}}(a^{i+1}), ..., \alpha^{m_i+\cdots+m_{i-1}}(a_i)) \\ &\quad = \sum_{k=1}^{m_i}{}' \phi\circ\alpha^{-k+m_i+m_{i+1}+\cdots+m_n}(\tilde{a}_0, \tilde{a}_1, ..., \tilde{a}_n) \\ &\quad = \sum_{k=1}^{m_i}{}' \phi\circ\alpha^{-k-m_0-\cdots-m_{i-1}}(\tilde{a}_0, \tilde{a}_1, ..., \tilde{a}_n). \end{aligned}$$

But then

$$\begin{aligned}(n+1)\, & N\bar{\pi}\phi(x_0, x_1, ..., x_n)\\ &= \sum_{i=0}^{n} (T^{-i}\bar{\pi}\phi)(x_0, x_1, ..., x_n)\\ &= \left(\sum_{i=0}^{n} \sum_{k=1}^{m_i}{}' \phi \circ \alpha^{-k-m_0-\cdots-m_{i-1}}\right)(\tilde{a}_0, \tilde{a}_1, ..., \tilde{a}_n),\end{aligned}$$

and the double sum gives zero by (∗) and by the fact that $m_0 + \cdots + m_n = 0$.

(IV) Note that $\pi b\phi = b\pi\phi$ and, moreover,

$$\bar{\pi}\phi = (\pi\phi)|_0.$$

Hence (II) gives an identity

$$(1-T)\,\pi\phi = b'\bar{\pi}\phi + \bar{\pi}b\phi.$$

Using this and the identities (I), (II), (III), it is now easy to check the identities

$$\begin{gathered}(1-T)\tilde{\#}\,\phi = b'N\bar{\pi}\phi + N\bar{\pi}b\phi,\\ \tilde{\#}\,bN\phi = bN\,\tilde{\#}\,N\phi,\\ \tilde{\#}\,b\phi = b\,\tilde{\#}\,\phi.\end{gathered}$$

This shows that $\tilde{\#}$ has the desired properties.

Let us now go back to our special case, when $\mathscr{A} = C^\infty(X)$.

THEOREM 6.4. $\mathbb{E}^n_\infty(C^\infty(X)\hat{\times}_\alpha \mathbb{Z}) \cong H^{n-1}_{\text{coeq}}(\mathscr{A}) \oplus H^n_{\text{eq}}(\mathscr{A})$.

Proof. Using the splitting of $\mathbb{E}^n_1(\bar{\mathscr{A}})$ given in Lemma 6.2, the fact that $\tilde{\#}$ is homotopic to π, and the equality $(\pi\phi)\circ h = \phi$, we can now write, for any cocycle ϕ representing an element in $\mathbb{E}^n_1(\bar{\mathscr{A}})$,

$$\phi = \tilde{\#}\,\phi_1 + \tilde{\phi}_2 \qquad \text{in} \quad \mathbb{E}^n_1(\bar{\mathscr{A}}).$$

Since both ϕ_1 and ϕ_2 are cyclic, so are $\tilde{\#}\phi_1$ (6.3(IV)), and $\tilde{\phi}_2$ (the lifting given by the formula (1)). This means that every element of $\mathbb{E}^n_1(\bar{\mathscr{A}})$ can be represented by a cyclic cocycle. Going back to the spectral sequence considered at the beginning of Section 5 we get, since all the boundary operators $d_0, d_1, d_2, ...$ kill cyclic cocycles, the isomorphisms

$$\mathbb{E}^n_1(\bar{\mathscr{A}}) \cong \mathbb{E}^n_2(\bar{\mathscr{A}}) \cong \cdots \cong \mathbb{E}^n_\infty(\mathscr{A}).$$

By Lemma 6.2 this gives the desired result.

Corollary 6.5. *Suppose $X = \mathbb{T}$ and α preserves orientation. Then*

$$\dim HC^{\mathrm{ev}}(\bar{\mathscr{A}}) = \dim HC^{\mathrm{odd}}(\bar{\mathscr{A}}) = 2.$$

Proof. Denote by $H_n(X)$ the homology groups of X computed as

$$H_n(X) = \frac{\operatorname{Ker} d_0 \mid \mathscr{D}'_n(x)}{\operatorname{Im} d_0 \mid \mathscr{D}'_{n-1}(x)}.$$

Since the groups $H_n(X)$, $H^n_{\mathrm{eq}}(\mathscr{A})$, $H^n_{\mathrm{coeq}}(\mathscr{A})$ are all computed by complexes of n-currents on X, it is easy to see that the identity map $\phi \mapsto \phi$ (on the cochain level) in cohomology descends to the maps

$$\alpha_n : H^n_{\mathrm{eq}}(\mathscr{A}) \to H_n(X)^{\alpha}$$

$$\beta_n : H_n(X)/\delta H_n(X) \to H^n_{\mathrm{coeq}}(\mathscr{A}).$$

Define the maps

$$s_1 : \operatorname{Coker} \alpha_{n-1} \to \operatorname{Coker} \beta_n,$$

$$s_2 : \operatorname{Coker} \beta_n \to \operatorname{Ker} \alpha_{n-2},$$

$$s_3 : \operatorname{Ker} \alpha_{n-2} \to \operatorname{Ker} \beta_{n-1}.$$

s_1, s_2, s_3 are given as follows:

(1) Starting with $\phi \in H_{n-1}(X)^{\alpha}$, we have $\delta\phi = d_0\omega$ and we set $s_1\phi =$ class of ω in $\operatorname{Coker} \beta_n$.

(2) Given $\phi \in H^n_{\mathrm{coeq}}(\mathscr{A})$, $d_0\phi = \delta\omega$ and then

$$\delta(d_0\omega) = d_0(\delta\omega) = 0.$$

This says that $d_0\omega \in H^{n-2}_{\mathrm{eq}}(X)$ and we set $s_2(\phi) =$ class of $d_0\omega$ in $\operatorname{Ker} \alpha_{n-2}$.

(3) Given $\phi \in \operatorname{Ker} \alpha_{n-2}$, $d_0\phi = \delta\phi = 0$ and $\phi = d_0\omega$ for some $n-1$ current ω. Then $\delta\omega$ gives an $n-1$-current on X and moreover $d_0\delta\omega = \delta d_0\omega = \delta\phi = 0$. We set $s_3(\phi) =$ class of $\delta\omega$.

It is a straightforward if rather lengthy diagram chase to check that these maps are well defined and that the following sequence is eaxact:

$$0 \longrightarrow \operatorname{Coker} \alpha_{n-1} \xrightarrow{s_1} \operatorname{Coker} \beta_n \xrightarrow{s_2} \operatorname{Ker} \alpha_{n-2} \xrightarrow{s_3} \operatorname{Ker} \beta_{n-1} \longrightarrow 0.$$

Applying it to the case when $X = \mathbb{T}$ we get the following

(4) $n = 0$: $H_0(\mathbb{T})/\delta H_0(\mathbb{T}) \to H^0_{\mathrm{coeq}}(\mathscr{A})$ is surjective, i.e.,

$$H^0_{\mathrm{coeq}}(\mathscr{A}) \cong H_0(\mathbb{T}) = \mathbb{C}.$$

(5) $n = 1$: $H^0_{\mathrm{eq}}(\mathscr{A}) \to H_0(\mathbb{T})$ is surjective since $H_0(\mathbb{T}) = \mathbb{C}$ and the generator can be represented by any α-invariant measure on the circle, and

$$H_1(\mathbb{T})/\delta H_1(\mathbb{T}) \to H^1_{\mathrm{coeq}}(\mathscr{A}) \text{ is surjective.}$$

(6) $n=2$: $H^1_{\mathrm{eq}}(\mathscr{A} \to H_1(\mathbb{T})$ is surjective, in fact α preserves orientation of $\mathbb{T}$, and

$$\operatorname{Ker}(H_1(\mathbb{T})/\delta H_1(\mathbb{T})) \to H^1_{\mathrm{coeq}}(\mathbb{T}) \cong \operatorname{Ker}(H^0_{\mathrm{eq}}(\mathscr{A}) \to H_0(\mathbb{T})).$$

Let μ denote an α-invariant probability measure on the unit circle and let τ be the fundamental class of $\mathbb{T}$, i.e.,

$$\tau(f, g) = \int f\, dg.$$

We can now write

$$H_0(\mathbb{T}) = H^0_{\mathrm{coeq}}(\mathscr{A}) = \mathbb{C}\mu$$
$$H_1(\mathbb{T}) = H^1_{\mathrm{eq}}(\mathscr{A}) = \mathbb{C}\tau.$$

There are two possibilities.

(1) τ is non-zero in $H^1_{\mathrm{coeq}}(\mathscr{A})$. Then, by (6),

$$H^0_{\mathrm{eq}}(\mathscr{A}) = H_0(\mathbb{T}) = \mathbb{C}\mu.$$

This gives

$$\dim \mathbb{E}^0_\infty(\bar{\mathscr{A}}) = \dim \mathbb{E}^2_\infty(\bar{\mathscr{A}}) = 1.$$

(2) $\tau = \delta\omega$, where ω is a 1-current on $\mathbb{T}$. Then $H^1_{\mathrm{coeq}}(\mathscr{A}) = 0$, $H^0_{\mathrm{eq}}(\mathscr{A}) = \mathbb{C}\mu \oplus \mathbb{C} d_0 \omega$ and

$$\dim \mathbb{E}^0_\infty(\bar{\mathscr{A}}) = 2,$$
$$\dim \mathbb{E}^2_\infty(\bar{\mathscr{A}}) = 0.$$

It is easy to check that in both cases

$$\dim \mathbb{E}^1_\infty(\bar{\mathscr{A}}) = 2.$$

Since $\mathbb{E}^n_\infty(\mathscr{A})$ are the graded groups of the filtration of $HC(\bar{\mathscr{A}})$ by dimension, the result follows now from Theorem 5.7 and the fact that $H^n_{\mathrm{eq}}(\mathbb{T}) = H^n_{\mathrm{coeq}}(\mathbb{T}) = 0$ for $n > 1$.

7. CYCLIC COHOMOLOGY OF THE CROSSED PRODUCT: OUTLINE OF THE COMPUTATION

The rest of this paper is devoted to the computation of the cyclic cohomology of the crossed product $\mathscr{A} \mathbin{\hat{\times}_\alpha} \mathbb{Z}$. Let us briefly describe the procedure.

The first step consists of looking at the following part of the exact sequence of Theorem 4.1:

$$\cdots \xrightarrow{\delta} H^{n-1}(\mathscr{A}, \mathscr{A}^*) \xrightarrow{\pi} H^n(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*)_{\mathrm{hom}} \xrightarrow{i} H^n(\mathscr{A}, \mathscr{A}^*) \xrightarrow{\delta} \cdots.$$

Let us look at the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\delta"] & H^{n-1}(\mathscr{A}, \mathscr{A}^*) \arrow[r, "\pi"] & H^n(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*)_{\mathrm{hom}} \arrow[r, "i"] & H^n(\mathscr{A}, \mathscr{A}^*) \arrow[r, "\delta"] & \cdots \\
\cdots \arrow[r, "\delta"] & H_\lambda^{n-1}(\mathscr{A}) \arrow[u, "I"] \arrow[r, dashed, "\#_\alpha"] & H_\lambda^n(\bar{\mathscr{A}})_{\mathrm{hom}} \arrow[u, "I"] \arrow[r, "i"] & H_\lambda^n(\mathscr{A}) \arrow[u, "I"] \arrow[r, "\delta"] & \cdots \quad (1) \\
\cdots \arrow[r, "\delta"] & H_\lambda^{n-3}(\mathscr{A}) \arrow[u, "S"] \arrow[r, dashed, "\#_\alpha"] & H_\lambda^{n-2}(\bar{\mathscr{A}})_{\mathrm{hom}} \arrow[u, "S"] \arrow[r, "i"] & H_\lambda^{n-2}(\mathscr{A}) \arrow[u, "S"] \arrow[r, "\delta"] & \cdots.
\end{tikzcd}$$

We will construct the map $\#_\alpha : H_\lambda^{n-1}(\mathscr{A}) \to H_\lambda^n(\bar{\mathscr{A}})_{\mathrm{hom}}$ so as to make the above diagram commutative and then prove that the middle row is exact.

The main ingredient of the proof is a cochain map

$$\eta : C^{n+1}(\bar{\mathscr{A}}, \mathscr{A}^*)_{\mathrm{hom}} \to C^n(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*)_{\mathrm{hom}},$$

which satisfies

$$b\eta = \eta b + \delta,$$
$$N\eta N = \eta N.$$

Such a map η induces maps as follows, by passage to quotients and restriction to $\mathscr{A}$:

$$\text{cocycles vanishing on } \mathscr{A}/\text{coboundaries} \to H(\mathscr{A}, \mathscr{A}^*),$$
$$\text{cyclic cocycles vanishing on } \mathscr{A}/\text{cyclic coboundaries} \to H_\lambda(\mathscr{A}).$$

It will be easy to check that both of these maps are actually defined on Ker i in cohomology:

$$\begin{tikzcd}
\operatorname{Ker} i \arrow[r, "\eta"] \arrow[d, phantom, "\cap"] & H^{n-1}(\mathscr{A}, \mathscr{A}^*) \\
H^n(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*)_{\mathrm{hom}} &
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{Ker} i \arrow[r, "\eta"] \arrow[d, phantom, "\cap"] & H_\lambda^{n-1}(\mathscr{A}) \\
H_\lambda^n(\bar{\mathscr{A}})_{\mathrm{hom}}. &
\end{tikzcd}$$

Next we shall show that

$$\eta\pi = \mathrm{id}$$

in Hochschild cohomology. This, together with the commutativity of the diagram (1), will allow us to conclude that

$$\operatorname{Ker} i \subseteq \operatorname{Im} \#_\alpha \qquad \text{in } H_\lambda(\bar{\mathscr{A}})_{\text{hom}}.$$

It will then be relatively easy to show that the sequence

$$\cdots \xrightarrow{\delta} H_\lambda^{n-1}(\mathscr{A}) \xrightarrow{\#_\alpha} H_\lambda^n(\bar{\mathscr{A}})_{\text{hom}} \xrightarrow{i} H_\lambda^n(\mathscr{A}) \xrightarrow{\delta} \cdots$$

is exact.

As the final step we shall combine this with Corollary 5.6, and construct a six-term exact sequence describing the periodic cyclic cohomology of the crossed product $\mathscr{A} \hat{\times}_\alpha \mathbb{Z}$.

8. CONSTRUCTION OF

We shall start with a differential graded algebra (E, d) defined as

$$E = E_0 \oplus E_1,$$

where

$$E_0 = \left\{ \text{finite sums } \sum_n a_n u^n;\ n \in \mathbb{Z},\ a_n \in \mathbb{C} \right\},$$

$$E_1 = \left\{ \text{finite sums } \sum_{n,m} a_{n,m} u^n (du)\, u^m;\ n, m \in \mathbb{Z},\ a_{n,m} \in \mathbb{C} \right\}.$$

The product structure is given by the formulas

$$u^0 = 1,$$
$$u^i u^j = u^{i+j}$$
$$u^i (u^j (du)\, u^k)\, u^l = u^{i+j} (du)\, u^{k+l},$$
$$(du)\, u^i (du) = 0,$$

and the graded differential d is defined by

$$du^n = \sum_{i=1}^{n}{}' u^i (u^{-1}\, du)\, u^{n-i},$$

$$d \,|\, E_1 = 0.$$

The fact that (E, d) does become a differential graded algebra with the above definitions follows from its representation as a quotient of the universal differential graded algebra $\Omega(\mathbb{C}[u, u^{-1}])$ by the graded ideal generated by $\bigoplus_{i \geqslant 2} \Omega_i$ together with $d1 \in \Omega_1$ (here $1 \in \Omega_0$ denotes the unit of $\mathbb{C}[u, u^{-1}]$).

Now suppose that we are given a cycle $(\Omega, \tilde{d}, \phi)$ (as defined in [2]) and an action of $\mathbb{Z}$ on Ω, i.e., an automorphism α of Ω commuting with $\tilde{d}$. We define a crossed product cycle

$$(E \otimes_\alpha \Omega, d, \#_\alpha \phi)$$

as follows:

(1) $E \otimes_\alpha \Omega = E \otimes \Omega$ as an algebraic tensor product of graded vector spaces;

(2) the product stucture is induced by the relations

$$(1 \otimes \omega)(du \otimes 1) = u \otimes \alpha^{-1}(\omega),$$
$$(1 \otimes \omega)(du \otimes 1) = (-1)^{\deg \omega}\, du \otimes \alpha^{-1}(\omega);$$

(3) the differential d is given by

$$d(\omega_1 \otimes \omega_2) = d\omega_1 \otimes \omega_2 + (-1)^{\deg \omega_1} \omega_1 \otimes \tilde{d}\omega_2;$$

(4) the closed graded trace $\#_\alpha \phi$ is given by

$$\#_\alpha \phi(u^i(u^{-1}\, du)\, u^j \otimes \omega) = \phi(\alpha^j(\omega))\, \delta_{i+j,0},$$
$$\#_\alpha \phi \mid E_0 \otimes \Omega = 0.$$

Let us check that $\#_\alpha \phi$ is indeed a closed graded trace.

$$\begin{aligned} &\#_\alpha \phi(d(u^n \otimes \omega)) \\ &\quad = \#_\alpha \phi \left({\sum_{i=1}^{n}}' u^i(u^{-1}\, du)\, u^{n-i} \otimes \omega + u^n \otimes \tilde{d}\omega \right) \\ &\quad = {\sum_{i=1}^{n}}' \phi(\alpha^{n-i}(\omega))\, \delta_{n,0} + \phi(\tilde{d}\omega) = 0, \end{aligned}$$

$$\begin{aligned} &\#_\alpha \phi(d(u^i(u^{-1}\, du)\, u^j \otimes \omega)) \\ &\quad = -\#_\alpha \phi(u^i(u^{-1}\, du)\, u^j \otimes \tilde{d}\omega) \\ &\quad = -\phi(\alpha^j(\tilde{d}\omega))\, \delta_{i+j,0} = -\phi(\tilde{d}\alpha^j(\omega))\, \delta_{i+j,0} = 0; \end{aligned}$$

the equalities follow from the fact that $\sum'{}_{i=1}^{n}=0$ for $n=0$, and the fact that ϕ is closed. Hence

$$\#_\alpha\phi(d\,\cdot)=0,$$

i.e., $\#_\alpha\phi$ is closed. To check the graded trace property it is sufficient to look at

$$\begin{aligned}&\#_\alpha\phi((u^{-n-m-1}(du)\,u^n\otimes\omega)(u^m\otimes\omega'))\\&\quad=\#_\alpha\phi(u^{-n-m-1}(du)\,u^{n+m}\otimes\alpha^{-m}(\omega)\omega')\\&\quad=\phi(\alpha^n(\omega)\,\alpha^{n+m}(\omega'));\\&\#_\alpha\phi((u^m\otimes\omega')(u^{-n-m-1}(du)\,u^n\otimes\omega))\\&\quad=(-1)^{\deg\omega'}\#_\alpha\phi(u^{-n-1}(du)\,u^n\otimes\alpha^m(\omega')\omega)\\&\quad=(-1)^{\deg\omega'}\phi(\alpha^{n+m}(\omega')\,\alpha^n(\omega)).\end{aligned}$$

Since ϕ is a graded trace on Ω, we get the equality

$$\begin{aligned}&\#_\alpha\phi((u^{-n-m-1}(du)\,u^n\otimes\omega)(u^m\otimes\omega'))\\&\quad=(-1)^{(\deg\omega+1)\deg\omega'}\#_\alpha\phi((u^m\otimes\omega')(u^{-n-m-1}(du)\,u^n\otimes\omega)).\end{aligned}$$

Since

$$\deg(u^{-n-m-1}(du)\,u^n\otimes\omega)=1+\deg\omega,$$

this shows that $\#_\alpha\phi$ is a graded trace.

Now, let ϕ be a cyclic cocycle on $\mathscr{A}$, and let us apply this construction to the cycle $(\Omega(\mathscr{A}), d, \hat{\phi})$ (see [2, Proposition 1]), with the action of $\mathbb{Z}$ defined by

$$\alpha(x_0\,dx_1\cdots dx_n)=\alpha(x_0)\,d\alpha(x_1)\cdots d\alpha(x_n).$$

Here $\hat{\phi}$ denotes the associated graded trace

$$\hat{\phi}(x_0\,dx_1\cdots dx_n)=\phi(x_0, x_1, ..., x_n).$$

Then $\#_\alpha\phi$ is a closed graded trace on $E\otimes_\alpha\Omega(\mathscr{A})$. To go from this to a cocycle on $\bar{\mathscr{A}}$ we shall use the homomorphism

$$\begin{aligned}\rho\colon \mathscr{A}_\alpha[u, u^{-1}]&\to E\otimes_\alpha\Omega(\mathscr{A})_0\\u^m x&\mapsto u^m\otimes x,\end{aligned}$$

thus

$$\#_\alpha\phi(x_0, x_1, ..., x_n)=\#_\alpha\phi(\rho(x_0)\,d\rho(x_1)\cdots d\rho(x_n)).$$

Note that we use the same symbol $\#_\alpha\phi$ to denote both the closed graded trace on $E\otimes_\alpha\Omega(\mathscr{A})$ and the corresponding cyclic cocycle on $\bar{\mathscr{A}}$.

We shall need some explicit formulas later on, so let us fix the notation

$$\begin{gathered}
x_i = u^{m_i}a_i, \qquad i=1, ..., n, m_i \in \mathbb{Z}, a_i \in \mathscr{A}, \\
D(e\otimes\omega) = de\otimes\omega, \qquad e\in E, \omega\in\Omega(\mathscr{A}), \\
\gamma = u^{-1}\, du\otimes 1 \in E\otimes_\alpha\Omega(\mathscr{A}).
\end{gathered}$$

Note that γ is closed and D is a derivation of $E\otimes_\alpha\Omega(\mathscr{A})$ anticommuting with d. Let us set

$$\begin{aligned}
\pi_i\phi(x_0, x_1, ..., x_n) &= \#_\alpha\phi(x_0\, dx_1 \cdots dx_{i-1}\, Dx_i\, dx_{i+1} \cdots dx_n), \\
\bar{\pi}_i\phi(x_0, x_1, ..., x_{n-1}) &= \pi_{i+1}\phi(1, x_0, x_1, ..., x_{n-1}), \\
\rho_i\phi(x_0, x_1, ..., x_{n-1}) &= \#_\alpha\phi(x_0\, dx_1 \cdots dx_{i-1}\gamma\, dx_i \cdots dx_{n-1}).
\end{aligned}$$

LEMMA 8.1. *Suppose that ϕ is a cyclic cocycle on $\mathscr{A}$. Then the following identities hold*:

(1) $\#_\alpha\phi = \sum_{i=1}^{n} \pi_i\phi$,

(2) $b\bar{\pi}_i\phi = \pi_{i+1}\phi - \pi_i\phi$ *for* $i>0$, *and* $b\bar{\pi}_0\phi = \pi_1\phi - \pi_n\phi$,

(3) $b\rho_i\phi = (x_0, x_1, ..., x_n)$

$$= (-1)^i\,\#_\alpha\phi(x_0\, dx_1 \cdots dx_{i-1}\,[x_i, \gamma]\, dx_{i+1} \cdots dx_n),$$

(4) $T\rho_i\phi = -\rho_{i-1}\phi$.

Proof. Ad 1. From the definition of the product in $E\otimes_\alpha\Omega$ we get

$$\begin{aligned}
x_0\, dx_1 \cdots dx_n = \sum_{i=1}^{n} & x_0(u^{m_1}\, da_1)\cdot \cdots \cdot(u^{m_{i-1}}\, da_{i-1}) \\
& \times(du^{m_i})\, a_i(u^{m_{i+1}}\, da_{i+1})\cdot \cdots \cdot(u^{m_n}\, da_n) \\
& + x_0u^{m_1}(da_1)\, u^{m_2}(da_2)\cdot \cdots \cdot u^{m_n}\, da_n \\
= \sum_{i=1}^{n} & x_o\, dx_1 \cdots dx_{i-1}\, Dx_i\, dx_{i+1} \cdots dx_n \\
& + x_0u^{m_1}(da_1)\, u^{m_2}(da_2)\cdot \cdots \cdot u^{m_n}\, da_n,
\end{aligned}$$

and $\#_\alpha\phi$ is zero on the second term.

Ad 2, 3. These follow by a straightforward computation using the facts that both D and d are derivations of $E \otimes_\alpha \Omega(\mathscr{A})$, and that $\#_\alpha \phi$ is a graded trace.

Ad 4. We have

$$\begin{aligned}
&(T\rho_i \phi)(x_0, x_1, ..., x_{n-1}) \\
&\quad = (-1)^{n-1} \#_\alpha \phi(x_{n-1}\, dx_0 \cdots dx_{i-2} \gamma\, dx_{i-1} \cdots dx_{n-2}) \\
&\quad = (-1)^{n-1} \#_\alpha \phi(dx_0\, dx_1 \cdots dx_{i-2} \gamma\, dx_{i-1} \cdots dx_{n-2} x_{n-1}) \\
&\quad = (-1)^{n-1} (-1)^n \#_\alpha \phi(x_0\, dx_1 \cdots dx_{i-2} \gamma\, dx_{i-1} \cdots dx_{n-1}),
\end{aligned}$$

where the last equality follows from the identity

$$\begin{aligned}
&d(x_0 (dx_1 \cdots dx_{i-2} \gamma\, dx_{i-1} \cdots dx_{n-2})\, x_{n-1}) \\
&\quad = (dx_0\, dx_1 \cdots dx_{i-2} \gamma\, dx_{i-1} \cdots dx_{n-2})\, x_{n-1} \\
&\qquad + (-1)^{n-1} x_0\, dx_1 \cdots dx_{i-2} \gamma\, dx_{i-1} \cdots dx_{n-1},
\end{aligned}$$

together with closedness of $\#_\alpha \phi$.
Recalling now the map

$$\tilde{\#} : M^*_{n-1} \to (\bar{M}_n)^*_{\mathrm{hom}},$$

defined by (1), Section 6, we have the following important result.

Proposition 8.2. (1) $\#_\alpha = (-1)^n n \tilde{\#} \mid H^{n-1}_\lambda(\mathscr{A})$,

(2) $\#_\alpha$ *commutes with* S,

(3) $\#_\alpha \delta = 0$ *in cyclic cohomology.*

Proof. Ad 1. Let ϕ be a cyclic cocycle on $\mathscr{A}$. Using the trace property of $\#_\alpha \phi$ it is easy to check the identity

$$T \bar{\pi}_i \phi = \bar{\pi}_{i-1} \phi \qquad (i \operatorname{mod}(n-1)),$$

and hence, by Lemma 8.1,

$$\begin{aligned}
nbR\bar{\pi}_0 \phi &= b(n\bar{\pi}_0 \phi + (n-1)\, \bar{\pi}_{n-1} \phi + \cdots + \bar{\pi}_1 \phi) \\
&= n(\pi_1 \phi - \pi_n \phi) + (n-1)(\pi_n \phi - \pi_{n-1} \phi) + \cdots + (\pi_2 \phi - \pi_1 \phi) \\
&= n\pi_1 \phi - \sum_{i=1}^{n} \pi_n \phi = n\pi_1 \phi - \#_\alpha \phi.
\end{aligned}$$

But it is a straightforward consequence of the definitions that $\pi = (-1)^n \pi_1$, $\bar{\pi} = (-1)^n \bar{\pi}_0$ on $(n-1)$-cochains, and hence

$$\begin{aligned} \#_\alpha \phi &= (-1)^n n(\pi_1 \phi - bR\bar{\pi}_0 \phi - R\bar{\pi}_0 b\phi) \\ &= (-1)^n n \tilde{\#} \phi. \end{aligned}$$

Ad 2. Let us look at the diagram

$$\begin{tikzcd}
\Omega(\mathscr{A}_\alpha[u, u^{-1}]) \arrow[r, phantom, "="] \arrow[d, "\approx"] & \Omega(\mathscr{A}_\alpha[u, u^{-1}]) \arrow[d, "\approx"] \\
\Omega((\mathscr{A} \otimes \mathbb{C})_\alpha[u, u^{-1}] \arrow[d] & \Omega(\mathscr{A}_\alpha[u, u^{-1}] \otimes \mathbb{C}) \arrow[d] \\
E \otimes_\alpha \Omega(\mathscr{A} \otimes \mathbb{C}) \arrow[d] & \Omega(\mathscr{A}_\alpha[u, u^{-1}]) \otimes \Omega(\mathbb{C}) \arrow[d] \\
E \otimes_\alpha (\Omega(\mathscr{A}) \otimes \Omega(\mathbb{C})) \arrow[r, "\approx"] & (E \otimes_\alpha \Omega(\mathscr{A}) \otimes \Omega(\mathbb{C})).
\end{tikzcd}$$

The vertical arrows are given by the universality of the Ω-construction, and the obvious isomorphism at the bottom gives

$$\#_\alpha(\phi \# \omega) = (\#_\alpha \phi) \# \omega,$$

for any closed graded trace on $\Omega(\mathbb{C})$. If we take for ω the generator of $H_\lambda^2(\mathbb{C})$ given by $\omega(1d1d1) = 2\pi i$, the left-hand column computes $\#_\alpha S\phi$, while the right-hand column computes $S\#_\alpha\phi$, i.e., $S\#_\alpha = \#_\alpha S$.

Ad 3. Let us compute $\#_\alpha \delta\phi$ for a cyclic cocycle ϕ. We have

$$\begin{aligned} \pi_i(\delta\phi)(x_0, x_1, ..., x_n) &= \#_\alpha \phi(x_0\, dx_1 \cdots dx_{i-1}\, Dx_1\, dx_{i+1} \cdots dx_n) \\ &\quad - \#_\alpha \phi \circ \alpha(x_0\, dx_1 \cdots dx_{i-1}\, Dx_i\, dx_{i+1} \cdots dx_n) \\ &= \#_\alpha \phi(x_0\, dx_1 \cdots dx_{i-1} (du^{m_i})\, a_i\, dx_{i+1} \cdots dx_n) \\ &= \#_\alpha \phi(x_0\, dx_1 \cdots dx_{i-1} u^{-1} (du^{m_i})\, ua_i\, dx_{i+1} \cdots dx_n). \end{aligned}$$

But

$$\begin{aligned} &(du^{m_i})\, a_i - u^{-1}(du^{m_i})\, ua_i \\ &\quad = \sum_{i=1}^{m_i} (u^{-i-1}\, du\, u^i - u^{-i-2}\, du\, u^{i+1})\, a_i \\ &\quad = [u^{m_i}, u^{-1}\, du]\, a_i = [x_i, \gamma], \end{aligned}$$

and hence, according to Lemma 8.1(1) and (3), we get

$$\#_\alpha \delta\phi = \sum_{i=1}^{n} \pi_i \delta\phi = b\left(\sum_{i=1}^{n} (-1)^i \rho_i \phi\right). \tag{1}$$

Then Lemma 8.1(4) shows that $\sum_{i=1}^{n}(-1)^i \rho_i \phi$ is a cyclic cochain on $\bar{\mathscr{A}}$ and completes the proof.

PROPOSITION 8.3. *All the cochains constructed in this section extend by continuity to all of* $\bar{\mathscr{A}}$.

Proof. We shall consider $\pi_1 \phi$ (all the other cases follow the same pattern).

Suppose $x_0, x_1, ..., x_n \in \bar{\mathscr{A}}$ are the monomials

$$x_i = u^{m_i} a_i, \qquad a_i \in \mathscr{A}, m_0 + \cdots + m_n = 0,$$

and set

$$\tilde{a}_j = \alpha^{m_1 + \cdots + m_j}(a_j).$$

Then

$$\pi_1 \phi(x_0, x_1, ..., x_n) = \sum_{j=1}^{m_1}{}' \phi \circ \alpha^{-j}(\tilde{a}_0 \tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n).$$

Given that ϕ satisfies an estimate of the form

$$|\phi(a_0, a_1, ..., a_{n-1})| \leqslant c \, \|a_0\|_k \, \|a_1\|_k \cdots \|a_{n-1}\|_k,$$

we get

$$|\pi_1 \phi| \leqslant c \sum_{j=1}^{m_1} \|\alpha^{-j}\|_k \, \|\alpha^{m_1}\|_k^n \, \|\alpha^{m_2}\|_k^{n-1} \cdot \cdots \cdot \|\alpha^{m_n}\|_k$$
$$\cdot \|a_0\|_{k'} \, \|a_1\|_{k'} \, \|a_2\|_k \cdot \cdots \cdot \|a_n\|_k,$$

where we used the fact that $\|ab\|_k \leqslant \|a\|_{k'} \, \|b\|_{k'}$ for some $k' \in \mathbb{N}$. But then, choosing $k'' = \max(k, k') + n + 1$,

$$|\pi_1 \phi| \leqslant c \rho_{k''}(m_0) \, \rho_{k''}(m_1) \cdot \cdots \cdot \rho_{k''}(m_n) \, \|a_0\|_{k''} \, \|a_1\|_{k''} \cdot \cdots \cdot \|a_n\|_{k''}.$$

If now x_i are given as finite sums of monomials

$$x_i = \sum_{m_i} u^{m_i} a_{i, m_i},$$

we get from above, using Lemma 1.1,

$$
\begin{aligned}
&|\pi_1 \phi(x_0, x_1, ..., x_n)| \\
&\quad \leqslant C \sum_{m_0} \cdots \sum_{m_n} \frac{1}{m_0^2} \cdots \frac{1}{m_n^2} \rho_{k''+2}(m_0) \cdot \cdots \\
&\qquad \cdot \rho_{k''+2}(m_n) \, \|a_{0,m_0}\|_{k''} \cdots \|a_{n,m_n}\| \\
&\quad \leqslant C \cdot 2^n \left(\sum_{m>0} \frac{1}{m^2} \right)^n \|x_0\|_{k''+2} \cdot \cdots \cdot \|x_n\|_{k''+2},
\end{aligned}
$$

and hence the continuity of $\pi_1 \phi$ is established.

DEFINITION 8.4. The cochain map

$$\# : C^{n-1}(\mathscr{A}, \mathscr{A}^*) \to C^n(\bar{\mathscr{A}}, \bar{\mathscr{A}}^*)_{\text{hom}}$$

is defined by

$$\# \phi = (-1)^n n \tilde{\#} \phi.$$

9. A SECTION FOR

(A) Given $\phi \in \bar{M}_n^*$ and $x_0, x_1, ..., x_{n-1}, a \in \bar{\mathscr{A}}$, we set

$$
\begin{aligned}
&h_a \phi(x_0, x_1, ..., x_{n-1}) \\
&\quad = \sum_{i=0}^{n-1} (-1)^{n-i-1} \phi(x_0, ..., x_i, a, x_{i+1}, ..., x_{n-1}).
\end{aligned}
$$

We have

$$
\begin{aligned}
&bh_a \phi(x_0, x_1, ..., x_n) \\
&\quad = h_a b\phi(x_0, x_1, ..., x_n) + (-1)^{n-1} \sum_{i=0}^{n} \phi(x_0, ..., [a, x_i], ..., x_n), \\
&((1-T)\, h_a \phi(x_0, x_1, ..., x_{n-1})) \\
&\quad = (h_a(1-T)\phi)(x_0, x_1, ..., x_{n-1}) \\
&\qquad + ((-1)^{n-1}(1-T)\phi)(a, x_0, x_1, ..., x_{n-1}).
\end{aligned}
$$

The proof of these relations consists of a straightforward computation, which will be omitted.

(B) We will work inside $\mathscr{A}\otimes M_2$ and replace ϕ by $\phi \# \operatorname{Tr}$. Set

$$\mathscr{U}=\begin{pmatrix}0 & 1\\ u & 0\end{pmatrix}, \qquad R_\theta=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \qquad \mathscr{I}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

$$\alpha_\theta=\operatorname{Ad}\mathscr{U}R_\theta\in\operatorname{Aut}(\mathscr{A}\otimes M_2).$$

Since

$$\frac{d}{d\theta}\alpha_\theta(x)=\alpha_\theta([\mathscr{I},x]),$$

we have, for any n-cochain ψ on $\mathscr{A}\otimes M_2$,

$$\begin{aligned}\psi\circ\alpha_{\pi/2}&(x_0,x_1,...,x_n)-\psi\circ\alpha_0(x_0,x_1,...,x_n)\\ &=\int_0^{\pi/2}\frac{d}{d\theta}\psi\circ\alpha_\theta(x_0,x_1,...,x_n)\,d\theta\\ &=\sum_{i=0}^{n}\int_0^{\pi/2}\psi\circ\alpha_\theta(x_0,x_1,...,[\mathscr{I},x_i],...,x_n)\,d\theta.\end{aligned}$$

(C) Define, given $\phi\in(\bar{M}_n^*)_{\text{hom}}$,

$$\bar{\eta}\phi=\int_0^{\pi/2}h_{\mathscr{I}}((\phi\#\operatorname{Tr})\circ\alpha_\theta)\,d\theta\,|_{\mathscr{A}\otimes e_{11}}.$$

Applying (A) and (B) above we get

$$b\bar{\eta}\phi=\bar{\eta}b\phi+(-1)^{n-1}\delta\phi,$$

$$\begin{aligned}(1-T)\,&\bar{\eta}\phi(x_0,x_1,...,x_{n-1})\\ &=\bar{\eta}(1-T)\,\phi(x_0,x_1,...,x_{n-1})\\ &\quad+(-1)^{n-1}\int_0^{\pi/2}(1-T)(\phi\#\operatorname{Tr}\circ\alpha_\theta)(\mathscr{I},\bar{x}_0,\bar{x}_1,...,\bar{x}_{n-1})\,d\theta,\end{aligned}$$

where $\bar{x}_i=x_i\otimes e_{11}$.

(D) Finally, note that if we define

$$\eta\phi=\bar{\eta}\phi\,|\,\mathscr{A},$$

then

$$\phi\,|_{\mathscr{A}}=0\Rightarrow b\eta\phi=\eta b\phi,$$

$$(1-T)\phi=0\Rightarrow(1-T)\,\eta\phi=0.$$

Thus we can consider η as the map

$$\eta\colon (Q_n^*)_{\mathrm{hom}} \to (M_{n-1})^*,$$

which commutes with the coboundary operator b and sends cyclic cochains to cyclic cochains.

(E) Note now that, given any cochain ϕ on $\mathscr{A}$, the formula

$$\phi((\lambda + a_0)\, da_1 \cdots da_n) = \phi(a_0, a_1, ..., a_n)$$

defines a linear functional on $\Omega(\mathscr{A})$. Moreover the construction of $\#_\alpha \phi$ extends to this more general case and gives us the map

$$\begin{aligned} \#_\alpha \colon (M_{n-1})^* &\to (\bar{M}_n^*)_{\mathrm{hom}}, \\ \phi &\mapsto \#_\alpha \phi. \end{aligned}$$

LEMMA 9.1. *Let* $\omega_0, \omega_1, \omega_2 \in \Omega(\mathscr{A})$, $a \in \mathscr{A}$, *and suppose that* ϕ *is a cochain on* $\mathscr{A}$. *Set*

$$\begin{aligned} \bar{\omega}_i &= \omega_i \otimes e_1 \in \Omega(\mathscr{A}) \otimes \Omega(M_2), \\ \bar{a} &= a \otimes e_{11} \in \Omega(\mathscr{A}) \otimes \Omega(M_2). \end{aligned}$$

The following identities hold.

$$(1) \quad \int_0^{\pi/2} d\theta((\#_\alpha \phi) \# \mathrm{Tr}) \circ \alpha_\theta(\bar{\omega}_0 (D\mathscr{J}) \bar{\omega}_1) = (-1)^{\deg \omega_0 + 1} \phi(\omega_0 \omega_1),$$

$$(2) \quad \int_0^{\pi/2} d\theta((\#_\alpha \phi) \# \mathrm{Tr}) \circ \alpha_\theta(\bar{\omega}_0 (d\mathscr{J}) \bar{\omega}_1 (D\bar{a}) \omega_2) = 0,$$

$$(3) \quad \int_0^{\pi/2} d\theta((\#_\alpha \phi) \# \mathrm{Tr}) \circ \alpha_\theta(\bar{\omega}_0 (D\bar{a}) \bar{\omega}_1 (d\mathscr{J}) \bar{\omega}_2) = 0,$$

$$(4) \quad \int_0^{\pi/2} d\theta((\#_\alpha \phi) \# \mathrm{Tr}) \circ \alpha_\theta(\bar{\omega}_0 (d\mathscr{J})(D\bar{a}) \bar{\omega}_1)$$
$$= (-1)^{\deg \omega_0 + 1} \frac{1}{2} \phi(\omega_0 (d1)\, a \omega_1),$$

$$(5) \quad \int_0^{\pi/2} d\theta((\#_\alpha \phi) \# \mathrm{Tr}) \circ \alpha_\theta(\bar{\omega}_0 (D\bar{a})(d\mathscr{J}) \bar{\omega}_1)$$
$$= (-1)^{\deg \omega_0} \frac{1}{2} \phi(\omega_0 a (d1)\, \omega_1).$$

Proof. Since all five integrals are computed in the same way, we will only write down the computations for the first one. We have, setting $\sin\theta = s$ and $\cos\theta = c$,

$$\begin{aligned}
I_\theta &= ((\#_\alpha\phi)\ \#\ \mathrm{Tr})(\alpha_\theta(\omega_0 \otimes e_{11})\, D(\alpha_\theta(\mathscr{J}))\, \alpha_\theta(\omega_1 \otimes e_{11}))\\
&= ((\#_\alpha\phi)\ \#\ \mathrm{Tr})\left(\begin{pmatrix} \omega_0 s^2 & \omega_0 u^{-1} sc \\ u\omega_0 sc & \alpha(\omega_0)\, c^2 \end{pmatrix} D\begin{pmatrix} 0 & u^{-1} \\ -u & 0 \end{pmatrix}\begin{pmatrix} \omega_1 s^2 & \omega_1 u^{-1} sc \\ u\omega_1 sc & \alpha(\omega_1) c^2 \end{pmatrix}\right)\\
&= (\#_\alpha\phi)\ \#\ \mathrm{Tr}\left(\begin{pmatrix} \omega_0 s^2 & \omega_0 u^{-1} sc \\ u\omega_0 sc & \alpha(\omega_0) c^2 \end{pmatrix}\begin{pmatrix} 0 & Du^{-1} \\ -Du & 0 \end{pmatrix}\begin{pmatrix} \omega_1 s^2 & \omega_1 u^{-1} sc \\ u\omega_1 sc & \alpha(\omega_1) c^2 \end{pmatrix}\right)\\
&= \#_\alpha\phi(-\omega_0 u^{-1}\, Du\omega_1 s^3 c + \omega_0\, Du^{-1} u\omega_1 s^3 c\\
&\quad - \alpha(\omega_0)\, Du\omega_1 u^{-1} sc^3 + u\omega_0\, Du^{-1}\alpha(\omega_1)\, sc^3)\\
&= \#_\alpha\phi((-1)^{\deg\omega_0 + 1} u^{-1}\, Du\omega_0\omega_1 s^3 c + (-1)^{\deg\omega_0}\, Du^{-1} u\omega_0\omega_1 s^3 c\\
&\quad + (-1)^{\deg\omega_0 + 1}\, Du\, u^{-1}\alpha(\omega_0\omega_1)\, sc^3 + (-1)^{\deg\omega_0} u\, Du^{-1}\alpha(\omega_0\omega_1)\, sc^3)\\
&= (-1)^{\deg\omega_0 + 1}\phi(\omega_0\omega_1)\, 2\sin\theta\cos\theta,
\end{aligned}$$

and then

$$\int_0^{\pi/2} I_\theta\, d\theta = (-1)^{\deg\omega_0 + 1}\phi(\omega_0\omega_1).$$

PROPOSITION 9.2. (1) *For an arbitrary cochain* $\phi \in M^*_{n-1}$, *we have*

$$\eta\pi\psi = \psi + \tfrac{1}{2} b(\psi|_1) + \tfrac{1}{2}(b\psi)|_1.$$

(2) *For an arbitrary cyclic cocycle* $\phi \in C^{n-1}_\lambda(\mathscr{A}, \mathscr{A}^*)$, *we have*

$$\eta\ \#_\alpha\phi = (-1)^n n\phi.$$

Proof. Ad. 1. We have, according to Lemma 9.1 and since $\pi\phi = (-1)^n \pi_1\phi$,

$$\begin{aligned}
&\eta\pi\phi(a_0, a_1, ..., a_{n-1})\\
&\quad = (-1)^n(-1)^{n-1}\left[\int_0^{\pi/2} d\theta((\#_\alpha\psi)\ \#\ \mathrm{Tr})\circ\alpha_\theta(\bar{a}_0\, D\mathscr{J}\, d\bar{a}_1 \cdots d\bar{a}_{n-1})\right.\\
&\qquad \left. - \int_0^{\pi/2} d\theta((\#_\alpha\psi)\ \#\ \mathrm{Tr})\circ\alpha_\theta(\bar{a}_0\, D\bar{a}_1\, d\mathscr{J}\, d\bar{a}_2 \cdots d\bar{a}_{n-1})\right]\\
&\quad = (-1)(-\psi(a_0\, da_1 \cdots da_n) - \tfrac{1}{2}\psi(a_0 a_1\, d1\, da_1 \cdots da_{n-1}))\\
&\quad = \psi(a_0, a_1, ..., a_{n-1}) + \tfrac{1}{2}\psi(a_0 a_1, 1, a_2, ..., a_{n-1}).
\end{aligned}$$

But it is easy to check directly the identity

$$((b\psi)|_1 + b(\psi|_1))(a_0, ..., a_{n-1}) = \psi(a_0 a_1, 1, a_2, ..., a_{n-1}).$$

Ad. 2. Since $(\#_\alpha \phi) \# \mathrm{Tr}$ is a cyclic cocycle on $\mathscr{A} \otimes M_2$ we have, denoting by N_a the cyclic antisymmetrization operator in the a-variables, the equality

$$\eta \#_\alpha \phi(a_0, a_1, ..., a_{n-1}) = n N_a \int_0^{\pi/2} d\theta((\#_\alpha \phi) \# \mathrm{Tr}) \circ \alpha_\theta(a_0, ..., a_{n-1}, \mathscr{J}).$$

Using Lemma 9.1 we get for the integral the expression

$$\int_0^{\pi/2} d\theta((\#_\alpha \phi) \# \mathrm{Tr}) \circ \alpha_\theta(\bar{a}_0\, d\bar{a}_1 \cdots d\bar{a}_{n-1}\, D\mathscr{J} + \bar{a}_0\, d\bar{a}_1 \cdots d\bar{a}_{n-2}\, D\bar{a}_{n-1}\, d\mathscr{J})$$
$$= (-1)^n \phi(a_0\, da_1 \cdots da_{n-1}) + \tfrac{1}{2}(-1)^{n-2} \phi((a_0\, da_1 \cdots da_{n-2})\, a_{n-1}\, d1).$$

But

$$\phi((a_0\, da_1 \cdots da_{n-2})\, a^{n-1}\, d1)$$
$$= \phi(a_0\, da_1 \cdots da_{n-2}\, d(a^{n-i} 1)) - \phi(a_0\, da_1 \cdots da_{n-1} 1) = 0,$$

and hence

$$\eta \#_\alpha \phi = (-1)^n n N \phi = (-1)^n n \phi.$$

10. THE LONG EXACT SEQUENCE

Let us put all the pieces together. We have the four cochain maps

$$\pi : M^*_{n-1} \to (Q^*_n)_{\mathrm{hom}},$$
$$\tilde{\#} : M^*_{n-1} \to (Q^*_n)_{\mathrm{hom}},$$
$$h : (Q^*_n)_{\mathrm{hom}} \to M^*_{n-1},$$
$$\eta : (Q^*_n)_{\mathrm{hom}} \to M^*_{n-1}.$$

π and h were constructed in Section 3, $\tilde{\#}$ was constructed in Section 6, and η in Section 9. Moreover, in cohomology, the induced maps satisfy

$$h\pi = \eta\pi = \mathrm{id}, \qquad \pi = \tilde{\#}$$

(see Proposition 9.2(1) for $\eta\pi$, while $h\pi = \mathrm{id}$ on cochain level).

Now let us look at

$$(\pi h)\ \tilde{\#}\eta(\pi h) = \pi(h\tilde{\#})(\eta\pi)h = \pi h,$$

which holds on the cohomology level. Since πh is the identity on the cohomology level, and all maps considered commute with b as cochain maps, we can find a cochain homotopy

$$\bar{\rho}\colon (Q_n^*)_{\mathrm{hom}} \to (Q_{n-1}^*)_{\mathrm{hom}}, \qquad n > 1,$$

such that

$$\tilde{\#}\eta = \mathrm{id} - b\bar{\rho} - \bar{\rho}b. \tag{1}$$

Set

$$\bar{B}\colon (Q_n^*)_{\mathrm{hom}} \to (Q_{n-1}^*)_{\mathrm{hom}},$$

$$\phi \to \frac{1}{2\pi i n(n+1)} B\phi.$$

LEMMA 10.1. *Suppose that ϕ is a homogeneous cyclic cocycle on $\tilde{\mathscr{A}}$ which is zero when restricted to $\mathscr{A}$. Then*

$$\phi = \sum_{k=0}^{\infty} S^k \tilde{\#}\eta(\bar{B}\bar{\rho})^k\phi$$

on the cochain level.

Proof. By (1) we have

$$\phi = \tilde{\#}\eta\phi + b\bar{\rho}\phi.$$

In particular $b\bar{\rho}\phi$ is cyclic, and hence, according to Lemma 34 of [2],

$$b\bar{\rho}\phi = S\bar{B}\bar{\rho}\phi,$$

and

$$\phi = \tilde{\#}\eta\phi + S\bar{B}\bar{\rho}\phi.$$

But note that $\bar{B}\bar{\rho}\phi$ again is a homogeneous cyclic cocycle vanishing on $\mathscr{A}$. By induction on k we get

$$(\bar{B}\bar{\rho})^k\phi = \tilde{\#}\eta(\bar{B}\bar{\rho})^k\phi + S(\bar{B}\bar{\rho})^{k+1}\phi.$$

Acting on both sides with S^k and summing over $k \geqslant 0$ we get

$$\sum_{k\geqslant 0} S^k(\bar{B}\bar{\rho})^k\phi = \sum_{k\geqslant 0} S^k\tilde{\#}\pi(\bar{B}\bar{\rho})^k\phi + \sum_{k\geqslant 0} S^{k+1}(\bar{B}\bar{\rho})^{k+1}\phi,$$

where the sums are finite for dimensional reasons, and hence

$$\phi = \sum_{k \geqslant 0} S^k \tilde{\#} \eta (\bar{B}\bar{\rho})^k \phi.$$

Theorem 10.2. *The following long cohomology sequence is exact*:

$$\cdots \xrightarrow{\delta} H_\lambda^{n-1}(\mathscr{A}) \xrightarrow{\#} H_\lambda^n(\tilde{\mathscr{A}})_{\text{hom}} \xrightarrow{i} H_\lambda^n(\mathscr{A}) \xrightarrow{\delta} \cdots.$$

Proof. 1. $i\# = 0$. Follows directly from the definition of $\#$.

2. Ker $i \subset$ Im $\#$. Suppose ϕ is a homogeneous cyclic n-cocycle on $\tilde{\mathscr{A}}$ and $i[\phi] = 0$ in $H_\lambda^n(\mathscr{A})$. Then we can find a cyclic n-cochain λ on $\mathscr{A}$ such that

$$\phi \,|\, \mathscr{A} = b\lambda.$$

Set

$$\tilde{\lambda}(a_0 u^{m_0}, ..., a_n u^{m_n}) = \lambda(a_0, ..., a_n)\, \delta_{m_0,0} \cdot \cdots \cdot \delta_{m_n,0}, \qquad a_i \in \mathscr{A}.$$

Then $\tilde{\lambda}$ defines a cyclic element of $(Q_n^*)_{\text{hom}}$ and

$$(\phi - b\tilde{\lambda}) \,|\, \mathscr{A} = 0.$$

By Lemma 10.1,

$$\phi - b\tilde{\lambda} = \sum_{k \geqslant 0} S^k \tilde{\#} \eta (\bar{B}\bar{\rho})^k \phi.$$

But, by Proposition 9.2(2) and Definition 8.4,

$$\text{Im } S^k \tilde{\#} = \text{Im } S^k \# \subseteq \text{Im } \#, \qquad k \geqslant 0,$$

and thus

$$\phi - b\tilde{\lambda} \in \text{Im } \#.$$

3. $\#\delta = 0$. This is the content of Proposition 8.2(3).

4. Ker $\# \subset$ Im δ. Suppose that ϕ is a cyclic cocycle on $\mathscr{A}$ such that

$$\#\phi = b\lambda, \qquad \lambda \text{ cyclic}.$$

Since $\#\phi$ is homogeneous, we can assume that λ is homogeneous as well. But then, using Section 9(C) and Proposition 9.2(2), we have

$$\phi = \eta \# \phi = \eta b \lambda = b \eta \lambda \pm \delta(\lambda \,|\, \mathscr{A}).$$

5. $\delta i = 0$. This can be deduced from the fact that inner automorphisms act trivially on the level of cyclic cohomology [2]. Alternatively, given a cyclic cocycle ϕ on $\tilde{\mathscr{A}}$, we have, by Section 9(C), the equality

$$\delta(\phi \,|\, \mathscr{A}) = \pm b\eta\phi.$$

6. Ker $\delta \subset$ Im i. Suppose that ϕ is a cyclic cocycle on $\mathscr{A}$ such that

$$\delta\phi = b\lambda, \qquad \lambda \text{ cyclic cochain on } \mathscr{A}.$$

Set, in notation of Lemma 8.1,

$$\tilde{\phi} = \frac{1}{n}\left(\sum_i (-1)^i \rho_i \phi - \# \lambda\right).$$

Then $\tilde{\phi}$ is cyclic on $\tilde{\mathscr{A}}$, $\tilde{\phi} \,|\, \mathscr{A} = \phi$, and

$$\begin{aligned} nb\tilde{\phi} &= b\left(\sum (-1)^i \rho_i \phi\right) - b \# \lambda \\ &= \#_\alpha \delta\phi - \# b\lambda = 0, \end{aligned}$$

where we used the fact that $\# = \#_\alpha$ on cyclic cocycles, Lemma 8.1(4), and identity (1) of Section 8.

11. Periodic Cyclic Cohomology of the Crossed Product

Theorem 11.1. *The following sequence is exact*:

$$\begin{tikzcd} HC^{\mathrm{ev}}(\mathscr{A}) \arrow[r, "\#"] & HC^{\mathrm{odd}}(\mathscr{A} \hat{\times}_\alpha \mathbb{Z}) \arrow[r, "i"] & HC^{\mathrm{odd}}(\mathscr{A}) \arrow[d, "1-\alpha"] \\ HC^{\mathrm{ev}}(\mathscr{A}) \arrow[u, "1-\alpha"] & HC^{\mathrm{ev}}(\mathscr{A} \hat{\times}_\alpha \mathbb{Z}) \arrow[l, "i"'] & HC^{\mathrm{odd}}(\mathscr{A}). \arrow[l, "\#"'] \end{tikzcd}$$

Proof. According to Corollary 5.6 we have

$$HC(\tilde{\mathscr{A}}) = \varinjlim_k S^k H_\lambda(\mathscr{A}) = \varinjlim_k S^k H_\lambda(\mathscr{A})_{\mathrm{hom}},$$

and hence it suffices to look at the homogeneous cyclic cohomology of the crossed product $\tilde{\mathscr{A}}$. Let us look at the diagram

$$\begin{tikzcd} \cdots \arrow[r, "\delta"] & H_\lambda^{n-1+2k}(\mathscr{A}) \arrow[r, "\#"] & H_\lambda^{n+2k}(\tilde{\mathscr{A}})_{\mathrm{hom}} \arrow[r, "i"] & H_\lambda^{n+2k}(\mathscr{A}) \arrow[r, "\delta"] & \cdots \\ \cdots \arrow[r, "\delta"] & H_\lambda^{n-1}(\mathscr{A}) \arrow[r, "\#"] \arrow[u, "S^k"] & H_\lambda^{n}(\tilde{\mathscr{A}})_{\mathrm{hom}} \arrow[r] \arrow[u, "S^k"] & H_\lambda^{n}(\mathscr{A}) \arrow[r, "\delta"] \arrow[u, "S^k"] & \cdots. \end{tikzcd}$$

It is commutative by Proposition 8.2(2) and the rows are exact by the Theorem 10.2. Now a straightforward diagram chase proves the desired result.

12. COUPLING WITH K-THEORY

Let $\mathscr{A}$ be a Fréchet algebra, nuclear as a topological vector sace. We denote by $\mathscr{A}^+$ the algebra $\mathscr{A}$ with unit adjoined. In what follows we will use the following definition of K-groups of $\mathscr{A}$.

$K_0(\mathscr{A})$: when $\mathscr{A}$ has no unit, $K_0(\mathscr{A})$ is the group of stable equivalence classes of idempotents in matrix algebras over $\mathscr{A}^+$, while in the case when $\mathscr{A}$ has a unit, we take as $K_0(\mathscr{A})$ the kernel of the natural homomorphism induced by the injection of $\mathscr{A}$ as a closed ideal of $\mathscr{A}^+$.

$$K_1(\mathscr{A})\text{: we define } K_1(\mathscr{A}) \text{ as the quotients of}$$
$$GL_\infty(\mathscr{A}) = \{v \in GL_\infty(\mathscr{A}^+) \mid v \equiv 1 \bmod \mathscr{A}\}$$

by the continuous piecewise C^1 equivalence relation $\sim_{C^1}$, given by

$$\begin{aligned} w_1 \sim_{C^1} w_2 \Leftrightarrow \exists &\text{ continuous, piecewise } C^1 \text{ path} \\ &[0, 1] \ni t \to v_t \in GL_\infty(\mathscr{A}) \text{ such} \\ &\text{that } v_0 = w_1 \text{ and } v_1 = w_2. \end{aligned}$$

We will denote by $\langle , \rangle$ the pairing between $K_0(\mathscr{A})$ and $HC^{\text{ev}}(\mathscr{A})$ and between $GL_\infty(\mathscr{A})$ and $HC^{\text{odd}}(\mathscr{A})$ constructed in Propositions 14 and 15 of [2], where we extended cocycles on $\mathscr{A}$ to cocycles on $\mathscr{A}^+$ using the standard device of setting

$$\phi(1, a_1, ..., a_n) = 0.$$

LEMMA 12.1. *The pairing $\langle , \rangle$ descends to a pairing between $K_1(\mathscr{A})$ and $HC^{\text{odd}}(\mathscr{A})$.*

Proof. Suppose that ϕ is an odd-dimensional cyclic cocycle on $\mathscr{A}$ and that $t \to v_t$ is an continuous piecewise C^1 path of elements of $GL_\infty(\mathscr{A})$. By passing to a matrix algebra over $\mathscr{A}^+$ we can assume that $v_t \in \mathscr{A}^+$. It will be enough to show the equality

$$\frac{d}{dt}\phi(v_t^{-1}, v_t, v_t^{-1}, ..., v_t) = 0.$$

Let $\hat{\phi}$ denote the closed graded trace on $\Omega(\mathscr{A}^+)$ corresponding to ϕ. Then the left-hand side is equal to

$$\frac{d}{dt}\hat{\phi}(v_t^{-1}\,dv_t\,d(v_t^{-1})\cdots dv_t)$$
$$=\hat{\phi}((v_t^{-1})^{\cdot}\,dv_t\,d(v_t^{-1})\cdots dv_t)+\hat{\phi}(v_t^{-1}\,d\dot{v}_t\,d(v_t^{-1})\cdots dv_t)+\cdots.$$

Since ϕ is cyclic, it is enough to show that the sum of the first two terms is zero. But

$$\begin{aligned}
&\hat{\phi}((v_t^{-1})^{\cdot}\,dv_t\cdots dv_t)+(v_t^{-1}\,d\dot{v}_t\cdots dv_t)\\
&\quad=-\hat{\phi}(v_t^{-1}\dot{v}_t v_t^{-1}\,dv_t\cdots dv_t)-\hat{\phi}(d(v_t^{-1})\,\dot{v}_t\,d(v_t^{-1})\cdots dv_t)\\
&\quad=-\hat{\phi}(v_t^{-1}\dot{v}_t v_t^{-1}\,dv_t\cdots dv_t)\times\hat{\phi}(v_t^{-1}\,dv_t v_t^{-1}\dot{v}_t\,d(v_t^{-1})\cdots dv_t)\\
&\qquad-\hat{\phi}(d1v_t^{-1}\dot{v}_t\,d(v_t^{-1})\cdots dv_t)\\
&\quad=-\hat{\phi}(v_t^{-1}\dot{v}_t v_t^{-1}\,dv_t\cdots dv_t)+\hat{\phi}(v_t^{-1}\dot{v}_t\,d(v_t^{-1})\cdots dv_t v_t^{-1}\,dv_t)\\
&\qquad-\hat{\phi}(v_t^{-1}\dot{v}_t\,d(v_t^{-1})\cdots dv_t\,d1)=0,
\end{aligned}$$

where we use the identities

$$d(v_t^{-1})1=d1\cdot v_t^{-1}-v_t^{-1}\,dv_t v_t^{-1}$$
$$d(v_t^{-1})\,dv_t v_t^{-1}=d(v_t^{-1})\,d1-d1\,d(v_t^{-1})+v_t^{-1}\,dv_t\,d(v_t^{-1}),$$

and the equality

$$\phi(a_0,\ldots,1,\ldots,a_n)=0.$$

Note. The above proof is essentially a rearrangement of the proof of Lemma 5 of [3].

LEMMA 12.2. *Suppose that $(\mathscr{A},\alpha)$ satisfy the Assumption from Section* 1. *Let ϕ be a cyclic cocycle on $\mathscr{A}$ and ω a cyclic cocycle on $\mathscr{B}$, where $\mathscr{B}$ is either a finite matrix algebra or $C(\mathbb{T})$. Then*

$$\#_{\alpha\otimes\mathrm{id}}(\phi\,\#\,\omega)=(\#_\alpha\phi)\,\#\,\omega.$$

Proof. It is enough to note that both sides of the stated equality are computed by columns of the commutative diagram

$$\begin{tikzcd}
\Omega((\mathscr{A}\otimes_{\mathrm{alg}}\mathscr{B})_{\alpha\otimes\mathrm{id}}[u,u^{-1}]) \arrow[r, phantom, "="] \arrow[d] & \Omega((\mathscr{A}_\alpha[u,u^{-1}])\otimes_{\mathrm{alg}}\mathscr{B}) \arrow[d] \\
E\otimes_{\alpha\otimes\mathrm{id}}(\mathscr{A}\otimes_{\mathrm{alg}}\mathscr{B}) \arrow[d] & \Omega(\mathscr{A}_\alpha[u,u^{-1}])\otimes\Omega(\mathscr{B}) \arrow[d] \\
E\otimes_{\alpha\otimes\mathrm{id}}(\Omega(\mathscr{A})\otimes\Omega(\mathscr{B})) \arrow[r, "\simeq"] & (E\otimes_\alpha\Omega(\mathscr{A}))\otimes\Omega(\mathscr{B}).
\end{tikzcd}$$

The extension of both sides of (1) to continuous cocycles on respective algebras is handled as in Proposition 8.3.

Let us now introduce the Bott map $\bar{b}$ as follows:

$$\bar{b}\colon \text{idempotents in } M_k(\mathscr{A}^+) \to GL_k(\mathscr{A}^+ \otimes C_0^\infty(\mathbb{T}))$$

$$p \mapsto e^{2\pi i t} p + (1-p)$$

$$\bar{b}\colon GL_k(\mathscr{A}_\alpha) \to \text{idempotents in } M_{2k}(\mathscr{A}^+ \otimes C_0^\infty(\mathbb{T}))$$

$$v \mapsto v_t \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} v_t^{-1} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $t \to v_t$ is any continuous piecewise C^∞ path inside $M_2(\mathbb{C}[v, v^{-1}])$ connecting $\left(\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Note that $\bar{b}$ descends to homomorphisms of the K-groups of $\mathscr{A}$, in fact, if we set

$$\mathscr{S}\mathscr{A} = \mathscr{A} \otimes C_0^\infty(\mathbb{T}),$$

we get the homomorphisms

$$\bar{b}\colon K_i(\mathscr{A}) \to K_{i+1}(\mathscr{S}\mathscr{A}), \qquad i \bmod 2.$$

Definition 12.3. $K_i^p(\mathscr{A}) = \varinjlim_n K_i(\mathscr{S}^{2n}\mathscr{A})$, where we consider $\{K_i(\mathscr{S}^{2n}(\mathscr{A}))\}_{n \in N}$ as an inductive system of abelian groups connected by even powers of $\bar{b}$.

Proposition 12.4. *The pairing* $\langle\, , \rangle$ *extends to a bilinear pairing between* K^p*-groups of* $\mathscr{A}$ *and* $HC(\mathscr{A})$.

Proof. Applying Theorem 11.1 to the algebra $\mathscr{A}^+$ with the trivial action of $\mathbb{Z}$ we get short exact sequences

$$0 \to HC^i(\mathscr{A}^+) \to HC^i(\mathscr{A}^+ \otimes C^\infty(\mathbb{T})) \to HC^{i+1}(\mathscr{A}^+) \to 0.$$

These give us the maps

$$\#_{\mathrm{id}}\colon HC^i(\mathscr{A}) \to HC^{i+1}(\mathscr{S}\mathscr{A}).$$

Since, as easily seen from its definition, $\#_{\mathrm{id}}$ is given by the shuffle product with a generator of $H_\lambda^1(C_0^\infty(\mathbb{T}))$, Lemma 10 of [6] gives us the equality

$$\langle \phi, x \rangle = \langle \#_{\mathrm{id}} \phi, \bar{b} x \rangle. \tag{1}$$

Now an application of Lemmas 12.1 and 12.2 gives the desired result (we need Lemma 12.2 to assure that $\#_{\mathrm{id}}^2(\phi \# \mathrm{Tr}) = (\#_{\mathrm{id}}^2 \phi) \# \mathrm{Tr}$ and hence that

$\#^2_{\mathrm{id}}$ is compatible with the identifications involved in the construction of $GL_\infty(\mathscr{A})$ and $M_\infty(\mathscr{A})$).

We shall now apply the above results to the following situation.

$\mathscr{A}$ is a dense unital subalgebra of a unital C^*-algebra A, α is an automorphism of A, $\alpha(\mathscr{A}) = \mathscr{A}$, the pair $(\mathscr{A}, \alpha)$ satisfies the Assumption from Section 1, and the imbedding $\mathscr{A} \to A$ is continuous.

Note that in the category of C^*-algebras the map

$$\bar{b}^2 \colon K_i(A) \to K_i(\mathscr{S}^2 \mathscr{A})$$

is an isomorphism, and hence we have natural maps

$$K_i^p(\mathscr{A}) = \varinjlim_n K_i(\mathscr{S}^{2n} \mathscr{A}) \to \varinjlim_n K_i(\mathscr{S}^{2n} A) = K_i(A),$$

$$K_i^p(\mathscr{A} \hat{\times}_\alpha \mathbb{Z}) = \varinjlim_n K_i(\mathscr{S}^{2n}(\mathscr{A} \hat{\times}_\alpha \mathbb{Z})) \to K_i(A \times_\alpha \mathbb{Z}).$$

THEOREM 12.5. *Suppose that the maps*

$$K_i^p(\mathscr{A}) \to K_i(A), \qquad i = 1, 0,$$

are isomorphisms. Then the maps

$$K_i^p(\mathscr{A} \hat{\times}_\alpha \mathbb{Z}) \to K_i(A \times_\alpha \mathbb{Z}), \qquad i = 1, 0,$$

are surjective and the pairing $\langle\, , \rangle$ *described in Proposition* 12.4, *applied to the smooth crossed product, descends to a pairing between* $HC(\mathscr{A} \hat{\times}_\alpha \mathbb{Z})$ *and the K-groups of the* C^**-crossed product* $A \times_\alpha \mathbb{Z}$:

Proof. We shall start with the diagram

$$\begin{tikzcd}
{} \arrow[r] & K_1^p(\mathscr{A}) \arrow[r, "1-\alpha"] \arrow[d] & K_1^p(\mathscr{A}) \arrow[r] \arrow[d] & K_1^p(\mathscr{A} \hat{\times}_\alpha \mathbb{Z}) \arrow[r] \arrow[d] & K_0^p(\mathscr{A}) \arrow[r, "1-\alpha"] \arrow[d] & K_0^p(\mathscr{A}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & K_1(A) \arrow[r, "1-\alpha"] & K_1(A) \arrow[r] & K_1(A \hat{\times}_\alpha \mathbb{Z}) \arrow[r, "\partial"] & K_0(A) \arrow[r, "1-\alpha"] & K_0(A) \arrow[r] & {}.
\end{tikzcd} \tag{2}$$

The bottom row, being a part of the six-term exact sequence of Pimsner and Voiculescu [7], is exact. We will apply the following result, proved in [4].

The map

$$s \colon \left\{ (e, v) \;\middle|\; \begin{matrix} e \in \text{proj } A,\, v \in \text{unit } A \\ vev^{-1} = \alpha(e) \end{matrix} \right\} \to \{\text{unitaries in } A \times_\alpha \mathbb{Z}\}$$

$$(e, v) \mapsto ue + v(1 - e)$$

is a right inverse for ∂ and its range, after passing to matrix algebras over A, generates $K_1(A \times_\alpha \mathbb{Z})$ as an abelian group.

Since, according to our assumptions, we can always choose e, v in a matrix algebra over $\mathscr{S}^{2n}\mathscr{A}$ for some n, the surjectivity of

$$K_1^p(\mathscr{A} \hat{\times}_\alpha \mathbb{Z}) \to K_1(A \times_\alpha \mathbb{Z})$$

follows.

Suppose now that ϕ is an odd-dimensional cyclic cocycle on $\mathscr{A} \times_\alpha \mathbb{Z}$ and suppose moreover that $e, v, \bar{e}, \bar{v} \in \mathscr{A}$ are such that

$$s(e, v) \sim s(\bar{e}, \bar{v}) \qquad \text{in} \quad A \times_\alpha \mathbb{Z}. \tag{3}$$

Since then $[e] = \partial[s(e, v)] = \partial[s(\bar{e}, \bar{v})] = [\bar{e}]$ in $K_0(A)$, we can (after passing to matrix algebra over some suspension $\mathscr{S}^{2n}\mathscr{A}$) suppose that there exists an invertible element w in $\mathscr{A}$ such that

$$wew^{-1} = \bar{e}.$$

If we set

$$X = \alpha(e) + \alpha^{-1}(w^{-1})\, \bar{v}wv^{-1}(1 - \alpha(e)),$$

then a straightforward calculation gives

$$s(\bar{e}, \bar{v}) = \alpha^{-1}(w)\, Xs(e, v)\, w^{-1}. \tag{4}$$

Moreover, by (3),

$$X \sim 1 \qquad \text{in} \quad A \times_\alpha \mathbb{Z},$$

and hence $[X] \in 1m(1-\alpha) \mid K_1(A) = 1m(1-\alpha) \mid K_1^p(\mathscr{A})$. Since $\phi \mid \mathscr{A}$ is α-invariant in cyclic cohomology, we get, using (4), the equality

$$\langle \phi, s(e, v) \rangle = \langle \phi, s(\bar{e}, \bar{v}) \rangle.$$

This implies in a straightforward way that $\langle\, , \rangle$ does descend to $K_1(A \times_\alpha \mathbb{Z})$.

To deal with the K_0-case, note that the K^p-groups satisfy (by definition) the Bott isomorphism property

$$\bar{b}: K_i^p(\mathscr{A}) \xrightarrow{\cong} K_{i+1}^p(\mathscr{S}\mathscr{A}),$$

and hence it suffices to apply the K_1-case dealt with above to $\mathscr{S}\mathscr{A}$ in the diagram

$$\begin{tikzcd}
K_1^p(\mathscr{S}\mathscr{A} \hat{\times}_\alpha \mathbb{Z}) \arrow[r, "\cong"] \arrow[d] & K_0^p(\mathscr{A} \hat{\times}_\alpha \mathbb{Z}) \arrow[d] \\
K_1(\mathscr{S}\mathscr{A} \times_\alpha \mathbb{Z}) \arrow[r, "\cong"] & K(A \times_\alpha \mathbb{Z});
\end{tikzcd}$$

and note that the pairing $\langle \, , \rangle$ commutes with the Bott map (using Lemma 12.2 applied to $\mathscr{B} = C_0^\infty(\mathbb{T})$ and (1)).

Proposition 12.6. *Suppose that the assumptions of Theorem* 12.5 *are satisfied. Then the maps*

$$\partial \colon K_i(A \times_\alpha \mathbb{Z}) \to K_{i+1}(A), \qquad i = 1, 0 \bmod 2$$

and

$$\# : HC^i(\mathscr{A}) \to HC^{i+1}(\mathscr{A} \hat{\times}_\alpha \mathbb{Z})$$

are dual to each other.

Proof. Since the pairing $\langle \, , \rangle$ is compatible with the maps $\bar{b}$ and $\#_{\mathrm{id}}$ and since, by Lemma 12.2, $\# \#_{\mathrm{id}} = \#_{\mathrm{id}} \#$ ($\#_{\mathrm{id}}$ is a shuffle product), it is enough to show that

$$\frac{1}{2\pi i} \langle \#\phi, [s(e, v)] \rangle$$

holds for $\phi \in H_\lambda^{2n+1}(\mathscr{A})$ and $e, v \in \mathscr{A}$. Suppose first that $v = 1$, i.e.,

$$ueu^{-1} = e.$$

Then, if we set

$$X = ue + 1 - e$$

we have

$$\begin{aligned} &\#\phi(X^{-1} - 1, X - 1, ..., X - 1) \\ &\quad = \sum_{k=0}^{n-1} \#\phi((u^{-1} - 1)^{k+1}(u - 1)^k(du) \\ &\qquad \times (u^{-1} - 1)^{n-k}(u - 1)^{n-k} \, e(de)^{2k} \, e(de)^{2(n-k)}) \\ &\qquad + \sum_{k=1}^{n} \#\phi((u^{-1} - 1)^k(u - 1)^{k+1}(du^{-1}) \\ &\qquad \times (u - 1 - 1)^{n-k}(u - 1)^{n-k} \, e(de)^{2k-1} \, e(de)^{2(n-k)+1}). \end{aligned}$$

Using the identities $e(de)^{2k}e = d(de)^{2k}$, $e(de)^{2k+1}e = 0$, and the α-invariance of e we get

$$\begin{aligned} &\phi(X^{-1} - 1, X - 1, ..., X - 1) \\ &\quad = (n + 1) \times_\alpha \phi((u^{-1} - 1)^{n+1}(u - 1)^n \, due(de)^{2n}) \\ &\quad = (n + 1) \binom{2n + 1}{n} \phi(e, ..., e), \end{aligned}$$

and the result follows in this case.

In general, since $\# \phi \mid \mathscr{A} = 0$ as a cochain, we can suppose that X has the form

$$X = uve + 1 - e, \qquad uve = euv$$

and that there is a C^∞-path of invertibles $v_t \in \mathscr{A}$ such that $v_0 = v$ and $v_1 = 1$. Applying the homotopy invariance of cyclic cohomology proved in Section 4(C) of [2] to the family of homomorphisms

$$\begin{aligned} \rho_t \colon \mathscr{A}_\alpha [u, u^{-1}] &\to \mathscr{A} \times_\alpha \mathbb{Z}, \\ u &\mapsto uv_t, \\ a &\mapsto v_t^{-1} a v_t, \end{aligned}$$

we get, by the above case,

$$\frac{1}{2\pi i} \langle \# \phi, X \rangle = \frac{1}{2\pi i} \langle \# \phi, [v^{-1} ev] \rangle = \langle \phi, [e] \rangle.$$

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