Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 256 (2009) 1618–1641
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Smooth solutions for the motion of a ball in an incompressible perfect fluid

Carole Rosier [a], Lionel Rosier [b,*]

[a] *Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, Université du Littoral, 50 rue F. Buisson, B.P. 699, 62228 Calais Cedex, France*

[b] *Institut Elie Cartan, Université Henri Poincaré Nancy 1, B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France*

Received 28 May 2008; accepted 28 October 2008

Available online 28 November 2008

Communicated by J. Coron

Abstract

In this paper we investigate the motion of a rigid ball surrounded by an incompressible perfect fluid occupying $\mathbb{R}^N$. We prove the existence, uniqueness, and persistence of the regularity for the solutions of this fluid-structure interaction problem.

Keywords: Euler equations; Fluid-rigid body interaction; Exterior domain; Classical solutions

1. Introduction

We consider a homogeneous rigid body occupying a ball $B(t) \subset \mathbb{R}^N$ ($N \geqslant 2$) of radius one and which is surrounded by a homogeneous incompressible perfect fluid. We denote by $\Omega(t) = \mathbb{R}^N \setminus \overline{B(t)}$ the domain occupied by the fluid, and write merely $B = B(0) = \{x;\ |x| < 1\}$ and $\Omega = \Omega(0) = \{x;\ |x| > 1\}$. The equations modeling the dynamics of the system read

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = g, \quad \text{in } \Omega(t) \times [0, T], \tag{1.1}$$

$$\operatorname{div} u = 0, \quad \text{in } \Omega(t) \times [0, T], \tag{1.2}$$

* Corresponding author.
E-mail addresses: rosier@lmpa.univ-littoral.fr (C. Rosier), rosier@iecn.u-nancy.fr (L. Rosier).

doi:10.1016/j.jfa.2008.10.024

$$u \cdot n = \big(h' + r \times (x-h)\big) \cdot n, \quad \text{on } \partial\Omega(t) \times [0,T], \tag{1.3}$$

$$\lim_{|x|\to\infty} u(x,t) = u_\infty, \tag{1.4}$$

$$mh'' = \int\limits_{\partial\Omega(t)} pn\, d\sigma + f_{rb}, \quad \text{in } [0,T], \tag{1.5}$$

$$Jr' = \int\limits_{\partial\Omega(t)} (x-h) \times pn\, d\sigma + T_{rb}, \quad \text{in } [0,T], \tag{1.6}$$

$$u(x,0) = a(x), \quad x \in \Omega, \tag{1.7}$$

$$h(0) = 0 \in \mathbb{R}^N, \qquad h'(0) = b \in \mathbb{R}^N. \tag{1.8}$$

In the above equations, u (respectively p) is the velocity field (respectively the pressure) of the fluid, g is the external force field applied to the fluid (assumed for simplicity to be defined on $\mathbb{R}^N \times [0, T_0]$), f_{rb} (respectively T_{rb}) stands for the external force (respectively the external torque) applied to the rigid body, m (respectively J) is the mass (respectively the inertia matrix) of the ball, h denotes the position of the center of the ball, assumed to be 0 at $t = 0$, r is the angular velocity of the ball, n is the unit outward normal vector to $\partial\Omega$, and u_∞ is a given constant vector. As $x - h = -n$ on $\partial\Omega(t)$, (1.3) reduces to

$$u \cdot n = h' \cdot n, \tag{1.9}$$

whereas (1.6) simplifies into $Jr' = T_{rb}$. It follows that the dynamics of r, which has no influence on the dynamics of u and h, may be ignored.

As in most of fluid-structure interaction problems, one of the main difficulties in proving the wellposedness of (1.1)–(1.8) comes from the fact that the domain occupied by the fluid is *variable* and *not a priori known*. If in the last decade a large number of papers have been devoted to the wellposedness of fluid-structure interaction problems involving a viscous fluid (that is, governed by Navier–Stokes equations), the motion of a rigid body in a (not potential) Eulerian flow has been investigated only in a few papers. In [11], the existence and uniqueness of a (global) classical solution of (1.1)–(1.8) was established when $N = 2$. A result in the same vein was obtained in [12] for a body of arbitrary form, again for $N = 2$. The aim of this paper is to extend the results of [11] to a space of arbitrary dimension N ($N \in \{2, 3\}$ in practice), and to any order of smoothness. We shall for instance establish the existence of C^∞ smooth (global) solutions when $N = 2$. Moreover, the fluid considered here will have a (not necessary null) limit at infinity, and will undergo the action of a force. It is clear that a suitable wellposedness theory is required if we have in mind to prove control results in the spirit of those in [5]. Notice that another application concerns inverse problems. In [4], is was proved that a moving ball surrounded by a potential fluid occupying a bounded domain in $\mathbb{R}^2$ can be detected thanks to a measurement at some time of the velocity of the fluid on some part of the boundary of the domain.

In this paper, the wellposedness of (1.1)–(1.8) is tackled in a direct way, without proving a similar result for Navier–Stokes equations as in [11,12]. This results in a direct and shorter proof. The method of proof combines the study of a variant of Leray projector designed to eliminate the pressure and to take into account the dynamics of the solid, to the classical approach for the wellposedness of Euler equations due to R. Temam [13,14], T. Kato [7], and Kato, Lai [8], which is based upon certain a priori estimates and a Galerkin method. For the sake of shortness, we shall derive the existence of smooth solutions from an abstract result given in [8], although a direct proof as in [13] could certainly be done.

To state the results, we introduce the usual solution $v_\infty = v_\infty(y)$ of the system

$$\operatorname{curl} v_\infty = 0, \quad \text{in } \Omega,$$
$$\operatorname{div} v_\infty = 0, \quad \text{in } \Omega,$$
$$v_\infty \cdot n = 0, \quad \text{on } \partial\Omega,$$
$$\lim_{|y|\to\infty} v_\infty(y) = u_\infty.$$

Simple calculations give

$$v_\infty(y) = u_\infty + \frac{1}{(N-1)|y|^{N+2}}\big(|y|^2 u_\infty - N(u_\infty \cdot y)y\big). \tag{1.10}$$

Notice that $v_\infty(\cdot) - u_\infty \in W^{s,p}(\Omega)$ for all $s \geqslant 0$ and all $p \in (1, +\infty]$. In order to write the equations of the fluid in a *fixed* domain, we perform a change of coordinates. For any $y \in \Omega = \Omega(0)$ and any $t \in [0, T]$, we set $v(y,t) = u(y + h(t), t) - v_\infty(y)$, $q(y,t) = p(y + h(t), t)$, $f(y,t) = g(y + h(t), t)$, and $l(t) = h'(t)$.

Then, the functions (v, q, l) satisfy the following system:

$$\frac{\partial v}{\partial t} + (v_\infty + v - l) \cdot \nabla(v_\infty + v) + \nabla q = f, \quad \text{in } \Omega \times [0, T], \tag{1.11}$$
$$\operatorname{div} v = 0, \quad \text{in } \Omega \times [0, T], \tag{1.12}$$
$$v \cdot n = l \cdot n, \quad \text{on } \partial\Omega \times [0, T], \tag{1.13}$$
$$\lim_{|y|\to\infty} v(y,t) = 0, \tag{1.14}$$
$$ml' = \int_{\partial\Omega} qn\, d\sigma + f_{rb}, \quad \text{in } [0, T], \tag{1.15}$$
$$v(y, 0) = a(y) - v_\infty(y), \quad y \in \Omega, \tag{1.16}$$
$$l(0) = b. \tag{1.17}$$

For the sake of shortness, if H denotes any space of real-valued functions, we write $v \in H$ when each component v_i of v belongs to H. For any $s \geqslant 1$, we denote by $\widehat{H}^s(\Omega)$ the homogeneous Sobolev space

$$\widehat{H}^s(\Omega) = \big\{q \in L^2_{loc}(\overline{\Omega}) \bigm| \nabla q \in H^{s-1}(\Omega)\big\},$$

where $q \in L^2_{loc}(\overline{\Omega})$ means that $q \in L^2(\Omega \cap B_0)$ for all open balls $B_0 \subset \mathbb{R}^N$ with $B_0 \cap \Omega \neq \emptyset$. Throughout the paper, s_0 will denote the number

$$s_0 = [N/2] + 2,$$

so that $H^{s-1}(\Omega) \subset L^\infty(\Omega)$ for each $s \geqslant s_0$. (s is assumed to be an integer.) The main result in this paper is the following one.

Theorem 1.1. *Let $s \geqslant s_0$, $u_\infty \in \mathbb{R}^N$, $a \in v_\infty + H^s(\Omega)$, where v_∞ is given by* (1.10)*, $b \in \mathbb{R}^N$, $f \in C([0,T_0]; H^s(\Omega))$, and $f_{rb} \in C([0,T_0])$. Assume that* $\operatorname{div} a = 0$ *and $(a-b)\cdot n_{|\partial\Omega} = 0$. Then there exist a time $T \leqslant T_0$ and a solution (v,q,l) of* (1.11)–(1.17) *such that $v \in C([0,T]; H^s(\Omega))$, $q \in C([0,T]; \widehat{H}^s(\Omega))$ and $l \in C^1([0,T]; \mathbb{R}^N)$. Such a solution is unique up to an arbitrary function of t which may be added to q. Furthermore, T does not depend on s.*

Remark 1.2. (1) It follows from (1.11) that $v \in C^1([0,T]; H^{s-1}(\Omega))$.

(2) Theorem 1.1 can be extended to the case when the external force field f has a nonzero limit at infinity (e.g. if f stands for the gravity force). Let $f_\infty(t) := \lim_{|y|\to\infty} f(y,t)$ and $\tilde{f}(y,t) := f(y,t) - f_\infty(t)$. If $\tilde{f} \in C([0,T_0]; H^s(\Omega))$ and $(\tilde{v}, \tilde{q}, \tilde{l})$ is the solution of (1.11)–(1.17) corresponding to $a, b, \tilde{f}$ and $\tilde{f}_{rb} = f_{rb} + \int_{\partial\Omega}(f_\infty(t)\cdot y)n\,d\sigma$, then $(v,q,l) = (\tilde{v}, \tilde{q} + f_\infty(t)\cdot y, \tilde{l})$ solves (1.11)–(1.17) with the forcing terms f, f_{rb} in (1.11) and (1.15), respectively.

(3) It is sufficient to prove Theorem 1.1 with $f_{rb} \equiv 0$. Indeed, introducing a function $q_{rb} \in C([0,T_0]; H^{s+1}(\Omega))$ with $\int_{\partial\Omega} q_{rb}(y,t)n\,d\sigma = f_{rb}(t)$ and setting $\hat{q} = q(y,t) + q_{rb}(y,t)$, $\hat{f} = f + \nabla q_{rb}$, then (1.11) and (1.15) hold with $(\hat{q}, \hat{f}, 0)$ substituted to (q, f, f_{rb}). We shall assume thereafter that $f_{rb} \equiv 0$.

Finally, the existence of global smooth solutions can be asserted when $N = 2$.

Corollary 1.3. *Assume that $N = 2$ and that s, T_0, u_∞, a, b, f and f_{rb} are as in Theorem* 1.1*, with* $\operatorname{curl} a \in L^p(\Omega)$ *and* $\operatorname{curl} f \in L^1(0,T_0; L^p(\Omega))$ *for some $p \in [1,2)$. Then we can pick $T = T_0$ in Theorem* 1.1.

We stress that Corollary 1.3 does not follow from [11], since there is a gap between the regularity of the solutions provided in [11] (namely, $v \in C^1(\overline{\Omega}) \cap H^1(\Omega)$) and the minimal regularity required in Theorem 1.1 ($v \in H^3(\Omega)$). To prove Corollary 1.3, we use the well-known fact (see e.g. [2]) that a solution remains smooth as long as its vorticity is uniformly bounded.

If we compare the results in this paper with the ones in [9–12], we notice that no *weighted* Sobolev space is involved here. This follows from the crucial observation that ∇v can be estimated in function of the vorticity $\omega = \operatorname{curl} v$ in the same usual Sobolev space $H^s(\Omega)$, without incorporating any weight.

The paper is outlined as follows. Section 2 provides some background on Kato–Lai theory. Section 3 is concerned with the proof of Theorem 1.1. It begins with the study of the projector which has to be substituted to Leray projector in order to take into account the motion of the rigid ball. Then we apply Kato-Lai theory to a certain abstract system, and we check that the solution provided by that theory is indeed a solution of the original fluid-structure interaction problem. Section 3 is concerned with the proof of Corollary 1.3. It contains the proof of several a priori estimates relating the velocity to the vorticity in an exterior domain.

2. Proof of Theorem 1.1

2.1. Kato–Lai theory

In this section we review briefly Kato–Lai theory and introduce some notations. The reader is referred to [8] for more details. Let V, H, X be three real separable Banach spaces. We say that the family $\{V, H, X\}$ is an *admissible triplet* if the following conditions hold.

(i) $V \subset H \subset X$, the inclusions being dense and continuous.
(ii) H is a Hilbert space, with inner product $(.,.)_H$ and norm $\|\cdot\|_H = (.,.)_H^{\frac{1}{2}}$.
(iii) There is a continuous, nondegenerate bilinear form on $V \times X$, denoted by $\langle .,. \rangle$, such that

$$\langle v, u \rangle = (v, u)_H \quad \text{for all } v \in V \text{ and } u \in H. \tag{2.1}$$

Recall that the bilinear form $\langle v, u \rangle$ is *continuous and nondegenerate* when

$$\bigl|\langle v, u \rangle\bigr| \leqslant C\|v\|_V \|u\|_X \quad \text{for some constant } C > 0; \tag{2.2}$$

$$\langle v, u \rangle = 0 \quad \text{for all } u \in X \text{ implies } v = 0; \tag{2.3}$$

$$\langle v, u \rangle = 0 \quad \text{for all } v \in V \text{ implies } u = 0. \tag{2.4}$$

A map $A : [0,T] \times H \to X$ is said to be *sequentially weakly continuous* if $A(t_n, v_n) \rightharpoonup A(t, v)$ in X whenever $t_n \to t$ and $v_n \rightharpoonup v$ in H. We denote by $C_w([0,T]; H)$ the space of sequentially weakly continuous functions from $[0,T]$ to H, and by $C_w^1([0,T]; X)$ the space of the functions $u \in W^{1,\infty}(0,T; X)$ such that $du/dt \in C_w([0,T], X)$.

We are concerned with the Cauchy problem

$$\frac{dv}{dt} + A(t, v) = 0, \quad t \geqslant 0, \qquad v(0) = v_0. \tag{2.5}$$

The Kato–Lai existence result for abstract evolution equations is as follows.

Theorem 2.1. *(See [8, Theorem A].) Let $\{V, H, X\}$ be an admissible triplet. Let A be a sequentially weakly continuous map from $[0,T] \times H$ into X such that*

$$\bigl\langle v, A(t, v)\bigr\rangle \geqslant -\beta\bigl(\|v\|_H^2\bigr) \quad \text{for } t \in [0,T],\ v \in V, \tag{2.6}$$

where $\beta(r) \geqslant 0$ is a continuous nondecreasing function of $r \geqslant 0$. Then for any $u_0 \in H$ there is a time $T > 0$, $T \leqslant T_0$, and a solution v of (2.5) in the class

$$v \in C_w\bigl([0,T]; H\bigr) \cap C_w^1\bigl([0,T]; X\bigr). \tag{2.7}$$

Moreover, one has

$$\bigl\|v(t)\bigr\|_H^2 \leqslant \gamma(t), \quad t \in [0,T], \tag{2.8}$$

where γ solves the ODE $\gamma'(t) = 2\beta(\gamma(t))$, $\gamma(0) = \|v_0\|_H^2$.

3. Proof of Theorem 1.1

In this section, we put system (1.11)–(1.17) (with $f_{rb} \equiv 0$) in the form (2.5) in order to apply Theorem 2.1.

Pick $s \geqslant s_0$ and define the (uniform) density of the ball as $\rho = m/|B|$, where $|B|$ stands for the Lebesgue measure of the ball B. Let $X = L^2(\mathbb{R}^N)$ be endowed with the scalar product

$$(u, v)_X = \int_\Omega u(x)v(x)\,dx + \rho \int_B u(x)v(x)\,dx.$$

We introduce the (closed) subspace

$$X_* = \{u \in X;\ \operatorname{div} u = 0 \text{ on } \mathbb{R}^N \text{ and } u = const \text{ on } B\}.$$

For any $u \in X_*$, we denote by l_u the unique vector in $\mathbb{R}^N$ such that $u(x) = l_u$ a.e. on B. Let $H = \{u \in X; u_{|\Omega} \in H^s(\Omega)\} = H^s(\Omega) \oplus L^2(B)$ be endowed with the scalar product

$$(u_1, u_2)_H = (u, v)_{H^s(\Omega)} + \rho(u_1, u_2)_{L^2(B)}.$$

Finally, following Kato–Lai, we define V as the space of functions $v \in H$ such that $v_{|\Omega}$ belongs to $\mathcal{D}(S)^N$, where S is the nonnegative selfadjoint operator $S : \mathcal{D}(S) \subset L^2(\Omega) \to L^2(\Omega)$ defined by

$$(Sf, g)_{L^2(\Omega)} = (f, g)_{H^s(\Omega)} \quad \forall f \in \mathcal{D}(S),\ \forall g \in H^s(\Omega).$$

Recall that S is the elliptic operator $Sf = \sum_{|\alpha| \leqslant s} (-1)^{|\alpha|} \partial^{2\alpha}$ with Neumann boundary conditions, and that $\mathcal{D}(S) \subset H^{2s}(\Omega)$. V is endowed with the scalar product

$$(v_1, v_2)_V = (v_1, v_2)_{H^{2s}(\Omega)} + \rho(v_1, v_2)_{L^2(B)}.$$

To emphasize the dependence in s, at some places we shall write X_s, H_s, V_s instead of X, H, V. Clearly, X, H and V are Hilbert spaces, and the inclusions in $V \subset H \subset X$ are continuous and dense. Introduce the bilinear form on $V \times X$

$$\langle v, u \rangle = \left(\sum_{|\alpha| \leqslant s} (-1)^{|\alpha|} \partial^{2\alpha} v, u \right)_{L^2(\Omega)} + \rho(v, u)_{L^2(B)}.$$

Notice that

$$\langle v, u \rangle = (v, u)_H \quad \text{for all } v \in V,\ u \in H.$$

Clearly, the conditions (2.2) and (2.3) are satisfied. (2.4) follows from the self-adjointness of S.

3.1. Determination of the projector

Let P denote the orthogonal projection from the space $X = L^2(\mathbb{R}^N)$, endowed with the scalar product $(.,.)_X$, onto X_*, and $Q = 1 - P$. To prove that $P(H) \subset H$, we need to compute explicitly $P(u)$ for any $u \in X$. This is done in the following proposition.

Proposition 3.1.

(i) *Pick any* $u \in X$. *Then*

$$P(u) = \begin{cases} u - \nabla\varphi & \text{in } \Omega, \\ l & \text{in } B, \end{cases}$$

where $\varphi(x) = \varphi_u(x) + (l \cdot x)(N-1)^{-1}|x|^{-N}$, φ_u *is the unique solution in* $\widehat{H}^1(\Omega)$ *of the elliptic problem*

$$\begin{cases} \Delta\varphi_u = \operatorname{div} u & \text{in } \Omega, \\ \frac{\partial\varphi_u}{\partial n} = u \cdot n & \text{on } \partial\Omega, \end{cases}$$

and

$$l = \left(|B| + \frac{|\partial B|}{N(N-1)}\right)^{-1} \left(\int_B u(x)\,dx + \int_{\partial B} \varphi_u n\,dx\right).$$

(ii) P *maps* H_s *into* H_s *continuously for any* $s \geqslant 1$.

Proof. (i) We write $X_* = X_1 \cap X_2$, where $X_1 := \{u \in X;\ u = const \text{ in } B\}$ and $X_2 := \{u \in X; \operatorname{div} u = 0 \text{ in } \mathbb{R}^N\}$. Obviously $X_1^\perp + X_2^\perp \subset X_*^\perp$. Clearly

$$X_1^\perp = \left\{v \in X;\ v = 0 \text{ in } \Omega \text{ and } \int_B v(x)\,dx = 0\right\}.$$

We claim that

$$X_2^\perp = \left\{v = 1_\Omega \nabla\varphi_\Omega + 1_B \nabla\varphi_B;\ \varphi_\Omega \in \widehat{H}^1(\Omega),\ \varphi_B \in H^1(B) \text{ with } \varphi_\Omega - \rho\varphi_B = 0 \text{ on } \partial\Omega\right\},$$

where 1_Ω and 1_B denote the characteristic functions of Ω and B, respectively. Indeed, if $v \in X_2^\perp$, then by a classical result (see e.g. [15]) there exist two functions $\varphi_\Omega \in \widehat{H}^1(\Omega)$ and $\varphi_B \in H^1(B)$ such that $v = \nabla\varphi_\Omega$ in Ω and $v = \nabla\varphi_B$ in B. Pick any $u \in X_2 \cap H^1(\mathbb{R}^N)$. Then we have

$$\begin{aligned} 0 &= (u, v)_X \\ &= \int_\Omega u \cdot \nabla\varphi_\Omega\,dx + \rho \int_B u \cdot \nabla\varphi_B\,dx \\ &= \int_{\partial\Omega} \varphi_\Omega u \cdot n\,d\sigma + \rho \int_{\partial B} \varphi_B u \cdot n\,d\sigma \\ &= \int_{\partial\Omega} (\varphi_\Omega - \rho\,\varphi_B) u \cdot n\,d\sigma. \end{aligned}$$

This yields $\varphi_\Omega - \rho\varphi_B = 0$ on $\partial\Omega$. The other inclusion is obvious.

We aim to construct two functions $u_1, u_2 \in X$ satisfying

$$u = u_1 + u_2, \tag{3.1}$$

$$\operatorname{div} u_1 = 0 \quad \text{in } \Omega, \tag{3.2}$$

$$u_1 \cdot n = l \cdot n \quad \text{on } \partial\Omega, \tag{3.3}$$

$$u_1 = l \quad \text{in } B, \tag{3.4}$$

$$u_2 = \nabla\varphi_\Omega \quad \text{in } \Omega, \tag{3.5}$$

$$u_2 = \nabla\varphi_B + v \quad \text{in } B \tag{3.6}$$

for some vector $l \in \mathbb{R}^N$, some functions $\varphi_\Omega \in \widehat{H}^1(\Omega)$, $\varphi_B \in H^1(B)$ with $\varphi_\Omega - \rho\varphi_B = 0$ on $\partial\Omega$, and some function $v \in L^2(B)$ with $\int_B v(x)\,dx = 0$. With such a pair (u_1, u_2) at hand, it is clear that $P(u) = u_1$, for $u_1 \in X_*$ and $u_2 \in X_*^\perp$.

We first determine the function φ_Ω. From (3.1)–(3.3) and (3.5), we infer that φ_Ω has to solve

$$\Delta\varphi_\Omega = \operatorname{div} u \quad \text{in } \Omega, \tag{3.7}$$

$$\frac{\partial\varphi_\Omega}{\partial n} = u \cdot n - l \cdot n \quad \text{on } \partial\Omega. \tag{3.8}$$

We seek φ_Ω in the form $\varphi_\Omega = \varphi_u - \varphi_l$, where φ_u and φ_l solve respectively

$$\Delta\varphi_u = \operatorname{div} u \quad \text{in } \Omega, \tag{3.9}$$

$$\frac{\partial\varphi_u}{\partial n} = u \cdot n \quad \text{on } \partial\Omega, \tag{3.10}$$

$$\Delta\varphi_l = 0 \quad \text{in } \Omega, \tag{3.11}$$

$$\frac{\partial\varphi_l}{\partial n} = l \cdot n \quad \text{on } \partial\Omega. \tag{3.12}$$

Clearly, for a very general function $u \in L^2(\mathbb{R}^N)$, the trace $u \cdot n$ on $\partial\Omega$ does not make sense. However, we may define a generalized solution of (3.9)–(3.10) by using a variational formulation. Scaling in (3.9) by $\theta \in \widehat{H}^1(\Omega)$ and integrating by parts, we arrive to

$$\int_\Omega \nabla\varphi_u \cdot \nabla\theta\,dx = \int_\Omega u \cdot \nabla\theta\,dx \quad \text{for all } \theta \in \widehat{H}^1(\Omega). \tag{3.13}$$

According to Riesz representation theorem, for any $u \in L^2(\mathbb{R}^N)$ there exists a unique function $\varphi_u \in \widehat{H}^1(\Omega)$ satisfying (3.13).

Simple computations show that the function

$$\varphi_l(x) = -\frac{1}{N-1}\frac{l \cdot x}{|x|^N} \tag{3.14}$$

is the unique solution of (3.11)–(3.12) in the class $\widehat{H}^1(\Omega)$. Thus $\varphi_\Omega = \varphi_u - \varphi_l$ is the unique solution of (3.7)–(3.8) in $\widehat{H}^1(\Omega)$.

The function u_1 is defined as

$$u_1(x) = \begin{cases} u(x) - \nabla\varphi_\Omega(x) & \text{if } x \in \Omega, \\ l & \text{if } x \in B. \end{cases} \tag{3.15}$$

From (3.7)–(3.8), we infer that $\operatorname{div} u_1 = 0$ in $\mathbb{R}^N$ and that $u_1 \cdot n = l \cdot n$ on $\partial\Omega$, hence $u_1 \in X_*$. Since $\varphi_{\Omega|\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega)$, we may pick a function $\varphi_B \in H^1(B)$ such that

$$\rho\varphi_B = \varphi_\Omega \quad \text{on } \partial\Omega = \partial B. \tag{3.16}$$

Let $v : B \to \mathbb{R}^N$ be defined by $v(x) = u(x) - l - \nabla\varphi_B(x)$ for any $x \in B$. The value of l is imposed by the constraint $\int_B v(x)\,dx = 0$, i.e.,

$$\int_B u(x)\,dx - l|B| - \int_B \nabla\varphi_B\,dx = 0.$$

Note that, by (3.14)–(3.16),

$$\begin{aligned} \int_B \nabla\varphi_B\,dx &= \int_{\partial B} \varphi_B\, n\,d\sigma \\ &= -\rho^{-1}\left(\int_{\partial\Omega} \varphi_u n\,d\sigma - \int_{\partial\Omega} \varphi_l n\,d\sigma\right) \\ &= -\rho^{-1}\left(\int_{\partial\Omega} \varphi_u n\,d\sigma - \frac{|\partial B|}{N(N-1)} l\right). \end{aligned}$$

Therefore

$$l = \left(|B| + \frac{|\partial B|}{\rho N(N-1)}\right)^{-1}\left(\int_B u(x)\,dx + \rho^{-1}\int_{\partial\Omega} \varphi_u n\,d\sigma\right). \tag{3.17}$$

Notice that, for u sufficiently small at infinity, $\int_{\partial\Omega} \varphi_u n\,d\sigma = \int_\Omega u\,dx$, as it can be seen by letting $\theta = x_i$ in (3.13).

Let us proceed to the proof of (ii). Pick any $u \in H_s$ ($s \geqslant 1$), and consider $P(u) = u_1$ where u_1 is defined in (3.15). Clearly, $P(u) \in X$, and to prove that $P(u) \in H^s(\Omega)$, it is sufficient to show that $\nabla\varphi_u \in H^s(\Omega)$. Observe that φ_u is defined up to an additive constant. To fix that constant we may impose the condition

$$\int_{1<|x|<2} \varphi_u(x)\,dx = 0. \tag{3.18}$$

Introduce first a cutoff function $\rho_1 \in C^\infty(\mathbb{R}^N; [0,1])$ such that

$$\rho_1(x) = 1 \quad \text{for } |x| \leqslant 2 \quad \text{and} \quad \rho_1(x) = 0 \quad \text{for } |x| \geqslant 3. \tag{3.19}$$

The function $\varphi_1(x) = \rho_1(x)\varphi_u(x)$ solves the system

$$\Delta\varphi_1 = (\operatorname{div} u)\rho_1 + 2\nabla\rho_1 \cdot \nabla\varphi_u + \Delta\rho_1\varphi_u \quad \text{for } 1 < |x| < 4, \tag{3.20}$$

$$\frac{\partial\varphi_1}{\partial n} = u \cdot n \quad \text{for } |x| = 1, \tag{3.21}$$

$$\frac{\partial\varphi_1}{\partial n} = 0 \quad \text{for } |x| = 4. \tag{3.22}$$

$$\int_{1<|x|<2} \varphi_1(x)\,dx = 0. \tag{3.23}$$

From (3.9) and classical (interior) elliptic regularity, we have that

$$\begin{aligned}\|\varphi_u\|_{H^{s+1}(\{\frac{3}{2}<|x|<\frac{7}{2}\})} &\leqslant C\big(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|\varphi_u\|_{L^2(\{\frac{3}{2}<|x|<\frac{7}{2}\})}\big)\\ &\leqslant C\big(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|u\|_{L^2(\Omega)}\big).\end{aligned}$$

By (boundary) elliptic regularity applied to the system (3.20)–(3.22), we obtain that $\varphi_1 \in H^{s+1}(\{1 < |x| < 4\})$ with

$$\|\varphi_1\|_{H^{s+1}(\{1<|x|<4\})} \leqslant C\big(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|u\|_{L^2(\Omega)} + \|u\cdot n\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\big).$$

This implies that $\varphi_u \in H^{s+1}(\{1 < |x| < 2\})$ with

$$\|\varphi_u\|_{H^{s+1}(\{1<|x|<2\})} \leqslant C\big(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|u\|_{L^2(\Omega)} + \|u\cdot n\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\big). \tag{3.24}$$

Next, we introduce a function $\rho_2 \in C^\infty(\mathbb{R}^N; [0, 1])$ such that

$$\rho_2(x) = 0 \quad \text{for } |x| < 5/4 \quad \text{and} \quad \rho_2(x) = 1 \quad \text{for } |x| > 3/2. \tag{3.25}$$

Then the function $\varphi_2(x) = \rho_2(x)\varphi_u(x)$ belongs to $\widehat{H}^1(\mathbb{R}^N)$ and it solves

$$\Delta\varphi_2 = f_2 := (\operatorname{div} u)\rho_2 + 2\nabla\rho_2 \cdot \nabla\varphi_u + \Delta\rho_2\,\varphi_u. \tag{3.26}$$

Notice that $f_2 \in H^{s-1}(\mathbb{R}^N)$ with

$$\begin{aligned}\|f_2\|_{H^{s-1}(\mathbb{R}^N)} &\leqslant C\big(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|\varphi_u\|_{H^s(\{1<|x|<2\})}\big)\\ &\leqslant C\big(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|u\|_{L^2(\Omega)} + \|u\cdot n\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\big)\end{aligned} \tag{3.27}$$

by virtue of (3.24). Using Fourier transform we obtain

$$\begin{aligned}\|\nabla\varphi_2\|_{H^s(\mathbb{R}^N)} &\leqslant C\big(\|\Delta\varphi_2\|_{H^{s-1}(\mathbb{R}^N)} + \|\nabla\varphi_2\|_{L^2(\mathbb{R}^N)}\big)\\ &\leqslant C\big(\|f_2\|_{H^{s-1}(\mathbb{R}^N)} + \|\varphi_u\|_{L^2(\{1<|x|<2\})} + \|\nabla\varphi_u\|_{L^2(\Omega)}\big)\\ &\leqslant C\big(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|u\|_{L^2(\Omega)} + \|u\cdot n\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\big)\end{aligned} \tag{3.28}$$

by (3.27), (3.18) and (3.13). We conclude that $\nabla\varphi_u \in H^s(\Omega)$, with

$$\|\nabla\varphi_u\|_{H^s(\Omega)} \leqslant C\big(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|u\|_{L^2(\Omega)} + \|u\cdot n\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\big). \tag{3.29}$$

The proof of Proposition 3.1 is complete. □

3.2. Definition of the map $A(t,v)$

For $u \in L^2_{loc}(\overline{\Omega})$, $v \in H$ and $t \in [0,T_0]$, we set $F(u,v) = (u\cdot\nabla)v = u(\nabla v)$ (where $\nabla v = (\frac{\partial v}{\partial x})^T$) and

$$\begin{aligned} A(t,v) = {} & 1_\Omega F(v_\infty + Pv - l_{Pv}, v_\infty + v) \\ & - Q\big[1_\Omega F(v_\infty + Pv - l_{Pv}, v_\infty + Pv)\big] - P\big[1_\Omega f(t)\big] \end{aligned} \tag{3.30}$$

where $P : X \to X$ and $Q = I - P : X \to X$ are the projectors from X to X_* and to $X_*^\perp$, respectively, and 1_Ω denotes the characteristic function of the set Ω.

We first check that $A(t,v) \in X$ for $v \in H$, and that $A(t,v) \in H$ for $v \in V$. We focus on the first term in the right-hand side of (3.30), the second one being similar and the last one causing no trouble. Assume first that $v \in H$ only. Then $Pv \in H$, hence the function

$$F(v_\infty + Pv - l_{Pv}, v_\infty + v) = \underbrace{(v_\infty + Pv - l_{Pv})}_{\in W^{s,\infty}(\Omega)+H^s(\Omega)+\mathbb{R}^N}\ \underbrace{\nabla(v_\infty + v)}_{\in H^{s-1}(\Omega)}$$

belongs to $H^{s-1}(\Omega)$, for $H^{s-1}(\Omega) \subset L^\infty(\Omega)$. This proves in particular that $1_\Omega F(v_\infty + Pv - l_{Pv}, v_\infty + v) \in X$. The same argument, with $H^{2s}(\Omega)$ substituted to $H^s(\Omega)$, gives that $1_\Omega F(v_\infty + Pv - l_{Pv}, v_\infty + v) \in H$ for $v \in V$. The weak continuity of the map A from $[0,T_0]\times H$ to $X = L^2(\mathbb{R}^N)$ is clear. Indeed, $P[1_\Omega f] \in C([0,T];H)$ and if $v_n \rightharpoonup v$ in H, then Pv_n and ∇v_n converge weakly in $H^{s-1}(\Omega)$, hence locally uniformly, towards Pv and ∇v, respectively. Therefore

$$(v_\infty + Pv_n - l_{Pv_n})\cdot\nabla(v_\infty + v_n) \to (v_\infty + Pv - l_{Pv})\cdot\nabla(v_\infty + v) \quad \text{in } \mathcal{D}'(\Omega),$$

and the weak convergence in $L^2(\Omega)$ also holds.

It remains to check that the condition (2.6) is satisfied. Pick any pair $(t,v) \in [0,T_0]\times V$. Then

$$\begin{aligned} \big|\langle v, A(t,v)\rangle\big| &= \big|\big(v, A(t,v)\big)_H\big| \\ &\leqslant \big|\big(P[1_\Omega f], v\big)_H\big| + \big|\big(1_\Omega F(v_\infty + Pv - l_{Pv}, v_\infty), v\big)_H\big| \\ &\quad + \big|\big(1_\Omega F(v_\infty + Pv - l_{Pv}, v), v\big)_H\big| \\ &\quad + \big|\big(Q\big[1_\Omega F(v_\infty + Pv - l_{Pv}, v_\infty + Pv)\big], v\big)_H\big| \\ &= |I_1| + |I_2| + |I_3| + |I_4|. \end{aligned}$$

Clearly,

$$|I_1| \leqslant C\|f\|_{L^\infty(0,T;H^s(\Omega))}\|v\|_H. \tag{3.31}$$

On the other hand

$$|I_2| \leqslant \big\| F(v_\infty + Pv - l_{Pv}, v_\infty)\big\|_{H^s(\Omega)} \|v\|_{H^s(\Omega)} \\ \leqslant C\big(1 + \|v\|_H\big)\|v\|_H. \tag{3.32}$$

To estimate I_3 and I_4 we introduce the notations $w = v_\infty + Pv$ and $l = l_{Pv}$. Then

$$|I_3| = \big|\big(F(w-l, v), v\big)_{H^s(\Omega)}\big| = \bigg| \sum_{|\alpha|\leqslant s,\ 1\leqslant i\leqslant N} \int_\Omega \partial^\alpha\big[(w-l)\cdot\nabla v_i\big]\partial^\alpha v_i\,dx\bigg|. \tag{3.33}$$

Any integral term in the right-hand side of (3.33) may be written as a sum of terms of the form

$$\int_\Omega \big[\partial^{\alpha_1}(w-l)\cdot\nabla\partial^{\alpha_2} v_i\big]\partial^\alpha v_i\,dx \tag{3.34}$$

with $|\alpha_1| + |\alpha_2| = |\alpha| =: s' \leqslant s$. We first note that the integral term in (3.34) vanishes when $\alpha_2 = \alpha$, since $\operatorname{div}(w-l) = 0$ in Ω and $(w-l)\cdot n = 0$ on $\partial\Omega$. We may therefore assume that $|\alpha_2| \leqslant s' - 1$, hence $|\alpha_1| \geqslant 1$. By combining a classical estimate (see e.g. [1]) to an extension argument, we infer that for any $\sigma \in \mathbb{N}$

$$\big\|\big(\partial^{\beta_1} f\big)\big(\partial^{\beta_2} g\big)\big\|_{L^2(\Omega)} \leqslant C\big(\|f\|_{H^\sigma(\Omega)}\|g\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\|g\|_{H^\sigma(\Omega)}\big) \tag{3.35}$$

for all functions $f, g \in H^\sigma(\Omega) \cap L^\infty(\Omega)$ and all multi-indices β_1, β_2 with $|\beta_1| + |\beta_2| = \sigma$. This gives

$$\big\|\partial^{\alpha_1}(w-l)\cdot\nabla\partial^{\alpha_2} v_i\big\|_{L^2(\Omega)} \leqslant \big(\|\nabla w\|_{H^{s'-1}(\Omega)}\|\nabla v\|_{L^\infty(\Omega)} + \|\nabla w\|_{L^\infty(\Omega)}\|\nabla v\|_{H^{s'-1}(\Omega)}\big).$$

We conclude that

$$|I_3| \leqslant C\big(1 + \|v\|_{H_{s_0}}\big)\|v\|_H^2 \tag{3.36}$$

where we recall that H_{s_0} denotes the space H for $s = s_0$.

Let us estimate $|I_4|$. Using still the notations $w = v_\infty + Pv$, $l = l_{Pv}$, we have that

$$|I_4| \leqslant \big\|Q\big[1_\Omega F(w-l, w)\big]\big\|_{H^s(\Omega)}\|v\|_{H^s(\Omega)} + C\big\|F(w-l, w)\big\|_{L^2(\Omega)}\|v\|_{L^2(B)}.$$

Noticing that $\|w\|_{L^\infty(\Omega)} \leqslant C(1 + \|v\|_{H_{s_0}})$, we infer that

$$\big\|F(w-l, w)\big\|_{L^2(\Omega)}\|v\|_{L^2(B)} \leqslant C\big(1 + \|v\|_{H_{s_0}}\big)^2\|v\|_{H_{s_0}}.$$

On the other hand, $Q[1_\Omega F(w-l, w)]_{|\Omega} = \nabla\varphi$, where φ solves the problem

$$\Delta\varphi = \operatorname{div}\big[(w-l)\cdot\nabla w\big] \quad \text{in } \Omega,$$
$$\frac{\partial\varphi}{\partial n} = \big[(w-l)\cdot\nabla w\big]\cdot n - L\cdot n \quad \text{on } \partial\Omega$$

for some vector $L \in \mathbb{R}^N$ with

$$|L| \leqslant C \big\| 1_\Omega F(w-l, w) \big\|_X \leqslant C\big(1 + \|v\|_{H_{s_0}}\big)^2.$$

According to (3.29) and (3.14), we have that

$$\|\nabla\varphi\|_{H^s(\Omega)} \leqslant C\Big(\big\|(w-l)\cdot\nabla w\big\|_{L^2(\Omega)} + \big\|\operatorname{div}\big[(w-l)\cdot\nabla w\big]\big\|_{H^{s-1}(\Omega)} + \big\|\big[(w-l)\cdot\nabla w\big]\cdot n\big\|_{H^{s-\frac{1}{2}}(\partial\Omega)} + |L|\Big).$$

Using the fact that $w-l$ is divergence-free, we obtain that

$$\begin{aligned}\big\|\operatorname{div}\big[(w-l)\cdot\nabla w\big]\big\|_{H^{s-1}(\Omega)} &= \Bigg\| \sum_{i,j=1}^N \frac{\partial w_i}{\partial x_j}\frac{\partial w_j}{\partial x_i} \Bigg\|_{H^{s-1}(\Omega)} \\ &\leqslant C \sum_{i,j=1}^N \bigg\|\frac{\partial w_i}{\partial x_j}\bigg\|_{H^{s-1}(\Omega)} \bigg\|\frac{\partial w_j}{\partial x_i}\bigg\|_{L^\infty(\Omega)} \\ &\leqslant C\big(1 + \|v\|_{H_{s_0}}\big)\big(1 + \|v\|_H\big). \end{aligned} \tag{3.37}$$

To estimate the boundary term, we proceed as in [3]. Let $\delta(x) = (|x|^2 - 1)/2$ for $x \in \Omega$, so that $\nabla\delta = -n$ on $\partial\Omega$. From $(w-l)\cdot\nabla\delta = 0$ on $\partial\Omega$ we obtain upon differentiation $(w-l)\cdot\nabla[(w-l)\cdot\nabla\delta] = 0$. Thus

$$\sum_{i,j=1}^N (w_i - l_i)\frac{\partial w_j}{\partial x_i} n_j = \sum_{i,j=1}^N (w_i - l_i)(w_j - l_j)\frac{\partial^2\delta}{\partial x_i \partial x_j}$$

and

$$\begin{aligned}\big\|\big[(w-l)\cdot\nabla w\big]\cdot n\big\|_{H^{s-\frac{1}{2}}(\partial\Omega)} &\leqslant C\|w-l\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\|w-l\|_{L^\infty(\partial\Omega)} \\ &\leqslant C\big(1 + \|v\|_{H_{s_0}}\big)\big(1 + \|v\|_H\big). \end{aligned} \tag{3.38}$$

We conclude that

$$|I_4| \leqslant C\big(1 + \|v\|_{H_{s_0}}\big)\big(1 + \|v\|_H\big)\|v\|_H. \tag{3.39}$$

Gathering together (3.31), (3.32), (3.36) and (3.39), we infer that

$$|I_1| + |I_2| + |I_3| + |I_4| \leqslant C\|v\|_H\big(1 + \|v\|_H\big)\big(1 + \|v\|_{H_{s_0}}\big). \tag{3.40}$$

The condition (2.6) is therefore satisfied with $\beta(r) = C\sqrt{r}(1+r)$.

According to Theorem 2.1, for any $v_0 \in H$ and any $f \in C([0, T_0], H^s(\Omega))$ there exist a time $T \leqslant T_0$ and a solution

$$v \in C_w\big([0,T]; H\big) \cap C_w^1\big([0,T]; X\big) \tag{3.41}$$

of the Cauchy problem

$$v_t + A(t, v) = 0, \quad t \geqslant 0, \tag{3.42}$$

$$v(0) = v_0. \tag{3.43}$$

3.3. Solution of the system (1.11)–(1.17)

Let u_∞, a and b be as in the statement of Theorem 1.1, and set

$$v_0(x) = \begin{cases} a(x) - v_\infty(x) & \text{if } x \in \Omega, \\ b & \text{if } x \in B. \end{cases}$$

Let v be the solution of (3.42)–(3.43) emanating from the initial state v_0. We check that it gives a solution to (1.11)–(1.17). We begin with the

Claim 1. $v(t) \in H_* := H \cap X_*$ *for all* $t \in [0, T]$.

Proof. Notice first that $v(0) = v_0 \in H_*$. Applying Q to each term in (3.42) and taking the inner product with $Qv(t)$ in X yields

$$0 = \frac{d}{dt}\frac{1}{2}\|Qv(t)\|_X^2 + \big(Qv, QA(t, v)\big)_X.$$

On the other hand

$$\begin{aligned}\big(Qv, QA(t, v)\big)_X &= \big(Qv, Q\big[1_\Omega F(v_\infty + Pv - l_{Pv}, v_\infty + v) \\ &\quad - 1_\Omega F(v_\infty + Pv - l_{Pv}, v_\infty + Pv)\big]\big)_X \\ &= \int_\Omega Qv \cdot \big[(v_\infty + Pv - l_{Pv}) \cdot \nabla(Qv)\big]\, dx \\ &= 0\end{aligned}$$

since $\operatorname{div}[v_\infty + Pv - l_{Pv}] = 0$ in Ω and $(v_\infty + Pv - l_{Pv}) \cdot n = 0$ on $\partial\Omega$. We infer that $\|Qv\|_X^2 = \|Qv(0)\|_X^2 = 0$, i.e., $v(t) \in X_*$ for all $t \in [0, T]$. □

Since $Pv(t) = v(t)$ for all t, (3.42) may be rewritten

$$v_t + P\big[1_\Omega F(v_\infty + v - l, v_\infty + v) - 1_\Omega f\big] = 0, \tag{3.44}$$

where $l(t) = l_{v(t)}$. The regularity of v depicted in (3.41) can be slightly improved by adapting an argument in [8]. First, we claim that the solution of (3.44)–(3.43) is *unique* in the class (3.41). Indeed, if v_1 and v_2 are two solutions, then we have for a.e. t

$$\frac{d}{dt}\|v_1 - v_2\|_X^2 = -2\big(v_1 - v_2, 1_\Omega\big[F(v_1 - l_1 - v_2 + l_2, v_\infty + v_1)\big]\big)_X \leqslant C\|v_1 - v_2\|_X^2.$$

(We used the fact that $v_i(t) \in X_*$ for $t \in [0,T]$ and $i = 1,2$.) An application of Gronwall lemma yields $v_1 \equiv v_2$. On the other hand, the solution of (3.44) is strongly continuous in H at $t = 0$, since $v(t) \rightharpoonup v_0$ in H as $t \to 0$ and $\limsup_{t\to 0} \|v(t)\|_H^2 \leqslant \lim_{t\to 0} \gamma(t) = \|v_0\|_H^2$. This implies that v is right-continuous at any t, by uniqueness. As the equation is time-reversible, v is continuous from $[0,T]$ to H. Therefore, using (3.44), we infer that

$$v \in C\big([0,T]; H_s\big) \cap C^1\big([0,T]; H_{s-1}\big). \tag{3.45}$$

Claim 2. (v,l) *solves* (1.11)–(1.17).

Proof. Taking the inner product in X of (3.44) with a test function $\phi \in X_*$ (hence $P\phi = \phi$) gives

$$ml' \cdot l_\phi + \int_\Omega \big(v' + (v_\infty + v - l)\cdot \nabla(v_\infty + v) - f\big)\cdot \phi\, dx = 0. \tag{3.46}$$

Pick first as a test function $\phi(x) = 1_\Omega(x)\psi(x)$, where ψ is any function in $C_0^\infty(\Omega)^N$ satisfying $\operatorname{div}\psi = 0$. Then (3.46) yields $\int_\Omega (v' + (v_\infty + v - l)\cdot\nabla(v_\infty + v) - f)\cdot\psi\, dx = 0$. It follows then that there exists a function $q \in C([0,T]; \widehat{H}^s(\Omega))$ such that

$$v' + (v_\infty + v - l)\cdot\nabla(v_\infty + v) + \nabla q = f \quad \text{in } \Omega \times [0,T]. \tag{3.47}$$

Thus (1.11) is fulfilled. (1.12) and (1.13) follow from the fact that $v(t) \in X_*$ for each t. Taking the scalar product in $L^2(\Omega)$ of (3.47) with $\phi \in X_*$ and comparing with (3.46), we obtain after some integration by part

$$-ml' \cdot l_\phi + l_\phi \cdot \int_{\partial\Omega} qn\, d\sigma = 0.$$

(1.15) (with $f_{rb} \equiv 0$) follows at once, by arbitrariness of l_ϕ. The uniqueness of (v,q,l) may be obtained as in [11] by energy estimates. Finally, the persistence follows from the a priori estimate

$$\big|\big(v, A(t,v)\big)_{H_s}\big| \leqslant C\big(1 + \|v(t)\|_{H_{s_0}}\big)\big(1 + \|v(t)\|_{H_s}^2\big)$$

which prevents any blow-up in H_s whereas the solution exists in H_{s_0}. □

4. Proof of Corollary 1.3

By virtue of the persistence of the regularity stated in Theorem 1.1, it is sufficient to prove the result for $s = s_0 = 3$. We shall establish several a priori estimates which will be used thereafter to show that the $H^2(\Omega)$ norm of the vorticity does not blow up in finite time.

We introduce a few additional notations. For any $y = (y_1, y_2) \in \mathbb{R}^2$, we set $y^\perp = (-y_2, y_1)$. n still denotes the unit outward vector to $\partial\Omega$, and $\tau = -n^\perp$ on $\partial\Omega$. For any vector field $v = (v_1, v_2)$, $\operatorname{curl} v = \partial v_2/\partial y_1 - \partial v_1/\partial y_2$, $\operatorname{div} v = \partial v_1/\partial y_1 + \partial v_2/\partial y_2$. Finally, for any scalar function ψ, $\operatorname{curl}\psi = -(\nabla\psi)^\perp = (\partial\psi/\partial y_2, -\partial\psi/\partial y_1)$.

Let us begin with an estimate related to the conservation of the kinetic energy of the system fluid + rigid in the absence of any forcing term.

Lemma 4.1. *Let $E(t)=\frac{1}{2}(m|l(t)|^2+\int_\Omega |v(y,t)|^2\,dy)$. Then there exists a constant $C=C(u_\infty)$ such that for all $t\in[0,T_0]$ it holds*

$$E(t)\leqslant e^{Ct}\left(E(0)+C\int_0^t e^{-Cs}\left(1+\|f(s)\|^2_{L^2(\Omega)}\right)ds\right). \tag{4.1}$$

Proof. Scaling in (1.11) by v, using (1.15) and the fact that $\operatorname{div}(v_\infty+v-l)=0$ in Ω and $(v_\infty+v-l)\cdot n=0$ on $\partial\Omega$, we arrive to

$$\begin{aligned}\frac{dE}{dt}&=\int_\Omega f\cdot v\,dy-\int_\Omega\left[(v_\infty+v-l)\cdot\nabla v_\infty\right]\cdot v\,dy\\&\leqslant C\left(E+1+\|f(t)\|^2_{L^2(\Omega)}\right)\end{aligned}$$

where C is a constant depending only on v_∞. (4.1) follows at once. □

Assume given a velocity field $v\in C(\overline{\Omega})$ satisfying the system

$$\operatorname{div} v=0\quad\text{in }\Omega, \tag{4.2}$$

$$\operatorname{curl} v=\omega\quad\text{in }\Omega, \tag{4.3}$$

$$v\cdot n=l\cdot n\quad\text{on }\partial\Omega, \tag{4.4}$$

$$\lim_{|y|\to\infty} v(y)=0, \tag{4.5}$$

$$\int_{\partial\Omega} v\cdot\tau\,d\sigma=\lambda, \tag{4.6}$$

where $l\in\mathbb{R}^2$ and $\lambda\in\mathbb{R}$ (the circulation) are given.

We aim to prove the following estimates.

(1) *Velocity versus vorticity.* For any $s\in\mathbb{N}^*$, any $p\in[1,2)$, and any smooth solution v of (4.2)–(4.6),

$$\|v\|_{L^\infty(\Omega)}+\|\nabla v\|_{H^s(\Omega)}\leqslant C\left(\|\omega\|_{L^p(\Omega)}+\|\omega\|_{H^s(\Omega)}+|l|+|\lambda|\right). \tag{4.7}$$

Furthermore, for any $q\in(1,+\infty)$

$$\|\nabla v\|_{L^q(\Omega)}\leqslant C\left(\|\omega\|_{L^q(\Omega)}+|l|+|\lambda|\right), \tag{4.8}$$

$$\sum_{1\leqslant|\alpha|\leqslant 2}\left\|\partial^\alpha v\right\|_{L^q(\Omega)}\leqslant C\left(\|\omega\|_{W^{1,q}(\Omega)}+|l|+|\lambda|\right). \tag{4.9}$$

(2) *Uniform bound of the gradient of the velocity.* For any $v\in H^3(\Omega)$ satisfying (4.2)–(4.6)

$$\|\nabla v\|_{L^\infty(\Omega)} \leqslant C\big(1 + \|\omega\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} + \big(1 + \ln^+ \|v\|_{H^3(\Omega)}\big)\|\omega\|_{L^\infty(\Omega)} + |l| + |\lambda|\big). \tag{4.10}$$

where $\ln^+ r = \ln r$ if $r \geqslant 1$, $\ln^+ r = 0$ if $r < 1$.

(4.8), which has been established in [11], is given here for the sake of completeness. It will not been used thereafter. (4.10) is a variant of a classical estimate from [2].

Proposition 4.2. *Let $s \geqslant 1$, $p \in [1,2)$ and*

$$X = \big\{v \in C(\overline{\Omega});\ \nabla v \in H^s(\Omega), \operatorname{curl} v \in L^p(\Omega)\big\}.$$

Then for any $\omega \in H^s(\Omega) \cap L^p(\Omega)$, any $l \in \mathbb{R}^2$ and any $\lambda \in \mathbb{R}$, there exists a unique $v \in X$ fulfilling (4.2)–(4.6)*, and* (4.7) *is satisfied. If, in addition, $\omega \in W^{1,q}(\Omega)$, then* (4.9) *holds. Finally, if $v \in H^3(\Omega)$, then* (4.10) *holds.*

Proof. The uniqueness comes from [9, Lemma 2.14], which asserts that the only $v \in C(\overline{\Omega})$ fulfilling (4.2)–(4.6) with $\omega \equiv 0$ and $(l,\lambda) = (0,0)$ is $v = 0$. Let us now prove the existence of v. Pick any $\omega \in H^s(\Omega) \cap L^p(\Omega)$ with a compact support in $\overline{\Omega}$ (say $\omega(y) = 0$ for $|y| > R$) and any $(l,\lambda) \in \mathbb{R}^3$. Let Λ be an extension operator, bounded from $H^s(\Omega)$ to $H^s(\mathbb{R}^2)$ and from $L^q(\Omega)$ to $L^q(\mathbb{R}^2)$ for any $q \in [1,+\infty]$ (see e.g. [6]), and let $\omega_1 = \Lambda(\omega)$. Then

$$\|\omega_1\|_{H^s(\mathbb{R}^2)} \leqslant C\|\omega\|_{H^s(\Omega)}, \qquad \|\omega_1\|_{L^p(\mathbb{R}^2)} \leqslant C\|\omega\|_{L^p(\Omega)}. \tag{4.11}$$

(Here and in what follows, C denotes a constant which may vary from line to line, but which is independent of the support of ω.) Set

$$\psi_1(y) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|y-z|\omega_1(z)\,dz \quad \text{and} \quad v_1(y) = \operatorname{curl}\psi_1(y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(y-z)^\perp}{|y-z|^2}\omega_1(z)\,dz.$$

Notice first that $-\Delta\psi_1 = \omega_1$, which yields $|\xi|^2\hat{\psi}_1(\xi) = \hat{\omega}_1(\xi)$ and

$$\widehat{(\partial_j\partial_k\partial^\alpha\psi_1)} = -\frac{\xi_j\xi_k}{|\xi|^2}\widehat{\partial^\alpha\omega_1} \quad \text{for } j,k \in \{1,2\},\ |\alpha| \leqslant s.$$

As $\nabla\partial^\alpha v_1$ is a 2×2 matrix with $\pm\partial_j\partial_k\partial^\alpha\psi_1$ as coefficients, and $|\xi_j\xi_k|/|\xi|^2 \leqslant const$, we infer that

$$\|\nabla v_1\|_{H^s(\mathbb{R}^2)} \leqslant C\|\omega_1\|_{H^s(\mathbb{R}^2)} \leqslant C\|\omega\|_{H^s(\Omega)}. \tag{4.12}$$

Obviously, $\operatorname{div} v_1 = 0$ and $\operatorname{curl} v_1 = \omega_1$ on $\mathbb{R}^2$. We claim that $v_1 \in L^\infty(\mathbb{R}^2)$ and that $v_1(y) \to 0$ as $|y| \to \infty$. Indeed, splitting the integral term defining v_1 into two parts, corresponding to $|y-z| \leqslant 1$ and $|y-z| > 1$, we arrive to

$$\left|v_1(y)\right| \leqslant C\left(\left(\int_0^1 r^{1-p}\,dr\right)^{1/p} \cdot \|\omega_1\|_{L^{p'}(\mathbb{R}^2)} + \left(\int_1^\infty r^{1-p'}\,dr\right)^{1/p'} \|\omega_1\|_{L^p(\mathbb{R}^2)}\right)$$
$$\leqslant C\big(\|\omega\|_{H^1(\Omega)} + \|\omega\|_{L^p(\Omega)}\big) < \infty$$

where $p' = p/(p-1) > 2$. On the other hand, $v_1(y)$ tends to 0 as $|y|$ tends to infinity when $\omega(y) = 0$ for $|y| > R$, and that property is preserved (by density) for any $\omega \in H^s(\Omega) \cap L^p(\Omega)$. In order to satisfy the conditions (4.4) and (4.6), we modify v_1 by using the functions φ_u (with $u = v_1$) and $\varphi_l(y) = -|y|^{-2}(l \cdot y)$ fulfilling (3.9)–(3.10) and (3.11)–(3.12), respectively. Since v_1 is not expected to belong to $L^2(\mathbb{R}^2)$, an additional work is needed to justify the existence of φ_u together with some estimates about it.

Lemma 4.3. *Let $u \in L^\infty(\Omega)$ be such that $\nabla u \in H^s(\Omega)$ $(s \geqslant 1)$ with $\operatorname{div} u = 0$ in Ω and $\int_{\partial\Omega} u \cdot n\,d\sigma = 0$. Then there exists a unique solution $\varphi_u \in \widehat{H}^{s+2}(\Omega)$ of* (3.9)–(3.10). *Furthermore*

$$\|\nabla\varphi_u\|_{H^{s+1}(\Omega)} \leqslant C\|u \cdot n\|_{H^{s+\frac{1}{2}}(\partial\Omega)} \leqslant C\big(\|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{H^s(\Omega)}\big). \tag{4.13}$$

Furthermore, for any $q \in (1, +\infty)$, we have for some constant C

$$\sum_{2\leqslant|\alpha|\leqslant 3} \left\|\partial^\alpha\varphi_u\right\|_{L^q(\Omega)} \leqslant C\|u \cdot n\|_{W^{2-1/q,q}(\partial\Omega)} \leqslant C \sum_{1\leqslant|\alpha|\leqslant 2} \left\|\partial^\alpha u\right\|_{L^q(\Omega)}. \tag{4.14}$$

Proof. The variational formulation of (3.9)–(3.10) when $\operatorname{div} u = 0$ reads

$$\int_\Omega \nabla\varphi_u \cdot \nabla\theta\,dx = \int_{\partial\Omega} (u \cdot n)\theta\,d\sigma \quad \forall\theta \in \widehat{H}^1(\Omega). \tag{4.15}$$

Clearly, (4.15) has a unique solution $\varphi_u \in \widehat{H}^1(\Omega)$. Using again the condition (3.18), we obtain along the same lines as for Proposition 3.1

$$\|\varphi_u\|_{H^{s+2}(\{1<|x|<2\})} \leqslant C\|u \cdot n\|_{H^{s+\frac{1}{2}}(\partial\Omega)} \leqslant C\big(\|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{H^s(\Omega)}\big). \tag{4.16}$$

Consider now the function $\varphi_2(x) := \rho_2(x)\varphi_u(x)$ where the function ρ_2 is a cutoff function fulfilling (3.25). Then

$$\Delta\varphi_2 = f_2 := 2\nabla\rho_2 \cdot \nabla\varphi_u + \Delta\rho_2\,\varphi_u.$$

Since (by construction) $\nabla\varphi_u \in L^2(\Omega)$, hence $\nabla\varphi_2 \in L^2(\mathbb{R}^2)$, it follows from (3.28) and (4.16) that

$$\|\nabla\varphi_2\|_{H^{s+1}(\mathbb{R}^2)} \leqslant C\big(\|u \cdot n\|_{H^{s+\frac{1}{2}}(\partial\Omega)} + \|\varphi_u\|_{L^2(\{1<|x|<2\})} + \|\nabla\varphi_u\|_{L^2(\Omega)}\big)$$
$$\leqslant C\big(\|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{H^s(\Omega)}\big).$$

Thus $\varphi_2 \in \widehat{H}^{s+2}(\mathbb{R}^2)$. Set $w_2 := \operatorname{curl} \varphi_2$. Then w_2 is given by the Biot–Savart law

$$w_2(x) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} f_2(y)\, dy.$$

As ∇w_2 (respectively $\nabla \partial_i w_2$) is obtained from f_2 (respectively $\partial_i f_2$) via a singular integral of Calderon–Zygmund type, we infer that for any $q \in (1, +\infty)$ and $i = 1, 2$ the following estimates hold:

$$\begin{aligned} \|\nabla w_2\|_{L^q(\mathbb{R}^2)} &\leqslant C\|f_2\|_{L^q(\mathbb{R}^2)} \leqslant C\|\varphi_u\|_{W^{1,q}(\{1<|x|<2\})}, \\ \|\nabla \partial_i w_2\|_{L^q(\mathbb{R}^2)} &\leqslant C\|\partial_i f_2\|_{L^q(\mathbb{R}^2)} \leqslant C\|\varphi_u\|_{W^{2,q}(\{1<|x|<2\})}. \end{aligned} \tag{4.17}$$

Proceeding as above, one can prove that

$$\|\varphi_u\|_{W^{3,q}(\{1<|x|<2\})} \leqslant C\|u \cdot n\|_{W^{2-\frac{1}{q},q}(\partial\Omega)}. \tag{4.18}$$

On the other hand, using the fact that $\int_{\partial\Omega} u \cdot n\, d\sigma = 0$, one easily obtain

$$\|u \cdot n\|_{W^{2-\frac{1}{q},q}(\partial\Omega)} \leqslant C \sum_{1 \leqslant |\alpha| \leqslant 2} \left\|\partial^\alpha u\right\|_{L^2(\{1<|x|<2\})}. \tag{4.19}$$

(4.14) follows from (4.17)–(4.19). This completes the proof of Lemma 4.3. □

Setting

$$\begin{aligned} v(y) &= v_1(y) - \nabla\varphi_{v_1}(y) + \nabla\varphi_l(y) + \frac{1}{2\pi}\left(\lambda - \int\limits_{\partial\Omega} v_1 \cdot \tau\, d\sigma\right)\frac{y^\perp}{|y|^2} \\ &=: v_1(y) + v_2(y) + v_3(y) + v_4(y) \end{aligned}$$

we see that (4.2)–(4.6) are satisfied. (4.7) has already been established for v_1, and follows from Lemma 4.3 (applied with $u = v_1$) for v_2. On the other hand, (4.7) trivially holds true for v_3 and v_4, since by Stokes' formula

$$\left|\int\limits_{\partial\Omega} v_1 \cdot \tau\, d\sigma\right| = \left|\int\limits_B \omega_1(x)\, dx\right| \leqslant C\|\omega\|_{H^s(\Omega)}.$$

Let us turn to the proof of (4.9). It is true for v_1 since v_1 is obtained from ω_1 via the Biot–Savart law. (Notice that $\|\nabla\omega_1\|_{L^q(\mathbb{R}^2)} \leqslant C\|\nabla\omega\|_{L^q(\Omega)}$ may be imposed as well.) (4.9) is true for v_2 according to Lemma 4.3, and it is clearly satisfied for v_3 and v_4.

It remains to prove (4.10). Assume given a field $v \in H^3(\Omega)$ satisfying (4.2)–(4.6). Replacing v by

$$\tilde{v} = v - \nabla\varphi_l - \frac{\lambda}{2\pi}\frac{y^\perp}{|y|^2},$$

we may assume that $l=(0,0)$ and that $\lambda=0$. To extend v to $\mathbb{R}^2$, we proceed in a different way than above in order to be able to control $\|v\|_{H^3(\mathbb{R}^2)}$ by $\|v\|_{H^3(\Omega)}$. Using (4.4) with $l=(0,0)$, we can deduce as in [11] the existence of a stream function $\psi\in C^2(\overline{\Omega})$ satisfying $\operatorname{curl}\psi=-(\nabla\psi)^\perp=v$ (hence $-\Delta\psi=\omega$) and $\psi=0$ on $\partial\Omega$. Extend ψ to $\mathbb{R}^2$ by setting

$$\psi(r,\theta)=-h(r)\psi\big(r^{-1},\theta\big)$$

where (r,θ) denote polar coordinates and $h\in C^\infty(\mathbb{R}^+;\mathbb{R}^+)$ is a cutoff function such that $h(r)=0$ if $r\leqslant 1/3$, and $h(r)=1$ if $r\geqslant 2/3$. Then $\psi\in C^2(\overline{\Omega})\cap W^{2,\infty}(B_2(0))$ with

$$\Delta\psi(r,\theta)=-\bigg[\big(h''(r)+h'(r)r^{-1}\big)\psi\big(r^{-1},\theta\big)-2h'(r)r^{-2}\frac{\partial\psi}{\partial r}\big(r^{-1},\theta\big)+h(r)r^{-4}\Delta\psi\big(r^{-1},\theta\big)\bigg] \tag{4.20}$$

for $0<r<1$, $\theta\in[0,2\pi)$. This yields

$$\|\Delta\psi\|_{L^p(B)}\leqslant C\big(\|\psi\|_{W^{1,p}(\{\frac{3}{2}<|y|<3\})}+\|\omega\|_{L^p(\Omega)}\big).$$

Set

$$v(y)=\operatorname{curl}\psi(y)\quad\text{for } y\in B.$$

Then $v\in W^{1,\infty}(\mathbb{R}^2)\cap H^3(\Omega)$, therefore v may be expressed in terms of the vorticity $\omega(y)=-\Delta\psi(y)$ through the Biot–Savart law,

$$v(y)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(y-z)^\perp}{|y-z|^2}\omega(z)\,dz.$$

Pick a cutoff function $h_\rho\in C^\infty(\mathbb{R}^2,[0,1])$ such that $h_\rho(y)=1$ for $|y|<\rho$, $h_\rho(y)=0$ for $|y|>2\rho$, and $|\nabla h_\rho|\leqslant C/\rho$, where $\rho\in(0,1/6]$ is a number to be chosen later on. Set $K(z)=z^\perp/(2\pi|z|^2)$. Split v into

$$\begin{aligned}v(y)&=\int_{\mathbb{R}^2}h_\rho(y-z)K(y-z)\omega(z)\,dz+\int_{\mathbb{R}^2}\big[1-h_\rho(y-z)\big]K(y-z)\omega(z)\,dz\\&=:v_1(y)+v_2(y).\end{aligned}$$

Then

$$\begin{aligned}\partial_{y_j}v_1&=\int_{|z|<1}\partial_{y_j}\big[h_\rho(y-z)K(y-z)\big]\omega(z)\,dz+\int_{|z|>1}\partial_{y_j}\big[h_\rho(y-z)K(y-z)\big]\omega(z)\,dz\\&=\int_{|z|\neq 1}h_\rho(y-z)K(y-z)\partial_{z_j}\omega(z)\,dz+2\int_{\partial\Omega}h_\rho(y-z)K(y-z)\omega(z)z_j\,d\sigma(z)\\&=I_1+2I_2,\end{aligned}$$

where we have integrated by parts in the first two integral terms and noticed that $\omega_{|\partial B} = -\omega_{|\partial\Omega}$ by (4.20). The function $h_\rho(y-\cdot)K(y-\cdot)$ is in $L^p(\mathbb{R}^2)$ for any $1 \leqslant p < 2$. Taking $p = 4/3$ and using Hölder inequality and (4.20), we obtain

$$
\begin{aligned}
|I_1| &\leqslant C\left(\int_0^{2\rho} r^{1-\frac{4}{3}}\,dr\right)^{\frac{3}{4}}\left(\int_{\mathbb{R}^2}|\nabla\omega|^4\right)^{\frac{1}{4}} \\
&\leqslant C\sqrt{\rho}\big(\|\nabla\omega\|_{L^4(\Omega)} + \|\nabla\omega\|_{L^4(B)}\big) \\
&\leqslant C\sqrt{\rho}\big(\|\nabla\omega\|_{L^4(\Omega)} + \|\psi\|_{W^{2,4}(\{\frac{3}{2}<|y|<3\})}\big) \\
&\leqslant C\sqrt{\rho}\|v\|_{H^3(\Omega)}. \qquad (4.21)
\end{aligned}
$$

On the other hand, identifying $\mathbb{R}^2$ with $\mathbb{C}$, we have that

$$
K\big(y-e^{i\theta}\big) = \frac{i}{2\pi}\frac{y-e^{i\theta}}{|y-e^{i\theta}|^2} = \frac{i}{2\pi}\frac{1}{\bar{y}-e^{-i\theta}},
$$

hence

$$
\overline{I_2} = -\frac{i}{2\pi}\int_0^{2\pi}\frac{h_\rho(y-e^{i\theta})}{y-e^{i\theta}}\omega\big(e^{i\theta}\big)c_j(\theta)\,d\theta,
$$

where $\omega(e^{i\theta})$ stands for $\lim_{r\to 1^+}\omega(re^{i\theta})$, and $c_j(\theta) = \cos\theta$ if $j = 1$, $\sin\theta$ if $j = 2$. Using the fact that $\omega \in H^2(\Omega)$, hence $\omega_{|\partial\Omega} \in H^{3/2}(\partial\Omega)$, we can integrate by parts in I_2. This gives

$$
\overline{I_2} = -\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Log}\big(y-e^{i\theta}\big)\frac{\partial}{\partial\theta}\big[h_\rho\big(y-e^{i\theta}\big)\omega\big(e^{i\theta}\big)c_j(\theta)e^{-i\theta}\big]\,d\theta
$$

(Log denoting a determination of the logarithm defined on $y - S^1$), hence

$$
\begin{aligned}
|I_2| &\leqslant C\left(\rho^{-1}\int_0^{2\rho}\big(1+|\ln r|\big)\,dr\,\|\omega\|_{L^\infty(\Omega)} + \sqrt{\rho}\left(\left\|\frac{\partial\omega}{\partial\theta}\right\|_{L^2(\partial\Omega)} + \|\omega\|_{L^2(\partial\Omega)}\right)\right) \\
&\leqslant C\big((1-\ln\rho)\|\omega\|_{L^\infty(\Omega)} + \sqrt{\rho}\|v\|_{H^3(\Omega)}\big). \qquad (4.22)
\end{aligned}
$$

Let us estimate ∇v_2. We write

$$
\begin{aligned}
\nabla_y v_2 &= \int_{\rho\leqslant|y-z|\leqslant\frac{1}{3}}\nabla_y\big[\big(1-h_\rho(y-z)\big)K(y-z)\big]\omega(z)\,dz \\
&\quad + \int_{|y-z|>\frac{1}{3}}\nabla_y\big[\big(1-h_\rho(y-z)\big)K(y-z)\big]\omega(z)\,dz \\
&= I_3 + I_4.
\end{aligned}
$$

For I_3, using the fact that $|\nabla_y[K(y-z)]| \leqslant C/|y-z|^2$ and $|\nabla_y[h_\rho(y-z)]| \leqslant (C/\rho)\chi_{\rho<|y-z|<2\rho}$, we obtain

$$\begin{aligned} |I_3| &\leqslant C\left(\int_\rho^{1/3} r^{-2} r\,dr + \int_\rho^{2\rho} r^{-1}\rho^{-1} r\,dr\right)\|\omega\|_{L^\infty(|z|>2/3)} \\ &\leqslant C(1-\ln\rho)\|\omega\|_{L^\infty(\Omega)}. \end{aligned} \tag{4.23}$$

For I_4 we notice that $h_\rho(y-z)=0$, since $2\rho \leqslant \frac{1}{3} < |y-z|$. Since ∇K is L^2 for $|y-z|>\frac{1}{3}$, we obtain that

$$|I_4| \leqslant C\|\omega\|_{L^2(\mathbb{R}^2)} \leqslant C\big(\|\omega\|_{L^2(\Omega)} + \|\psi\|_{H^1(\{\frac{3}{2}<|y|<3\})}\big). \tag{4.24}$$

Combining (4.21)–(4.23) to (4.24), we arrive to

$$\|\nabla v\|_{L^\infty(\Omega)} \leqslant C\big(\sqrt{\rho}\|v\|_{H^3(\Omega)} + (1-\ln\rho)\|\omega\|_{L^\infty(\Omega)} + \|\omega\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\big)\cdot \tag{4.25}$$

To derive (4.10), it is sufficient to pick in (4.25) $\rho = 1/6$ if $\|v\|_{H^3(\Omega)} \leqslant 1$ and $\rho = \frac{1}{6}\|v\|^{-2}_{H^3(\Omega)}$ if $\|v\|_{H^3(\Omega)} \geqslant 1$. The proof of Proposition 4.2 is complete. □

To complete the proof of Corollary 1.3, we investigate the dynamics of the (scalar) vorticity $\omega = \operatorname{curl} v = \partial v_2/\partial y_1 - \partial v_1/\partial y_2$. Applying the curl operator in (1.11), (1.16) results in

$$\frac{\partial\omega}{\partial t} + (v_\infty + v - l)\cdot\nabla\omega = \operatorname{curl} f, \tag{4.26}$$

$$\omega(0) = \omega_0 := \operatorname{curl} a. \tag{4.27}$$

As long as $v(t) \in H^3(\Omega) \subset C^1(\overline{\Omega})$, ω is given by

$$\omega(y,t) = \omega_0\big(U_{0,t}(y)\big) + \int_0^t \operatorname{curl} f\big(U_{s,t}(y), s\big)\,ds \tag{4.28}$$

where $U_{s,t}(y)$ is the flow associated with the velocity $v_\infty + v - l$; that is, the solution to the Cauchy problem

$$\frac{\partial}{\partial s}U_{s,t}(y) = v_\infty\big(U_{s,t}(y)\big) + v\big(U_{s,t}(y), s\big) - l(s),$$
$$U_{t,t}(y) = y.$$

It follows from (4.28) and the invariance of the Lebesgue measure by the flow that

$$\big\|\omega(.,t)\big\|_{L^q(\Omega)} \leqslant \|\omega_0\|_{L^q(\Omega)} + \int_0^t \big\|\operatorname{curl} f(.,s)\big\|_{L^q(\Omega)}\,ds \tag{4.29}$$

for all $t \in [0, T_0]$, and all $q \in [p, \infty]$. Recall that $\omega_0 \in H^2(\Omega) \subset L^\infty(\Omega)$.

We now derive an energy estimate for (4.26)–(4.27) involving $\|\nabla v\|_{L^\infty(\Omega)}$. Pick a multi-index α with $|\alpha| \leqslant 2$ and apply $\partial^\alpha = \partial^{|\alpha|}/\partial y^\alpha$ to (4.26). This gives

$$w_t + (v_\infty + v - l) \cdot \nabla w = F_1 + F_2 \tag{4.30}$$

where $w = \partial^\alpha \omega$, $F_1 = \partial^\alpha \operatorname{curl} f$ and $F_2 = (v_\infty + v - l) \cdot \nabla \partial^\alpha \omega - \partial^\alpha((v_\infty + v - l) \cdot \nabla \omega)$. Obviously, $F_2 = 0$ if $\alpha = (0, 0)$. If $|\alpha| = 1$, then $F_2 = -\partial^\alpha (v_\infty + v) \cdot \nabla \omega$ is estimated in $L^2(\Omega)$ by

$$\begin{aligned} \|F_2\|_{L^2(\Omega)} &\leqslant C\big(1 + \|\nabla v\|_{L^\infty(\Omega)}\big)\|\nabla \omega\|_{L^2(\Omega)} \\ &\leqslant C\big(1 + \ln^+ \|v\|_{H^3(\Omega)}\big)\|\omega\|_{H^1(\Omega)} \\ &\leqslant C \ln\big(e + \|\omega\|_{H^2(\Omega)}\big)\|\omega\|_{H^1(\Omega)} \end{aligned} \tag{4.31}$$

where we used the invariance of the circulation, (4.1), (4.7), (4.10), and (4.29). Assume finally that $|\alpha| = 2$. By Leibniz' rule, we obtain

$$F_2 = -\partial^\alpha (v_\infty + v) \cdot \nabla \omega + \sum_{\alpha_1 + \alpha_2 = \alpha, \alpha_i \neq (0,0)} c_{\alpha_1} \partial^{\alpha_1}(v_\infty + v) \cdot \nabla \partial^{\alpha_2} \omega =: -F_2^1 + F_2^2$$

where c_{α_1} denotes some coefficient. Then

$$\begin{aligned} \big\|F_2^1\big\|_{L^2(\Omega)} &\leqslant \big\|\partial^\alpha (v_\infty + v)\big\|_{L^4(\Omega)}\|\nabla \omega\|_{L^4(\Omega)} \\ &\leqslant C\big(1 + \|\nabla \omega\|_{L^4(\Omega)}\big)\|\omega\|_{L^\infty(\Omega)}^{\frac{1}{2}}\|\omega\|_{H^2(\Omega)}^{\frac{1}{2}} \\ &\leqslant C\big(1 + \|\omega\|_{H^2(\Omega)}\big) \end{aligned} \tag{4.32}$$

by Hölder inequality, Gagliardo–Nirenberg inequality, (4.9) and (4.29). On the other hand, for any pair of multi-indices (α_1, α_2) with $\alpha_1 + \alpha_2 = \alpha$ and $|\alpha_1| = 1$, we have that

$$\begin{aligned} &\big\|\partial^{\alpha_1}(v_\infty + v) \cdot \nabla \partial^{\alpha_2} \omega\big\|_{L^2(\Omega)} \\ &\quad \leqslant \big\|\partial^{\alpha_1}(v_\infty + v)\big\|_{L^\infty(\Omega)}\big\|\partial^{\alpha_2}\omega\big\|_{H^1(\Omega)} \\ &\quad \leqslant C\big(1 + \big(1 + \ln^+ \|v\|_{H^3(\Omega)}\big)\|\omega\|_{L^\infty(\Omega)} + \|\omega\|_{L^2(\Omega)}\big)\|\omega\|_{H^2(\Omega)} \\ &\quad \leqslant C \ln\big(e + \|\omega\|_{H^2(\Omega)}\big)\|\omega\|_{H^2(\Omega)} \end{aligned} \tag{4.33}$$

by virtue of (4.7) and (4.10). Gathering together (4.32), and (4.33), we conclude that

$$\|F_2\|_{L^2(\Omega)} \leqslant C \ln\big(e + \|\omega\|_{H^2(\Omega)}\big)\|\omega\|_{H^2(\Omega)}. \tag{4.34}$$

Scaling in (4.30) by $w = \partial^\alpha \omega$ and summing over α for $|\alpha| \leqslant 2$, we infer from (4.31)–(4.34) that

$$\frac{d}{dt}\|\omega\|_{H^2(\Omega)}^2 \leqslant C \ln\big(e + \|\omega\|_{H^2(\Omega)}^2\big)\big(e + \|\omega\|_{H^2(\Omega)}^2\big)$$

which gives

$$\|\omega\|^2_{H^2(\Omega)} \leqslant \exp\bigl(\ln\bigl(e+\|\omega_0\|^2_{H^2(\Omega)}\bigr)e^{Ct}\bigr).$$

Therefore, using (4.1) and (4.7), we arrive to

$$\|v\|^2_{H^3(\Omega)} \leqslant C\bigl(1+\exp\bigl[\ln\bigl(e+\|\omega_0\|^2_{H^2(\Omega)}\bigr)e^{Ct}\bigr]\bigr).$$

The proof of Corollary 1.3 is complete.

References

[1] Serge Alinhac, Patrick Gérard, Pseudo-differential Operators and the Nash–Moser Theorem, Grad. Stud. Math., vol. 82, Amer. Math. Soc., Providence, RI, 2007, translated from the 1991 French original by S.S. Wilson.
[2] J.T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, Comm. Math. Phys. 94 (1) (1984) 61–66.
[3] J.P. Bourguignon, H. Brezis, Remarks on the Euler equation, J. Funct. Anal. 15 (1974) 341–363.
[4] C. Conca, P. Cumsille, J. Ortega, L. Rosier, On the detection of a moving obstacle in an ideal fluid by a boundary measurement, Inverse Problems 24 (2008) 045001, 18 pp.
[5] J.-M. Coron, On the controllability of 2-D incompressible perfect fluids, J. Math. Pures Appl. (9) 75 (2) (1996) 155–188.
[6] Lawrence C. Evans, Partial Differential Equations, Grad. Stud. Math., vol. 19, Amer. Math. Soc., Providence, RI, 1998.
[7] Tosio Kato, Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$, J. Funct. Anal. 9 (1972) 296–305.
[8] Tosio Kato, Chi Yuen Lai, Nonlinear evolution equations and the Euler flow, J. Funct. Anal. 56 (1) (1984) 15–28.
[9] Keisuke Kikuchi, Exterior problem for the two-dimensional Euler equation, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 30 (1) (1983) 63–92.
[10] Keisuke Kikuchi, The existence and uniqueness of nonstationary ideal incompressible flow in exterior domains in $\mathbb{R}^3$, J. Math. Soc. Japan 38 (4) (1986) 575–598.
[11] J. Ortega, L. Rosier, T. Takahashi, Classical solutions for the equations modelling the motion of a ball in a bidimensional incompressible perfect fluid, M2AN Math. Model. Numer. Anal. 39 (1) (2005) 79–108.
[12] J. Ortega, L. Rosier, T. Takahashi, On the motion of a rigid body immersed in a bidimensional incompressible perfect fluid, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (1) (2007) 139–165.
[13] Roger Temam, On the Euler equations of incompressible perfect fluids, J. Funct. Anal. 20 (1) (1975) 32–43.
[14] R. Temam, Local existence of C^∞ solutions of the Euler equations of incompressible perfect fluids, in: Turbulence and Navier–Stokes Equations, Proc. Conf., Univ. Paris-Sud, Orsay, 1975, in: Lecture Notes in Math., vol. 565, Springer, Berlin, 1976, pp. 139–165.
[15] R. Temam, Navier–Stokes Equations, third ed., North-Holland, Amsterdam, 1984, theory and numerical analysis, with an appendix by F. Thomasset.