Journal of Combinatorial Theory, Series A 118 (2011) 1168–1189

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

New interpretations for noncrossing partitions of classical types

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ARTICLE INFO

Article history:
Received 12 April 2010
Available online 19 January 2011

Keywords:
Noncrossing partitions
Nonnesting partitions
Catalan tableaux
Type-preserving bijections

ABSTRACT

We interpret noncrossing partitions of type B and type D in terms of noncrossing partitions of type A. As an application, we get type-preserving bijections between noncrossing and nonnesting partitions of type B, type C and type D which are different from those in the recent work of Fink and Giraldo. We also define Catalan tableaux of type B and type D, and find bijections between them and noncrossing partitions of type B and type D respectively.

1. Introduction

A *partition* of a set U is a collection of mutually disjoint nonempty subsets of U, called *blocks*, whose union is equal to U. Let $\Pi(n)$ denote the set of partitions of $[n] = \{1, 2, \dots, n\}$. For $\pi \in \Pi(n)$, an *edge* of π is a pair (i, j) of integers i and j with $i < j$ such that i and j are in the same block of π and this does not contain any integer between them.

A partition $\pi \in \Pi(n)$ is called *noncrossing* (resp. *nonnesting*) if π does not have two edges (a, b) and (c, d) satisfying $a < c < b < d$ (resp. $a < c < d < b$). We denote by $\mathrm{NC}(n)$ (resp. $\mathrm{NN}(n)$) the set of noncrossing (resp. nonnesting) partitions of $[n]$.

Recently, noncrossing and nonnesting partitions have received great attention and have been generalized in many different ways both combinatorially and algebraically; we refer the reader to excellent expositions [1,19] and the references therein. Bessis [4], Brady and Watt [6] defined the set $\mathrm{NC}(W)$ of noncrossing partitions for each finite reflection group W where $\mathrm{NC}(A_{n-1})$ is the same as $\mathrm{NC}(n)$. Postnikov defined the set $\mathrm{NN}(W)$ of nonnesting partitions for each crystallographic reflection group W where $\mathrm{NN}(A_{n-1})$ is the same as $\mathrm{NN}(n)$; see [17, Remark 2].

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[1] The author was supported by the grant ANR08-JCJC-0011.

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2010.12.005

For each classical reflection group W, we have a combinatorial model for $\mathrm{NC}(W)$: the set $\mathrm{NC}_B(n)$ of noncrossing partitions of type B_n defined by Reiner [17] and the set $\mathrm{NC}_D(n)$ of noncrossing partitions of type D_n defined by Athanasiadis and Reiner [3]. Both $\mathrm{NC}_B(n)$ and $\mathrm{NC}_D(n)$ are subsets of the set $\Pi_B(n)$ of partitions of type B_n introduced by Reiner [17]. We also have combinatorial models for $\mathrm{NN}(W)$ introduced by Athanasiadis [2], which we will denote by $\mathrm{NN}_B(n)$, $\mathrm{NN}_C(n)$ and $\mathrm{NN}_D(n)$. All of these are again subsets of $\Pi_B(n)$.

The main purpose of this paper is to give new interpretations for $\mathrm{NC}_B(n)$, $\mathrm{NC}_D(n)$, $\mathrm{NN}_B(n)$, $\mathrm{NN}_C(n)$ and $\mathrm{NN}_D(n)$. To do this, we first interpret $\pi \in \Pi_B(n)$ as a triple (σ, X, Y), where $\sigma \in \Pi(n)$, X is a set of blocks of σ and Y is a maximal matching on X. As a consequence, we obtain the following formula for the cardinality of $\Pi_B(n)$:

$$\#\Pi_B(n) = \sum_{k=1}^{n} S(n,k) t_{k+1},$$

where $S(n,k)$ is the Stirling number of the second kind and t_n is the number of involutions on $[n]$.

Definition 1.1. For a partition $\sigma \in \Pi(n)$, a block B of σ is called *nonnested* (resp. *nonaligned*) if there is no edge (i,j) of σ with $i < \min(B) \leqslant \max(B) < j$ (resp. $\max(B) < i$). We denote by $\mathrm{NNBK}(\sigma)$ (resp. $\mathrm{NABK}(\sigma)$) the set of nonnested (resp. nonaligned) blocks of σ. We define

$$\mathrm{NC}^{\mathrm{NN}}(n) = \bigl\{(\sigma, X) \colon \sigma \in \mathrm{NC}(n),\ X \subset \mathrm{NNBK}(\sigma)\bigr\},$$
$$\mathrm{NC}^{\mathrm{NA}}(n) = \bigl\{(\sigma, X) \colon \sigma \in \mathrm{NC}(n),\ X \subset \mathrm{NABK}(\sigma)\bigr\},$$
$$\mathrm{NN}^{\mathrm{NA}}(n) = \bigl\{(\sigma, X) \colon \sigma \in \mathrm{NN}(n),\ X \subset \mathrm{NABK}(\sigma)\bigr\}.$$

We denote by $\mathrm{NC}^{\mathrm{NN}}_{\{0,\pm1\}}(n)$ (resp. $\mathrm{NC}^{\mathrm{NA}}_{\{0,\pm1\}}(n)$ and $\mathrm{NN}^{\mathrm{NA}}_{\{0,\pm1\}}(n)$) the set of triples (σ, X, ϵ), where (σ, X) is in $\mathrm{NC}^{\mathrm{NN}}(n)$ (resp. $\mathrm{NC}^{\mathrm{NA}}(n)$ and $\mathrm{NN}^{\mathrm{NA}}(n)$) and $\epsilon \in \{-1,0,1\}$ with the additional condition that if $X = \emptyset$ then $\epsilon = 0$.

By using our interpretation for $\Pi_B(n)$, we obtain a bijection between $\mathrm{NC}_B(n)$ (resp. $\mathrm{NN}_B(n)$, $\mathrm{NN}_C(n)$) and $\mathrm{NC}^{\mathrm{NN}}(n)$ (resp. $\mathrm{NN}^{\mathrm{NA}}(n)$, $\mathrm{NN}^{\mathrm{NA}}(n)$). Similarly we get a bijection between $\mathrm{NC}_D(n)$ (resp. $\mathrm{NN}_D(n)$) and $\mathrm{NC}^{\mathrm{NN}}_{\{0,\pm1\}}(n-1)$ (resp. $\mathrm{NN}^{\mathrm{NA}}_{\{0,\pm1\}}(n-1)$). Since $\mathrm{NC}^{\mathrm{NN}}(n)$ and $\mathrm{NC}^{\mathrm{NN}}_{\{0,\pm1\}}(n-1)$ concern only type A noncrossing partitions, our interpretations have the advantage of understanding $\mathrm{NC}_B(n)$ and $\mathrm{NC}_D(n)$ as easily as $\mathrm{NC}(n)$.

To make a connection between noncrossing and nonnesting partitions in our interpretations we find an involution on $\mathrm{NC}(n)$ which interchanges the nonnested blocks and the nonaligned blocks. Thus, as a byproduct, we get that the nonnested blocks and the nonaligned blocks have a joint symmetric distribution on $\mathrm{NC}(n)$, in other words,

$$\sum_{\pi \in \mathrm{NC}(n)} x^{\mathrm{nn}(\pi)} y^{\mathrm{na}(\pi)} = \sum_{\pi \in \mathrm{NC}(n)} x^{\mathrm{na}(\pi)} y^{\mathrm{nn}(\pi)},$$

where $\mathrm{nn}(\pi)$ (resp. $\mathrm{na}(\pi)$) denotes the number of nonnested (resp. nonaligned) blocks of π.

Combining our bijections together with the bijection between $\mathrm{NC}(n)$ and $\mathrm{NN}(n)$ due to Athanasiadis [2], we obtain type-preserving bijections, i.e. bijections preserving block sizes, between noncrossing and nonnesting partitions of classical types. Our type-preserving bijections are different from those of Fink and Giraldo [11].

We provide another interpretation for $\mathrm{NC}_B(n)$ and $\mathrm{NC}_D(n)$: a bijection between $\mathrm{NC}_B(n)$ and the set $\mathfrak{B}(n)$ of pairs (σ, x) where $\sigma \in \mathrm{NC}(n)$ and x is either $\emptyset$, an edge of σ or a block of σ, and a bijection between $\mathrm{NC}_D(n)$ and the set $\mathfrak{D}(n)$ of pairs (σ, x) where $\sigma \in \mathrm{NC}(n-1)$ and x is either $\emptyset$, an edge of σ, a block of σ or an integer in $[\pm(n-1)]$. In fact, $\mathfrak{B}(n)$ and $\mathfrak{D}(n)$ are essentially the same as $\mathrm{NC}(n) \times [n+1]$ and $\mathrm{NC}(n-1) \times [3n-2]$ respectively. Using these interpretations, we give another proof of the formula for the number of noncrossing partitions of type B_n and type D_n with given block sizes.

It is well known that $\mathrm{NC}(n)$ is in bijection with the set of Dyck paths, i.e. lattice paths from $(0,0)$ to (n,n) which do not go below the line $y=x$. Using $\mathrm{NC}^{\mathrm{NA}}(n)$ and $\mathrm{NC}^{\mathrm{NA}}_{\{0,\pm 1\}}(n-1)$ we find a bijection between $\mathrm{NC}_B(n)$ and the set $\mathrm{LP}(n)$ of lattice paths from $(0,0)$ to (n,n) and a bijection between $\mathrm{NC}_D(n)$ and the set $\overline{\mathrm{LP}}(n)$ of lattice paths in $\mathrm{LP}(n)$ which do not touch $(n-1,n-1)$ and $(n,n-1)$ simultaneously.

Permutation tableaux were first introduced by Postnikov [16] in the study of the totally nonnegative Grassmannian. Catalan tableaux are special permutation tableaux. Permutation tableaux and Catalan tableaux are respectively in bijection with permutations and noncrossing partitions; see [7,9,15,20]. Lam and Williams [14] defined permutation tableaux of type B_n. In this paper we define Catalan tableaux of type B_n and D_n which are special permutation tableaux of type B_n. Then we find bijections between them and $\mathrm{NC}_B(n)$ and $\mathrm{NC}_D(n)$.

The rest of this paper is organized as follows. In Section 2 we recall the definitions of noncrossing and nonnesting partitions of finite reflection groups and the combinatorial models for them for classical reflection groups. In Section 3 we define a map from $\Pi_B(n)$ to the set of certain triples. In Section 4 we give new interpretations for $\mathrm{NC}_B(n)$, $\mathrm{NC}_D(n)$, $\mathrm{NN}_B(n)$, $\mathrm{NN}_C(n)$ and $\mathrm{NN}_D(n)$. In Section 5 we find type-preserving bijections between noncrossing and nonnested partitions of classical types. In Section 6 we find a bijection between $\mathrm{NC}_B(n)$ (resp. $\mathrm{NC}_D(n)$) and $\mathfrak{B}(n)$ (resp. $\mathfrak{D}(n)$). In Section 7 we find a bijection between $\mathrm{NC}_B(n)$ (resp. $\mathrm{NC}_D(n)$) and $\mathrm{LP}(n)$ (resp. $\overline{\mathrm{LP}}(n)$). In Section 8 we define the sets $\mathrm{CT}_B(n)$ and $\mathrm{CT}_D(n)$ of Catalan tableaux of type B_n and type D_n, and find bijections between them and $\mathrm{NC}_B(n)$ and $\mathrm{NC}_D(n)$ respectively.

2. Preliminaries

In this section we recall the definitions noncrossing and nonnesting partitions of finite reflection groups and the combinatorial models $\mathrm{NC}_B(n)$, $\mathrm{NC}_D(n)$, $\mathrm{NN}_B(n)$, $\mathrm{NN}_C(n)$ and $\mathrm{NN}_D(n)$.

2.1. General definitions for noncrossing and nonnesting partitions

For a finite Coxeter system (W,S) with the set $T=\{wsw^{-1}\colon s\in S,\ w\in W\}$ of reflections, the *absolute length* $\ell_T(w)$ of an element $w\in W$ is defined to be the smallest integer i such that w can be written as a product of i reflections. The *absolute order* on W is defined as follows: $u\leqslant_T w$ if and only if $\ell_T(w)=\ell_T(u)+\ell_T(u^{-1}w)$. Then the noncrossing partition poset $\mathrm{NC}(W)$ is defined to be the interval $\{w\in W\colon 1\leqslant_T w\leqslant_T c\}$, where c is a Coxeter element. It turns out that $\mathrm{NC}(W)$ does not depend on the particular choice of c up to isomorphism.

Nonnesting partitions are defined for crystallographic reflection groups. Suppose W is a crystallographic reflection group and Φ^+ is a positive root system of W. The *root poset* $(\Phi^+,\leqslant)$ has the partial order $\alpha\leqslant\beta$ if and only if $\beta-\alpha$ can be written as a linear combination of the positive roots with nonnegative integer coefficients. A *nonnesting partition* of W is an antichain in the root poset $(\Phi^+,\leqslant)$. We denote by $\mathrm{NN}(W)$ the set of nonnesting partitions of W.

For classical types, we will use the following root posets:

$$\Phi^+(A_{n-1})=\{e_i-e_j\colon 1\leqslant i<j\leqslant n\},$$

$$\Phi^+(B_n)=\{e_i\pm e_j\colon 1\leqslant i<j\leqslant n\}\cup\{e_i\colon 1\leqslant i\leqslant n\},$$

$$\Phi^+(C_n)=\{e_i\pm e_j\colon 1\leqslant i<j\leqslant n\}\cup\{2e_i\colon 1\leqslant i\leqslant n\},$$

$$\Phi^+(D_n)=\{e_i\pm e_j\colon 1\leqslant i<j\leqslant n\}.$$

2.2. Combinatorial models

We use the definitions in [11]. For type D_n, our definitions are stated in a slightly different way from those in [11], but one can easily check that they are equivalent.

For a partition π of a finite set U and a total order $a_1\prec a_2\prec\cdots\prec a_n$ of U, the *standard representation of π with respect to* the order $a_1\prec a_2\prec\cdots\prec a_n$ is the drawing obtained as follows. Arrange

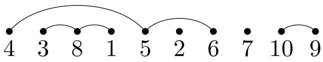


Fig. 1. The standard representation of $\{\{1,3,8\},\{2\},\{4,5,6\},\{7\},\{9,10\}\}$ with respect to the order $4 \prec 3 \prec 8 \prec 1 \prec 5 \prec 2 \prec 6 \prec 7 \prec 10 \prec 9$.

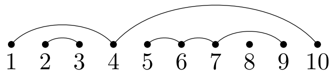


Fig. 2. A noncrossing partition of type A_9.

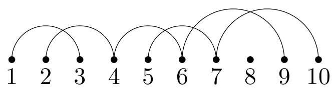


Fig. 3. A nonnesting partition of type A_9.

$a_1, a_2, \ldots, a_n$ in a horizontal line. Draw an arc between a_i and a_j for each pair (i, j) with $i < j$ such that $a_i, a_j \in B$ for a block B of π which does not contain a_t with $i < t < j$. See Fig. 1.

We say that π is *noncrossing* (resp. *nonnesting*) *with respect to* the order $a_1 \prec a_2 \prec \cdots \prec a_n$ if π satisfies the following condition: if $a_i, a_k \in B$ and $a_j, a_\ell \in B'$ (resp. $a_i, a_\ell \in B$ and $a_j, a_k \in B'$) for some blocks B and B' of π and for some integers $i < j < k < \ell$, then we have $B = B'$. In other words, π is noncrossing (resp. nonnesting) with respect to the order $a_1 \prec a_2 \prec \cdots \prec a_n$ if and only if the standard representation of π with respect to this order does not have two arcs which cross each other (resp. two arcs one of which nests the other). For example, the partition in Fig. 1 is noncrossing but not nonnesting with respect to the order written there.

A *noncrossing partition* (resp. *nonnesting partition*) is a partition of $[n]$ which is noncrossing (resp. nonnesting) with respect to the order $1 \prec 2 \prec \cdots \prec n$. See Figs. 2 and 3 for an example. We denote by $\mathrm{NC}(n)$ (resp. $\mathrm{NN}(n)$) the set of noncrossing (resp. nonnesting) partitions of type A_{n-1}.

There is a natural bijection between $\mathrm{NC}(A_{n-1})$ and $\mathrm{NC}(n)$. If we take $c = (1, 2, \ldots, n)$ for the Coxeter element, each element in $\mathrm{NC}(A_{n-1})$ can be written as a product of disjoint cycles of form $(a_1, a_2, \ldots, a_k)$ where $a_1 < a_2 < \cdots < a_k$. Then the bijection is simply changing each cycle $(a_1, a_2, \ldots, a_k)$ to the block $\{a_1, a_2, \ldots, a_k\}$. One can check that we alway get a noncrossing partition. For example, $(1, 4, 10)(2, 3)(5, 6, 7, 9)(8) \in \mathrm{NC}(A_9)$ corresponds to the noncrossing partition in Fig. 2. In fact, this bijection is a poset isomorphism if we order $\mathrm{NC}(n)$ by refinement. Thus we have $\mathrm{NC}(A_{n-1}) \cong \mathrm{NC}(n)$.

Similarly, there is a natural bijection between $\mathrm{NN}(A_{n-1})$ and $\mathrm{NN}(n)$. For an antichain π of $\Phi^+(A_{n-1})$, we construct the corresponding nonnesting partition by making the edge (i, j) for each element $e_i - e_j \in \pi$. For example, the nonnesting partition in Fig. 3 corresponds to

$$\{e_1 - e_3, e_2 - e_4, e_4 - e_6, e_6 - e_9, e_5 - e_7, e_7 - e_{10}\} \in \mathrm{NN}(A_9).$$

Thus we have $\mathrm{NN}(n) \cong \mathrm{NN}(A_{n-1})$.

In order to define combinatorial models for noncrossing and nonnesting partitions of other classical types, we need type B partitions introduced by Reiner [17]. There is a natural way to identify $\pi \in \Pi(n)$ with an intersection of a collection of the following reflecting hyperplanes of type A_{n-1}:

$$\{x_i - x_j = 0\colon 1 \leqslant i < j \leqslant n\}.$$

For example, $\{\{1,3,4\},\{2,6\},\{5\}\}$ corresponds to

$$\big\{(x_1, \ldots, x_6) \in \mathbb{R}^6\colon x_1 = x_3 = x_4,\ x_2 = x_6\big\}.$$

With this observation Reiner [17] defined a partition of type B_n to be an intersection of a collection of the following reflecting hyperplanes of type B_n:

$$\{x_i = 0\colon 1 \leqslant i \leqslant n\} \cup \{x_i \pm x_j = 0\colon 1 \leqslant i < j \leqslant n\}.$$

Note that we can also consider such an intersection as a partition of

$$[\pm n] = \{1, 2, \dots, n, -1, -2, \dots, -n\}.$$

For example, the intersection

$$\{(x_1, \dots, x_8) \in \mathbb{R}^8 \colon x_1 = -x_3 = x_6,\ x_5 = x_8,\ x_2 = x_4 = 0\}$$

corresponds to

$$\{\pm\{1, -3, 6\}, \{2, 4, -2, -4\}, \pm\{5, 8\}, \pm\{7\}\},$$

which means

$$\{\{1, -3, 6\}, \{-1, 3, -6\}, \{2, 4, -2, -4\}, \{5, 8\}, \{-5, -8\}, \{7\}, \{-7\}\}.$$

Equivalently, we define a partition of type B_n as follows.

A *partition of type* B_n is a partition π of $[\pm n]$ such that if B is a block of π then $-B = \{-x \colon x \in B\}$ is also a block of π, and there is at most one block, called a *zero block*, which satisfies $B = -B$. We denote by $\Pi_B(n)$ the set of partitions of type B_n.

Now we are ready to define combinatorial models for noncrossing and nonnesting partitions of other classical types.

A *noncrossing partition of type* B_n is a partition $\pi \in \Pi_B(n)$ which is noncrossing with respect to the order $1 \prec 2 \prec \cdots \prec n \prec -1 \prec -2 \prec \cdots \prec -n$. See Fig. 4 for an example. A *noncrossing partition of type* D_n is a partition $\pi \in \Pi_B(n)$ such that

1. if π has a zero block B, then $\{n, -n\} \subsetneq B$,
2. $\pi' \in \mathrm{NC}_B(n-1)$, where π' is the partition obtained from π by taking the union of the blocks containing n or $-n$ and removing n and $-n$.

See Fig. 5 for an example. We denote by $\mathrm{NC}_B(n)$ (resp. $\mathrm{NC}_D(n)$) the set of noncrossing partitions of type B_n (resp. type D_n). Like type A, we have $\mathrm{NC}_B(n) \cong \mathrm{NC}(B_n)$ and $\mathrm{NC}_D(n) \cong \mathrm{NC}(D_n)$. We note that $\mathrm{NC}_B(n)$ and $\mathrm{NC}_D(n)$ can also be defined using circular representation, see [3,12,17]. However, the standard representation is more suitable for our purpose.

A *nonnesting partition of type* B_n is a partition $\pi \in \Pi_B(n)$ such that π_0 is nonnesting with respect to the order $1 \prec \cdots \prec n \prec 0 \prec -n \prec \cdots \prec -1$, where π_0 is the partition of $[\pm n] \cup \{0\}$ obtained from π by adding 0 to the zero block if π has a zero block, and by adding the singleton $\{0\}$ otherwise. See Fig. 6 for an example. A *nonnesting partition of type* C_n is a partition $\pi \in \Pi_B(n)$ which is nonnesting with respect to the order $1 \prec \cdots \prec n \prec -n \prec \cdots \prec -1$. See Fig. 7 for an example. A *nonnesting partition of type* D_n is a partition $\pi \in \Pi_B(n)$ such that

1. if π has a zero block B, then $\{n, -n\} \subsetneq B$,
2. $\pi' \in \mathrm{NN}_B(n-1)$, where π' is the partition obtained from π by taking the union of the blocks containing n or $-n$ and removing n and $-n$.

See Fig. 8 for an example. We denote by $\mathrm{NN}_B(n)$ (resp. $\mathrm{NN}_C(n)$ and $\mathrm{NN}_D(n)$) the set of nonnesting partitions of type B_n (resp. type C_n and type D_n). Then we have $\mathrm{NN}_B(n) \cong \mathrm{NN}(B_n)$, $\mathrm{NN}_C(n) \cong \mathrm{NN}(C_n)$ and $\mathrm{NN}_D(n) \cong \mathrm{NN}(D_n)$.

3. Partitions of classical types

For $\pi \in \Pi_B(n)$ and a block B of π, let B^+ (resp. B^-) denote the set of positive (resp. negative) integers in B. Note that $(-B)^+ = -(B^-)$. We define $\alpha(\pi)$, $\beta(\pi)$ and $\gamma(\pi)$ as follows:

- $\alpha(\pi)$ is the partition in $\Pi(n)$ such that $A \in \alpha(\pi)$ if and only if $A = B^+$ for some $B \in \pi$,
- $\beta(\pi)$ is the set of blocks $A \in \alpha(\pi)$ such that π has a block containing A and at least one negative integer,

- $\gamma(\pi)$ is the matching on $\beta(\pi)$ such that $\{A_1, A_2\} \in \gamma(\pi)$ if and only if $A_1 \neq A_2$ and $A_1 \cup (-A_2)$ is a block of π.

Example 3.1. If $\pi = \{\pm\{1, -3, 6\}, \{2, 4, -2, -4\}, \pm\{5, 8\}, \pm\{7\}\}$, we have $\alpha(\pi) = \{\{1, 6\}, \{2, 4\}, \{3\}, \{5, 8\}, \{7\}\}$, $\beta(\pi) = \{\{1, 6\}, \{2, 4\}, \{3\}\}$ and $\gamma(\pi)$ is the matching on $\beta(\pi)$ with the only one matching pair $\{\{1, 6\}, \{3\}\}$.

Assume that a block $A \in \beta(\pi)$ is not matched in $\gamma(\pi)$. If B is the block of π with $A = B^+$, we have $B^+ \cap (-(B^-)) \neq \emptyset$ because otherwise A would be matched with another block $A' = (-B)^+ = -(B^-)$. Thus we have an integer i both in B^+ and $-(B^-)$, which implies $i, -i \in B$. Therefore B is a zero block of π, which is unique. This argument shows that $\gamma(\pi)$ is a maximal matching on $\beta(\pi)$. In other words, if $|\beta(\pi)|$ is even, then $\gamma(\pi)$ is a complete matching on $\beta(\pi)$; and if $|\beta(\pi)|$ is odd, then there is a unique unmatched block $A \in \beta(\pi)$ in $\gamma(\pi)$, and in this case, π has the zero block $A \cup (-A)$.

It is easy to see that π can be reconstructed from $(\alpha(\pi), \beta(\pi), \gamma(\pi))$. Thus we get the following proposition.

Proposition 3.1. *The map $\pi \mapsto (\alpha(\pi), \beta(\pi), \gamma(\pi))$ is a bijection between $\Pi_B(n)$ and the set of triples (σ, X, Y), where $\sigma \in \Pi(n)$, X is a set of blocks of σ and Y is a maximal matching on X.*

Now we define $\alpha_0(\pi) = \alpha(\pi) \cup \{\{0\}\}$, which is a partition of $[n] \cup \{0\}$, and $\gamma_0(\pi)$ to be the matching on the blocks of $\alpha_0(\pi)$ defined as follows. If $\gamma(\pi)$ is a complete matching, then the matching pairs of $\gamma(\pi)$ and $\gamma_0(\pi)$ are the same. If there is an unmatched block A in $\gamma(\pi)$, which is necessarily unique, then the matching pairs of $\gamma_0(\pi)$ are those in $\gamma(\pi)$ and $\{\{0\}, A\}$. Note that $\gamma_0(\pi)$ is not necessarily a maximal matching.

Example 3.2. If π is the partition in Example 3.1, we have

$$\alpha_0(\pi) = \big\{\{0\}, \{1, 6\}, \{2, 4\}, \{3\}, \{5, 8\}, \{7\}\big\},$$

and $\gamma_0(\pi)$ is the matching on $\alpha_0(\pi)$ with the two matching pairs $\{\{1, 6\}, \{3\}\}$ and $\{\{0\}, \{2, 4\}\}$.

Since $\gamma_0(\pi)$ determines $\beta(\pi)$ and $\gamma(\pi)$, we get the following.

Proposition 3.2. *The map $\pi \mapsto (\alpha(\pi), \gamma_0(\pi))$ is a bijection between $\Pi_B(n)$ and the set of pairs (σ, X) where $\sigma \in \Pi(n)$ and X is a matching on the blocks of the partition $\sigma \cup \{\{0\}\}$.*

If $\alpha(\pi)$ has k blocks, then $\alpha_0(\pi)$ has $k + 1$ blocks. Let $A_1, A_2, \dots, A_{k+1}$ be the blocks of $\alpha_0(\pi)$ with $\max(A_1) < \max(A_2) < \cdots < \max(A_{k+1})$. By identifying the block A_i with the integer i, we can consider $\gamma_0(\pi)$ as a matching on $[k+1]$ or an involution on $[k+1]$. Thus we get the following formula for the cardinality of $\Pi_B(n)$.

Corollary 3.3. *The cardinality of $\Pi_B(n)$ is equal to*

$$\sum_{k=1}^{n} S(n, k) t_{k+1},$$

where $S(n, k)$ is the Stirling number of the second kind and t_n is the number of involutions on $[n]$.

Note that the formula in Corollary 3.3 is a type B analog of $\#\Pi(n) = \sum_{k=1}^{n} S(n, k)$.

4. Interpretations for noncrossing and nonnesting partitions

The following terminologies will be used for the rest of this paper.

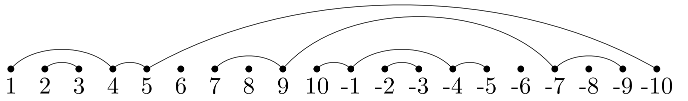


Fig. 4. The standard representation of an element in $\mathrm{NC}_B(10)$ with respect to the order $1 \prec 2 \prec \cdots \prec 10 \prec -1 \prec -2 \prec \cdots \prec -10$.

An *integer partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a weakly decreasing sequence of positive integers. Each λ_i is called *part* of λ and ℓ is called *length* of λ. We define $|\lambda|$ to be the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ of all parts of λ. We will also consider λ as the multiset $\{1^{m_1}, 2^{m_2}, \ldots\}$, where m_i is the number of parts equal to i in λ.

For two multisets A and B, let $A \uplus B$ denote the multiset union of A and B.

For a subset S of $[n]$ and a partition π of S, the *type* $\mathrm{type}(\pi)$ of π is the integer partition $\lambda = \{1^{m_1}, 2^{m_2}, \ldots\}$ such that m_i is equal to the number of blocks of size i in π. The *type* $\mathrm{type}(\pi)$ of a partition $\pi \in \Pi_B(n)$ is the integer partition $\lambda = \{1^{m_1}, 2^{m_2}, \ldots\}$ such that m_i is equal to the number of unordered pairs $(B, -B)$ of nonzero blocks of size i in π.

Recall the sets $\mathrm{NC}^{\mathrm{NN}}(n)$, $\mathrm{NC}^{\mathrm{NA}}(n)$, $\mathrm{NN}^{\mathrm{NA}}(n)$, $\mathrm{NC}^{\mathrm{NN}}_{\{0,\pm1\}}(n)$, $\mathrm{NC}^{\mathrm{NA}}_{\{0,\pm1\}}(n)$ and $\mathrm{NN}^{\mathrm{NA}}_{\{0,\pm1\}}(n)$ in Definition 1.1.

Notation. From now on, if we write $\{A_1, A_2, \ldots, A_k\}_<$, it is automatically assumed that A_i's are sorted in increasing order by their largest elements, that is, $\max(A_1) < \max(A_2) < \cdots < \max(A_k)$.

For a set $X = \{A_1, A_2, \ldots, A_{2k}\}_<$ of even number of blocks, we define *pairing*(X) to be the following multiset:

$$pairing(X) = \bigl\{|A_1 \cup A_{2k}|, |A_2 \cup A_{2k-1}|, \ldots, |A_k \cup A_{k+1}|\bigr\}.$$

4.1. Noncrossing partitions

Let $\pi \in \mathrm{NC}_B(n)$ and consider the map $\pi \mapsto (\alpha(\pi), \beta(\pi), \gamma(\pi))$ in the previous section. Since π is noncrossing with respect to the order $1 \prec 2 \prec \cdots \prec n \prec -1 \prec -2 \prec \cdots \prec -n$, one can easily see that $\alpha(\pi) \in \mathrm{NC}(n)$, all the blocks in $\beta(\pi)$ are nonnested, and the matching $\gamma(\pi)$ is uniquely determined by $\beta(\pi)$. For instance, if $\beta(\pi) = \{A_1, A_2, \ldots, A_k\}_<$, then $\gamma(\pi)$ is the matching consisting of $\{A_i, A_{k+1-i}\}$ for all $1 \leqslant i \leqslant \lfloor k/2 \rfloor$.

For $\pi \in \mathrm{NC}_B(n)$, we define $\phi_B^{\mathrm{NC}}(\pi) = (\alpha(\pi), \beta(\pi))$. In other words, $\phi_B^{\mathrm{NC}}(\pi)$ is the pair (σ, X) where σ is the partition obtained from π by removing all the negative integers and X is the set of blocks of σ which are properly contained in some blocks of π. Note that we have $\phi_B^{\mathrm{NC}}(\pi) \in \mathrm{NC}^{\mathrm{NN}}(n)$.

Example 4.1. If $\pi \in \mathrm{NC}_B(10)$ is the partition in Fig. 4, we have $\phi_B^{\mathrm{NC}}(\pi) = (\sigma, X)$, where

$$\sigma = \bigl\{\{1,4,5\}, \{2,3\}, \{6\}, \{7,9\}, \{8\}, \{10\}\bigr\}$$

and $X = \{\{1,4,5\}, \{7,9\}, \{10\}\}$.

From the construction, one can easily prove the following proposition.

Proposition 4.1. *The map $\phi_B^{\mathrm{NC}} : \mathrm{NC}_B(n) \to \mathrm{NC}^{\mathrm{NN}}(n)$ is a bijection. Moreover, if $\phi_B^{\mathrm{NC}}(\pi) = (\sigma, X)$ and $X = \{A_1, A_2, \ldots, A_k\}_<$, then*

$$\mathrm{type}(\pi) = \mathrm{type}(\sigma \setminus X) \uplus T,$$

where

$$T = \begin{cases} pairing(X), & \text{if } k \text{ is even,} \\ pairing\bigl(X \setminus \{A_{(k+1)/2}\}\bigr), & \text{if } k \text{ is odd.} \end{cases}$$

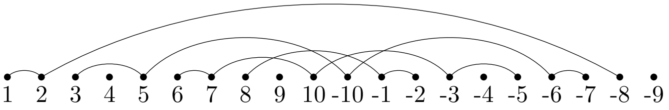


Fig. 5. The standard representation of an element in $\mathrm{NC}_D(10)$ with respect to the order $1 \prec 2 \prec \cdots \prec 10 \prec -10 \prec -1 \prec -2 \prec \cdots \prec -9$. Note that the locations of 10 and -10 are not important.

Now we consider $\pi \in \mathrm{NC}_D(n)$. Let π' be the partition obtained from π by taking the union of the blocks containing n or $-n$ and removing n and $-n$. Note that π is uniquely determined by π' and the block of π containing n. We define $\phi_D^{\mathrm{NC}}(\pi) = (\sigma, X, \epsilon)$, where σ, X and ϵ are obtained as follows.

1. If π has the blocks $\pm\{n\}$ or π has a zero block, then $(\sigma, X) = \phi_B^{\mathrm{NC}}(\pi')$ and $\epsilon = 0$.
2. Otherwise, the block of π containing n can be written as

$$\{a_1, a_2, \ldots, a_r, -b_1, -b_2, \ldots, -b_s, n\}$$

for some integers r, s, $a_1, \ldots, a_r$, $b_1, \ldots, b_s$ with $r, s \geqslant 0$, $r + s \geqslant 1$, $1 \leqslant a_1 < \cdots < a_r < n$ and $1 \leqslant b_1 < \cdots < b_s < n$. Let $\epsilon = 1$ if $s = 0$, or $r, s > 0$ and $a_r < b_s$; and $\epsilon = -1$ otherwise. Let σ be the partition of $[n-1]$ such that $A \in \sigma$ if and only if $A = B^+ \setminus \{n\}$ for some $B \in \pi$ with $B^+ \neq \emptyset$. Let X be the set of blocks of σ which are properly contained in some blocks of π.

Note that $\phi_D^{\mathrm{NC}}(\pi) \in \mathrm{NC}^{\mathrm{NN}}_{\{0,\pm1\}}(n-1)$.

Example 4.2. Let $\pi = \{\pm\{1, 2, -8\}, \pm\{-3, -5, 6, 7, 10\}, \pm\{4\}, \pm\{9\}\}$ as shown in Fig. 5. Then $\phi_D^{\mathrm{NC}}(\pi) = (\sigma, X, \epsilon)$ where $\sigma = \{\{1, 2\}, \{3, 5\}, \{4\}, \{6, 7\}, \{8\}, \{9\}\}$, $X = \{\{1, 2\}, \{3, 5\}, \{6, 7\}, \{8\}\}$ and $\epsilon = -1$.

Proposition 4.2. *The map $\phi_D^{\mathrm{NC}} : \mathrm{NC}_D(n) \to \mathrm{NC}^{\mathrm{NN}}_{\{0,\pm1\}}(n-1)$ is a bijection. Moreover, if $\phi_D^{\mathrm{NC}}(\pi) = (\sigma, X, \epsilon)$ and $X = \{A_1, A_2, \ldots, A_k\}_<$, then* $\mathrm{type}(\pi) = \mathrm{type}(\sigma \setminus X) \uplus T$, *where*

$$T = \begin{cases} \mathit{pairing}(X) \uplus \{1\}, & \text{if } \epsilon = 0 \text{ and } k = 2t, \\ \mathit{pairing}\big(X \setminus \{A_{t+1}\}\big), & \text{if } \epsilon = 0 \text{ and } k = 2t+1, \\ \mathit{pairing}\big(X \setminus \{A_t, A_{t+1}\}\big) \uplus \big\{|A_t| + |A_{t+1}| + 1\big\}, & \text{if } \epsilon \neq 0 \text{ and } k = 2t, \\ \mathit{pairing}\big(X \setminus \{A_{t+1}\}\big) \uplus \big\{|A_{t+1}| + 1\big\}, & \text{if } \epsilon \neq 0 \text{ and } k = 2t+1. \end{cases}$$

Proof. We will find the inverse map of ϕ_D^{NC}. Let $(\sigma, X, \epsilon) \in \mathrm{NC}^{\mathrm{NA}}_{\{0,\pm1\}}(n-1)$ and $\pi' = (\phi_B^{\mathrm{NC}})^{-1}(\sigma, X) \in \mathrm{NC}_B(n-1)$.

If $\epsilon = 0$, then $\pi \in \mathrm{NC}_D(n)$ is the partition obtained from π' by adding n and $-n$ to the zero block if π' has a zero block; and by adding the two singletons $\pm\{n\}$ otherwise.

Now assume $\epsilon \neq 0$. If $k = 2t$, then π' has the blocks $\pm(A_t \cup (-A_{t+1}))$. Then π is the partition obtained from π' by replacing $\pm(A_t \cup (-A_{t+1}))$ with $\pm(\epsilon(A_t \cup (-A_{t+1})) \cup \{n\})$. Here for a set B, the notation ϵB means the set $\{\epsilon \cdot x \colon x \in B\}$. If $k = 2t+1$, then π' has the blocks $\pm A_{t+1}$. Then π is the partition obtained from π' by replacing $\pm A_{t+1}$ with $\pm(\epsilon(A_{t+1}) \cup \{n\})$.

One can easily check that this is the inverse map of ϕ_D^{NC}. The 'moreover' statement is obvious from the construction of the inverse map. $\square$

4.2. Nonnesting partitions

As we did for noncrossing partitions, we can find interpretations for nonnesting partitions of classical types.

Consider the map $\pi \mapsto (\alpha(\pi), \beta(\pi), \gamma(\pi))$ for $\pi \in \mathrm{NN}_B(n)$. It is easy to see that $\alpha(\pi) \in \mathrm{NN}(n)$, all the blocks in $\beta(\pi)$ are nonaligned and $\gamma(\pi)$ is determined from $\beta(\pi)$ as follows. Let $\beta(\pi) = \{A_1, A_2, \ldots, A_{2k}\}_<$ if $\beta(\pi)$ has even number of blocks; and $\beta(\pi) = \{A_0, A_1, A_2, \ldots, A_{2k}\}_<$ otherwise. Then $\gamma(\pi)$ is the matching consisting of $\{A_i, A_{2k+1-i}\}$ for $i \in [k]$.

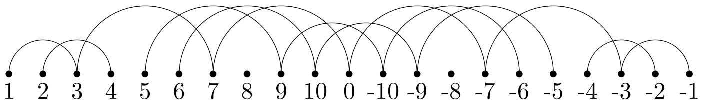


Fig. 6. The standard representation of π_0 for a $\pi \in \mathrm{NN}_B(10)$ with respect to the order $1 \prec 2 \prec \cdots \prec 10 \prec 0 \prec -10 \prec -9 \prec \cdots \prec -1$.

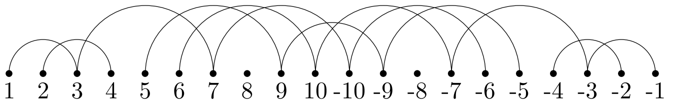


Fig. 7. The standard representation of an element in $\mathrm{NN}_C(10)$ with respect to the order $1 \prec 2 \prec \cdots \prec 10 \prec -10 \prec -9 \prec \cdots \prec -1$.

For $\pi \in \mathrm{NN}_B(n)$, we define $\phi_B^{\mathrm{NN}}(\pi) = (\alpha(\pi), \beta(\pi))$. In other words, $\phi_B^{\mathrm{NN}}(\pi)$ is the pair (σ, X) where σ is the partition obtained from π by removing all the negative integers and X is the set of blocks of σ which are properly contained in some blocks of π. Note that we have $\phi_B^{\mathrm{NN}}(\pi) \in \mathrm{NN}^{\mathrm{NA}}(n)$.

Example 4.3. Let $\pi = \{\{1,3,7,-7,-3,-1\}, \pm\{2,4\}, \pm\{5,9,-10,-6\}, \pm\{8\}\} \in \mathrm{NN}_B(10)$ as shown in Fig. 6. Then $\phi_B^{\mathrm{NN}}(\pi) = (\sigma, X)$ where $\sigma = \{\{1,3,7\}, \{2,4\}, \{5,9\}, \{6,10\}, \{8\}\}$ and $X = \{\{1,3,7\}, \{5,9\}, \{6,10\}\}$.

From the construction, one can easily prove the following proposition.

Proposition 4.3. *The map $\phi_B^{\mathrm{NN}} : \mathrm{NN}_B(n) \to \mathrm{NN}^{\mathrm{NA}}(n)$ is a bijection. Moreover, if $\phi_B^{\mathrm{NN}}(\pi) = (\sigma, X)$ and $X = \{A_1, A_2, \dots, A_k\}_<$, then*

$$\mathrm{type}(\pi) = \mathrm{type}(\sigma \setminus X) \uplus T,$$

where

$$T = \begin{cases} \mathit{pairing}(X), & \text{if } k \text{ is even}, \\ \mathit{pairing}\big(X \setminus \{A_1\}\big), & \text{if } k \text{ is odd}. \end{cases}$$

Similarly, we define $\phi_C^{\mathrm{NN}}(\pi) = (\alpha(\pi), \beta(\pi))$ for $\pi \in \mathrm{NN}_C(n)$. Then we have $\phi_C^{\mathrm{NN}}(\pi) \in \mathrm{NN}^{\mathrm{NA}}(n)$. Note that if $\pi \in \mathrm{NN}_C(n)$ and $\beta(\pi) = \{A_1, A_2, \dots, A_k\}_<$, then $\gamma(\pi)$ is the matching consisting of $\{A_i, A_{k+1-i}\}$ for all $i = 1, 2, \dots, \lfloor k/2 \rfloor$.

Example 4.4. Let $\pi = \{\pm\{1,3,7,-10,-6\}, \pm\{2,4\}, \{5,9,-9,-5\}, \pm\{8\}\} \in \mathrm{NN}_C(10)$ as shown in Fig. 7. Then $\phi_C^{\mathrm{NN}}(\pi) = (\sigma, X)$ where $\sigma = \{\{1,3,7\}, \{2,4\}, \{5,9\}, \{6,10\}, \{8\}\}$ and $X = \{\{1,3,7\}, \{5,9\}, \{6,10\}\}$.

Then we get the following proposition in the same way.

Proposition 4.4. *The map $\phi_C^{\mathrm{NN}} : \mathrm{NN}_C(n) \to \mathrm{NN}^{\mathrm{NA}}(n)$ is a bijection. Moreover, if $\phi_C^{\mathrm{NN}}(\pi) = (\sigma, X)$ and $X = \{A_1, A_2, \dots, A_k\}_<$, then*

$$\mathrm{type}(\pi) = \mathrm{type}(\sigma \setminus X) \uplus T,$$

where

$$T = \begin{cases} \mathit{pairing}(X), & \text{if } k \text{ is even}, \\ \mathit{pairing}\big(X \setminus \{A_{(k+1)/2}\}\big), & \text{if } k \text{ is odd}. \end{cases}$$

Now we consider nonnesting partitions of type D_n. Let $\pi \in \mathrm{NN}_D(n)$ and let π' be the partition obtained from π by unioning the blocks containing n or $-n$ and removing n and $-n$. Then $\phi_D^{\mathrm{NN}}(\pi)$

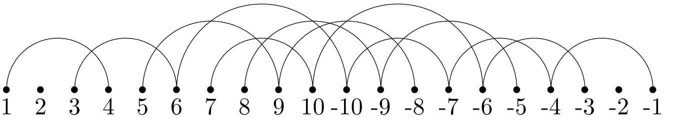


Fig. 8. The standard representation of an element in $\mathrm{NN}_D(10)$ with respect to the order $1 \prec 2 \prec \cdots \prec 10 \prec -10 \prec -9 \prec \cdots \prec -1$. Note that the locations of 10 and -10 are not important.

is defined in the same way as $\phi_D^{\mathrm{NC}}(\pi)$. That is, we define $\phi_D^{\mathrm{NN}}(\pi) = (\sigma, X, \epsilon)$, where σ and X are constructed as follows.

1. If π has the blocks $\pm\{n\}$ or π has a zero block, then $(\sigma, X) = \phi_B^{\mathrm{NN}}(\pi')$ and $\epsilon = 0$.
2. Otherwise, the block of π containing n can be written as

$$\{a_1, a_2, \dots, a_r, -b_1, -b_2, \dots, -b_s, n\}$$

for some integers r, s, $a_1, \dots, a_r$, $b_1, \dots, b_s$ with $r, s \geqslant 0$, $r + s \geqslant 1$, $1 \leqslant a_1 < \cdots < a_r < n$ and $1 \leqslant b_1 < \cdots < b_s < n$. Let $\epsilon = 1$ if $s = 0$ or $r, s > 0$ and $a_r < b_s$; and $\epsilon = -1$ otherwise. Let σ be the partition of $[n-1]$ such that $A \in \sigma$ if and only if $A = B^+ \setminus \{n\}$ for some $B \in \pi$ with $B^+ \neq \emptyset$. Let X be the set of blocks of σ which are properly contained in some blocks of π.

Note that $\phi_D^{\mathrm{NN}}(\pi) \in \mathrm{NN}_{\{0,\pm1\}}^{\mathrm{NA}}(n-1)$.

Example 4.5. Let $\pi = \{\pm\{1, 4, 7, -3, -6, 10\}, \pm\{2\}, \pm\{5, 9, -8\}\} \in \mathrm{NN}_D(10)$ as shown in Fig. 8. Then $\phi_D^{\mathrm{NN}}(\pi) = (\sigma, X, \epsilon)$ where $\sigma = \{\{1, 4, 7\}, \{2\}, \{3, 6\}, \{5, 9\}, \{8\}\}$, $X = \{\{3, 6\}, \{1, 4, 7\}, \{8\}, \{5, 9\}\}$ and $\epsilon = -1$.

Proposition 4.5. *The map $\phi_D^{\mathrm{NN}} : \mathrm{NN}_D(n) \to \mathrm{NN}_{\{0,\pm1\}}^{\mathrm{NA}}(n-1)$ is a bijection. Moreover, if $\phi_D^{\mathrm{NN}}(\pi) = (\sigma, X, \epsilon)$ and $X = \{A_1, A_2, \dots, A_k\}_<$, then* $\mathrm{type}(\pi) = \mathrm{type}(\sigma \setminus X) \uplus T$, *where*

$$T = \begin{cases} \mathit{pairing}(X) \uplus \{1\}, & \text{if } \epsilon = 0 \text{ and } k \text{ is even,} \\ \mathit{pairing}\big(X \setminus \{A_1\}\big), & \text{if } \epsilon = 0 \text{ and } k \text{ is odd,} \\ \mathit{pairing}\big(X \setminus \{A_1, A_2\}\big) \uplus \big\{|A_1| + |A_2| + 1\big\}, & \text{if } \epsilon \neq 0 \text{ and } k \text{ is even,} \\ \mathit{pairing}\big(X \setminus \{A_1\}\big) \uplus \big\{|A_1| + 1\big\}, & \text{if } \epsilon \neq 0 \text{ and } k \text{ is odd.} \end{cases}$$

Proof. The proof is similar to that of Proposition 4.2, hence we omit it. □

5. Type-preserving bijections

In the previous section we have interpreted noncrossing and nonnesting partitions of types B_n, C_n and D_n in terms of noncrossing and nonnesting partitions of type A_{n-1} or A_{n-2}. In this section we find type-preserving bijections between noncrossing and nonnesting partitions of types B_n, C_n and D_n using the following theorem as one of the building blocks.

Theorem 5.1. *(See [2, Theorem 3.1].) Suppose $\{A_1, A_2, \dots, A_k\}_<$ is the set of blocks of $\sigma \in \mathrm{NC}(n)$. Then there is a unique element $\sigma' \in \mathrm{NN}(n)$ such that $\{A'_1, A'_2, \dots, A'_k\}_<$ is the set of blocks of σ' with $\max(A_i) = \max(A'_i)$ and $|A_i| = |A'_i|$ for all $i \in [k]$.*

The above theorem follows from the observation that any partition in $\mathrm{NC}(n)$ or $\mathrm{NN}(n)$ is completely determined by the largest elements and the sizes of the blocks. For example, the largest elements (circled vertices) and the sizes (integers above vertices) of the blocks of the partition in Fig. 2 are represented below.

$$\begin{array}{cccccccccc} & & \mathbf{2} & & & & & \mathbf{1} & \mathbf{4} & \mathbf{3} \\ \bullet & \bullet & \odot & \bullet & \bullet & \bullet & \bullet & \odot & \odot & \odot \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

One can check that there are a unique noncrossing partition and a unique nonnesting partition whose largest elements and sizes of the blocks can represented as above. For instance, if it is a noncrossing partition, then 7 must be connected to 9 or 10, where it cannot be connected to 10 because the arc $(7,10)$ and the arc $(i,9)$ for some $i < 7$ will create a crossing. Thus 7 is connected to 9. In this way we can uniquely determine all arcs from the right. It is similar for a nonnesting partition. The unique nonnesting partition for the above diagram is the partition in Fig. 3.

For $\sigma \in \mathrm{NC}(n)$, let $\rho(\sigma)$ be the unique element $\sigma' \in \mathrm{NN}(n)$ in Theorem 5.1. For instance, if σ is the partition in Fig. 2, then $\rho(\sigma)$ is the one in Fig. 3. It is clear from Theorem 5.1 that the map $\rho : \mathrm{NC}(n) \to \mathrm{NN}(n)$ is a type-preserving bijection, which also preserves the largest elements of the blocks. We can naturally extend the map ρ to a map from $\mathrm{NC}^{\mathrm{NA}}(n)$ to $\mathrm{NN}^{\mathrm{NA}}(n)$. In order to do this, we need the following lemma.

Lemma 5.2. *Suppose $\{A_1, \dots, A_k\}_<$ and $\{A'_1, A'_2, \dots, A'_k\}_<$ are the sets of blocks of $\sigma \in \mathrm{NC}(n)$ and $\rho(\sigma) \in \mathrm{NN}(n)$ respectively. Then A_i is a nonaligned block of σ if and only if A'_i is a nonaligned block of $\rho(\sigma)$.*

Proof. By definition, A_i is aligned if and only if there is an integer t such that $\max(A_i) < t$ and $t \neq \max(A_j)$ for all $j \in [k]$. Thus A_{k-i} is nonaligned if and only if $\max(A_{k-i}) = n - i$. Since $\max(A_i) = \max(A'_i)$ for all $i \in [k]$, we are done. □

Now we define a map $\overline{\rho} : \mathrm{NC}^{\mathrm{NA}}(n) \to \mathrm{NN}^{\mathrm{NA}}(n)$. For $(\sigma, X) \in \mathrm{NC}^{\mathrm{NA}}(n)$, suppose that $\{A_1, A_2, \dots, A_k\}_<$ is the set of blocks of σ and $X = \{A_{i_1}, A_{i_2}, \dots, A_{i_r}\}_<$. Suppose also that $\{A'_1, A'_2, \dots, A'_k\}_<$ is the set of blocks of $\sigma' = \rho(\sigma)$ and $X' = \{A'_{i_1}, A'_{i_2}, \dots, A'_{i_r}\}_<$. Then we define $\overline{\rho}(\sigma, X) = (\sigma', X')$. In other words, if we identify a block A with its largest element $a = \max(A)$, then $\overline{\rho}(\sigma, (a_1, a_1, \dots, a_k)) = (\rho(\sigma), (a_1, a_1, \dots, a_k))$. For example, if σ is the partition in Fig. 2 and $X = \{\{8\}, \{1,4,10\}\}$ then $\overline{\rho}(\sigma, X) = (\sigma', X')$, where σ' is the partition in Fig. 3 and $X' = \{\{8\}, \{5,7,10\}\}$. Note that the largest elements of the blocks in X are exactly those in X'.

By Lemma 5.2, we have $\overline{\rho}(\sigma, X) \in \mathrm{NN}^{\mathrm{NA}}(n)$. Thus we get the following proposition.

Proposition 5.3. *The map $\overline{\rho} : \mathrm{NC}^{\mathrm{NA}}(n) \to \mathrm{NN}^{\mathrm{NA}}(n)$ is a bijection such that if $\overline{\rho}(\sigma, X) = (\sigma', X')$ and $X = \{A_1, A_2, \dots, A_k\}_<$, then $\mathrm{type}(\sigma) = \mathrm{type}(\sigma')$ and $X' = \{A'_1, A'_2, \dots, A'_k\}_<$ with $\max(A_i) = \max(A'_i)$ and $|A_i| = |A'_i|$ for all $i \in [k]$.*

5.1. Interchanging nonnested blocks and nonaligned blocks

In this subsection we will construct an involution on $\mathrm{NC}(n)$ which interchanges nonnested blocks and nonaligned blocks. In order to do this we need several definitions.

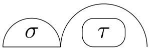
For $\pi \in \mathrm{NC}(n)$ and $S = \{a_1, a_2, \dots, a_k\}$ with $1 \leqslant a_1 < a_2 < \cdots < a_k \leqslant n$, we define $\pi \cap S$ to be the partition of $[k]$ obtained from π by removing all the integers not in S and replacing a_i with i for each $i \in [k]$.

For two partitions $\sigma \in \mathrm{NC}(n)$ and $\tau \in \mathrm{NC}(m)$, we define $\sigma \uplus \tau$ to be the partition in $\mathrm{NC}(n+m)$ obtained from σ by adding all the blocks of τ whose elements are increased by n. Ignoring the labels, the standard representation of $\sigma \uplus \tau$ looks as follows:

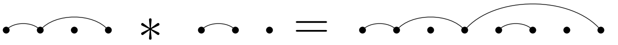
$$\sigma \uplus \tau = \boxed{\sigma} \quad \boxed{\tau}$$

If $\pi \in \mathrm{NC}(n)$ cannot be expressed as $\pi = \sigma \uplus \tau$ for some $\sigma \in \mathrm{NC}(r)$ and $\tau \in \mathrm{NC}(s)$ with $r, s \geqslant 1$, then we say that π is *connected*. Since $\pi \in \mathrm{NC}(n)$ is a noncrossing partition, π is connected if and only if 1 and n are in the same block.

For a connected partition $\sigma \in \mathrm{NC}(n)$ and any partition $\tau \in \mathrm{NC}(m)$, we define $\sigma * \tau$ to be the partition in $\mathrm{NC}(n+m+1)$ obtained from $\sigma \uplus \tau$ by adding $n+m+1$ to the block containing n. Thus the standard representation of $\sigma \uplus \tau$ looks as follows (here a half-circle means a connected partition and a round-rectangle means any partition):

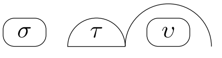
$\sigma * \tau =$ 

For example,



We also consider $\sigma * \tau$ when one (or both) of σ and τ is the empty partition $\emptyset$: $\emptyset * \emptyset$ is the unique partition $\{\{1\}\}$ in $\Pi(1)$, $\emptyset * \tau$ is $\tau \cup \{\{m+1\}\}$ and $\sigma * \emptyset$ is the partition obtained from σ by adding $n+1$ to the block containing n.

For $\pi \in \mathrm{NC}(n)$, we define two maps $decomp_1(\pi)$ and $decomp_2(\pi)$ as follows. If $\{n\}$ is not a block of π, then we can uniquely decompose π as $\pi = \sigma \uplus (\tau * \upsilon)$, see the diagram below.

$\pi =$ 

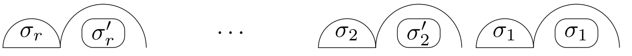
In this case, we define $decomp_1(\pi) = decomp_2(\pi) = (\sigma, \tau, \upsilon)$. If $\{n\}$ is a block of π, then we define $decomp_1(\pi) = (\pi \cap [n-1], \emptyset, \emptyset)$ and $decomp_2(\pi) = (\emptyset, \emptyset, \pi \cap [n-1])$. Note that if $decomp_1(\pi) = (\sigma, \tau, \upsilon)$ or $decomp_2(\pi) = (\sigma, \tau, \upsilon)$, we always have $\pi = \sigma \uplus (\tau * \upsilon)$. Moreover, if $decomp_1(\pi) = (\sigma, \tau, \upsilon)$ and $\tau = \emptyset$, then $\upsilon = \emptyset$, whereas, if $decomp_2(\pi) = (\sigma, \tau, \upsilon)$ and $\tau = \emptyset$, then $\sigma = \emptyset$.

Now we are ready to define a map $\xi : \mathrm{NC}(n) \to \mathrm{NC}(n)$. First, we assume that $\{n\}$ is not a block of $\pi \in \mathrm{NC}(n)$. Suppose also that π has r nonnested blocks and s nonaligned blocks.

For $i \in [r]$, let $decomp_1(\pi_i) = (\pi_{i+1}, \sigma_i, \sigma_i')$, where $\pi_1 = \pi$. Since π has r nonnested blocks, we have $\pi_i \neq \emptyset$ for $i \in [r]$ and $\pi_{r+1} = \emptyset$. Thus

$$\begin{aligned}
\pi = \pi_1 &= \pi_2 \uplus \left(\sigma_1 * \sigma_1'\right) \\
&= \pi_3 \uplus \left(\sigma_2 * \sigma_2'\right) \uplus \left(\sigma_1 * \sigma_1'\right) \\
&\ \ \vdots \\
&= \left(\sigma_r * \sigma_r'\right) \uplus \left(\sigma_{r-1} * \sigma_{r-1}'\right) \uplus \cdots \uplus \left(\sigma_1 * \sigma_1'\right).
\end{aligned}$$

Pictorially, the above decomposition of π can be represented as follows.

$\pi =$ 

Note that $\sigma_1 \neq \emptyset$, and for $2 \leqslant i \leqslant r$, if $\sigma_i = \emptyset$, the $\sigma_i' = \emptyset$. If $\{N_1, N_2, \ldots, N_r\}_<$ is the set of all nonnested blocks of π, then $|N_i| - 1$ is equal to the size of the block of σ_{r+1-i} containing the largest integer if $\sigma_{r+1-i} \neq \emptyset$; and 0 if $\sigma_{r+1-i} = \emptyset$.

Similarly, for $i \in [s]$, let $decomp_2(\upsilon_i) = (\tau_i', \tau_i, \upsilon_{i+1})$, where $\upsilon_1 = \pi$. Since π has s nonaligned blocks, we have $\upsilon_i \neq \emptyset$ for $i \in [s]$ and $\upsilon_{s+1} = \emptyset$. Thus

$$\begin{aligned}
\pi = \upsilon_1 &= \tau_1' \uplus (\tau_1 * \upsilon_2) \\
&= \tau_1' \uplus \left(\tau_1 * \left(\tau_2' \uplus (\tau_2 * \upsilon_3)\right)\right) \\
&\ \ \vdots \\
&= \tau_1' \uplus \left(\tau_1 * \left(\tau_2' \uplus \left(\tau_2 * \left(\tau_3' \uplus \cdots \left(\tau_s' \uplus (\tau_s * \emptyset)\right) \cdots\right)\right)\right)\right).
\end{aligned}$$

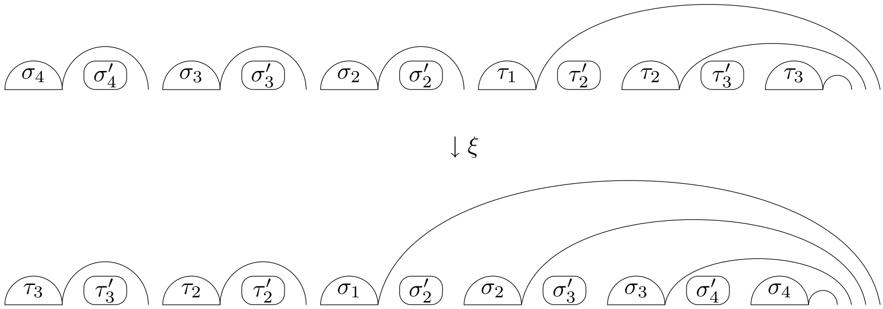


Fig. 9. Illustration of the map ξ. We have $\sigma_1 = \tau_1$.

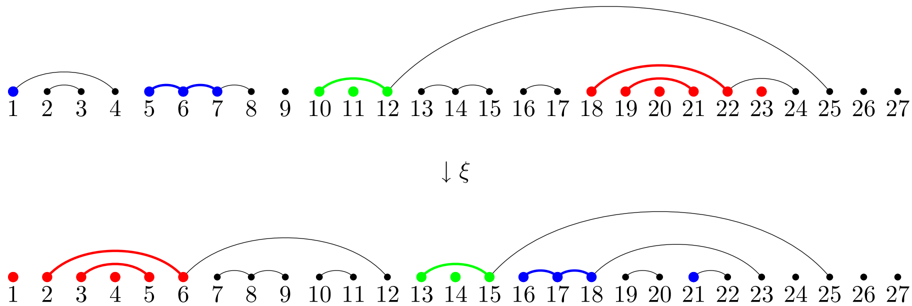
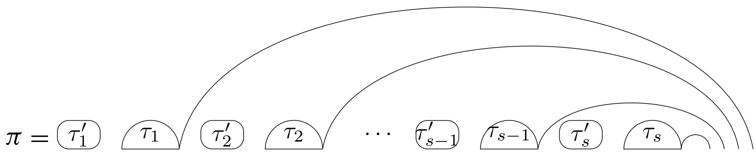


Fig. 10. An example of the map ξ. In the upper diagram, $\sigma_1 = \tau_1$ is colored green, σ_i's are colored blue, τ_i's are colored red for $i \geqslant 2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Pictorially, the above decomposition of π can be represented as follows.



Note that $\tau_1 \neq \emptyset$, and for $2 \leqslant i \leqslant s$, if $\tau_i = \emptyset$, the $\tau_i' = \emptyset$. If $\{A_1, A_2, \ldots, A_s\}_<$ is the set of all nonaligned blocks of π, then $|A_i| - 1$ is equal to the size of the block of τ_{s+1-i} containing the largest integer if $\tau_{s+1-i} \neq \emptyset$; and 0 if $\tau_{s+1-i} = \emptyset$.

Since $\{n\}$ is not a block of π, we have $decomp_1(\pi) = decomp_2(\pi)$, thus $\pi_2 = \tau_1'$, $\sigma_1 = \tau_1$ and $\sigma_1' = \upsilon_2$. Thus we get the following:

$$\pi = \left(\sigma_r * \sigma_r'\right) \uplus \cdots \uplus \left(\sigma_2 * \sigma_2'\right) \uplus \left(\tau_1 * \left(\tau_2' \uplus \left(\tau_2 * \left(\tau_3' \uplus \cdots \left(\tau_s' \uplus (\tau_s * \emptyset)\right) \cdots\right)\right)\right)\right).$$

Then we define

$$\xi(\pi) = \left(\tau_s * \tau_s'\right) \uplus \cdots \uplus \left(\tau_2 * \tau_2'\right) \uplus \left(\sigma_1 * \left(\sigma_2' \uplus \left(\sigma_2 * \left(\sigma_3' \uplus \cdots \left(\sigma_r' \uplus (\sigma_r * \emptyset)\right) \cdots\right)\right)\right)\right).$$

See Fig 9.

Now let π be any element in $\mathrm{NC}(n)$. If k is the largest integer such that $k \leqslant n$ and $\{k\}$ is not a block of π, we define $\xi(\pi)$ to be the partition obtained from $\xi(\pi \cap [k])$ by adding the blocks $\{k+1\}, \{k+2\}, \ldots, \{n\}$. See Fig. 10.

For $\pi \in \mathrm{NC}(n)$, let $\mathrm{nn}(\pi)$ (resp. $\mathrm{na}(\pi)$) denote the number of nonnested (resp. nonaligned) blocks of π. From the construction of ξ, it is easy to see that the following theorem holds.

Theorem 5.4. *The map ξ is a type-preserving involution on* $\mathrm{NC}(n)$ *satisfying* $\mathrm{nn}(\xi(\pi)) = \mathrm{na}(\pi)$ *and* $\mathrm{na}(\xi(\pi)) = \mathrm{nn}(\pi)$. *Moreover, if* $\{N_1, N_2, \ldots, N_r\}_<$, $\{N'_1, N'_2, \ldots, N'_s\}_<$, $\{A_1, A_2, \ldots, A_s\}_<$ *and* $\{A'_1, A'_2, \ldots, A'_r\}_<$ *are the set of nonnested blocks of* π *and* $\xi(\pi)$ *and the set of nonaligned blocks of* π *and* $\xi(\pi)$ *respectively, then* $|N_i| = |A'_i|$ *and* $|A_j| = |N'_j|$ *for all* $i \in [r]$ *and* $j \in [s]$.

The following corollary is an immediate consequence of Theorem 5.4.

Corollary 5.5. *We have*

$$\sum_{\pi \in \mathrm{NC}(n)} x^{\mathrm{nn}(\pi)} y^{\mathrm{na}(\pi)} = \sum_{\pi \in \mathrm{NC}(n)} x^{\mathrm{na}(\pi)} y^{\mathrm{nn}(\pi)}.$$

In fact, we can find a formula for the following generating function:

$$F(x, y, z) = \sum_{n \geqslant 0} \left(\sum_{\pi \in \mathrm{NC}(n)} x^{\mathrm{nn}(\pi)} y^{\mathrm{na}(\pi)} \right) z^n.$$

Let $\mathrm{NC}'(n)$ denote the set of connected partitions in $\mathrm{NC}(n)$. We define

$$C(z) = \sum_{n \geqslant 0} \# \mathrm{NC}(n) z^n = \frac{1 - \sqrt{1-4z}}{2z}, \qquad B(z) = \sum_{n \geqslant 1} \# \mathrm{NC}'(n) z^n,$$

$$A(x, z) = \sum_{n \geqslant 0} \left(\sum_{\pi \in \mathrm{NC}(n)} x^{\mathrm{nn}(\pi)} \right) z^n = \sum_{n \geqslant 0} \left(\sum_{\pi \in \mathrm{NC}(n)} x^{\mathrm{na}(\pi)} \right) z^n.$$

It is not difficult to see that

$$C(z) = \frac{1}{1 - B(z)}, \qquad A(x, z) = \frac{1}{1 - xB(z)}.$$

Using the decomposition $\pi = \sigma \uplus (\tau * \upsilon) \uplus \mu$, where μ is a partition consisting of singletons and τ is a connected partition, one can also show that

$$F(x, y, z) = \frac{1}{1 - xyz} \big(1 + xyzA(x, z)A(y, z)B(z) \big).$$

Solving the above equations, we get the following generating function.

Proposition 5.6. *We have*

$$F(x, y, z) = \frac{1}{1 - xyz} \left(1 + \frac{xyzC(C-1)}{((1-x)C + x)((1-y)C + y)} \right),$$

where $C = \frac{1 - \sqrt{1-4z}}{2z}$, *the generating function for the Catalan numbers* $\frac{1}{n+1}\binom{2n}{n}$.

We can naturally extend ξ to the map $\bar{\xi} : \mathrm{NC}^{\mathrm{NN}}(n) \to \mathrm{NC}^{\mathrm{NA}}(n)$ defined as follows. Let $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$ and $\sigma' = \xi(\sigma)$. Suppose $\{A_1, A_2, \ldots, A_k\}_<$ is the set of all nonnested blocks of σ and $\{A'_1, A'_2, \ldots, A'_k\}_<$ is the set of all nonaligned blocks of σ'. Then we can write $X = \{A_{i_1}, A_{i_2}, \ldots, A_{i_r}\}_<$. We define $\bar{\xi}(\sigma, X) = (\sigma', X')$, where $X' = \{A'_{i_1}, A'_{i_2}, \ldots, A'_{i_r}\}_<$. By Theorem 5.4, we get the following corollary.

Corollary 5.7. *The map* $\bar{\xi} : \mathrm{NC}^{\mathrm{NN}}(n) \to \mathrm{NC}^{\mathrm{NA}}(n)$ *is a bijection. Moreover, if* $\bar{\xi}(\sigma, X) = (\sigma', X')$, $X = \{A_1, \ldots, A_r\}_<$ *and* $X' = \{A'_1, \ldots, A'_s\}_<$, *then* $\mathrm{type}(\sigma) = \mathrm{type}(\sigma')$, $r = s$ *and* $|A_i| = |A'_i|$ *for all* $i \in [r]$.

5.2. Rearranging nonnested blocks

Let $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$. Suppose $\{A_1, A_2, \dots, A_\ell\}_<$ is the set of all nonnested blocks of σ, $X = \{A_{i_1}, A_{i_2}, \dots, A_{i_k}\}_<$, and $\sigma_j = \sigma \cap [\min(A_j), \max(A_j)]$. Then we have $\sigma = \sigma_1 \uplus \sigma_2 \uplus \cdots \uplus \sigma_\ell$. For a permutation $p = p_1 p_2 \cdots p_k$ of $[k]$, the *rearrangement of* (σ, X) *according to* p is defined to be the pair (σ', X') of $\sigma' = \sigma_{a_1} \uplus \sigma_{a_2} \uplus \cdots \uplus \sigma_{a_\ell}$ and $X = \{A'_{i_1}, A'_{i_2}, \dots, A'_{i_k}\}$, where $a_j = j$ if $j \notin \{i_1, i_2, \dots, i_k\}$; and $a_j = i_{p_t}$ if $j = i_t$, and $\{A'_1, A'_2, \dots, A'_\ell\}_<$ is the set of all nonnested blocks of σ'.

For $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$ with $|X| = k$, we define $\iota_B(\sigma, X)$ to be the rearrangement of (σ, X) according to

$$p = \begin{cases} 1\,2 \cdots k, & \text{if } k = 2t, \\ (t+1)1\,2 \cdots t(t+2)(t+3) \cdots (2t+1), & \text{if } k = 2t+1. \end{cases}$$

For $(\sigma, X, \epsilon) \in \mathrm{NC}^{\mathrm{NN}}_{\{0,\pm1\}}(n)$ with $|X| = k$, we define $\iota_D(\sigma, X, \epsilon)$ to be (σ', X', ϵ), where (σ', X') is the rearrangement of (σ, X) according to

$$p = \begin{cases} 1\,2 \cdots k, & \text{if } k = 2t \text{ and } \epsilon = 0, \\ t(t+1)1\,2 \cdots (t-1)(t+2)(t+3) \cdots (2t), & \text{if } k = 2t \text{ and } \epsilon \neq 0, \\ (t+1)1\,2 \cdots t(t+2)(t+3) \cdots (2t+1), & \text{if } k = 2t+1. \end{cases}$$

Clearly, $\iota_B : \mathrm{NC}^{\mathrm{NN}}(n) \to \mathrm{NC}^{\mathrm{NN}}(n)$ and $\iota_D : \mathrm{NC}^{\mathrm{NN}}_{\{0,\pm1\}}(n) \to \mathrm{NC}^{\mathrm{NN}}_{\{0,\pm1\}}(n)$ are type-preserving bijections.

By the properties of the bijections we have defined so far, we get the following theorem.

Theorem 5.8. *The composed maps $(\phi_B^{\mathrm{NN}})^{-1} \circ \overline{\rho} \circ \overline{\xi} \circ \iota_B \circ \phi_B^{\mathrm{NC}}$, $(\phi_C^{\mathrm{NN}})^{-1} \circ \overline{\rho} \circ \overline{\xi} \circ \phi_B^{\mathrm{NC}}$ and $(\phi_D^{\mathrm{NN}})^{-1} \circ \overline{\rho} \circ \overline{\xi} \circ \iota_D \circ \phi_D^{\mathrm{NC}}$ are type-preserving bijections between noncrossing partitions and nonnesting partitions of type B_n, C_n and D_n respectively*; *see Figs.* 14 *and* 15.

Remark 5.9. Our type-preserving bijections are different from those of Fink and Giraldo [11] because our bijections do not preserve certain statistics preserved by their bijections. In fact, they showed that their bijections are the unique ones preserving those statistics. There are other bijections between noncrossing and nonnesting partitions of classical types due to Rubey and Stump [18] for type B and Conflitti and Mamede [8] for type D. However their bijections preserve not the types but 'openers' and 'closers'.

6. Another interpretation for noncrossing partitions of type *B* and type *D*

We denote by $\mathfrak{B}(n)$ the set of pairs (σ, x), where $\sigma \in \mathrm{NC}(n)$ and x is either $\emptyset$, an edge or a block of σ. Note that if a partition σ of $[n]$ has i edges, then there are $n-i$ blocks in σ. For each $\sigma \in \mathrm{NC}(n)$, we have $n+1$ choices for x with $(\sigma, x) \in \mathfrak{B}(n)$. Hence, $\mathfrak{B}(n)$ is essentially the same as $\mathrm{NC}(n) \times [n+1]$.

We define a map $\varphi_B : \mathrm{NC}^{\mathrm{NN}}(n) \to \mathfrak{B}(n)$ as follows. For $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$ with $X = \{A_1, A_2, \dots, A_k\}_<$, $\varphi_B(\sigma, X)$ is defined to be (σ', x), where σ' is the partition obtained from σ by unioning A_i and A_{k+1-i} for $i = 1, 2, \dots, \lfloor k/2 \rfloor$, and

$$x = \begin{cases} \emptyset, & \text{if } k = 0, \\ \big(\max(A_t), \min(A_{t+1})\big), & \text{if } k \neq 0 \text{ and } k = 2t, \\ A_{t+1}, & \text{if } k = 2t+1. \end{cases}$$

Example 6.1. If $\sigma = \{\{1,2\}, \{3\}, \{4,7\}, \{5,6\}, \{8,9,10\}, \{11\}\}$ and $X = \{\{1,2\}, \{3\}, \{4,7\}, \{8,9,10\}, \{11\}\}$, then $\varphi_B(\sigma, X) = (\sigma', x)$, where $\sigma' = \{\{1,2,11\}, \{3,8,9,10\}, \{4,7\}, \{5,6\}\}$ and x is the block $\{4,7\}$.

Theorem 6.1. *The map $\psi_B = \varphi_B \circ \phi_B^{\mathrm{NC}}$ is a bijection between $\mathrm{NC}_B(n)$ and $\mathfrak{B}(n)$. Moreover, if $\psi_B(\pi) = (\sigma, x)$, then* $\mathrm{type}(\pi) = \mathrm{type}(\sigma)$ *if x is not a block*; *and* $\mathrm{type}(\pi) = \mathrm{type}(\sigma \setminus \{x\})$ *if x is a block.*

Proof. Since $\phi_B^{\mathrm{NC}}:\mathrm{NC}_B(n)\to\mathrm{NC}^{\mathrm{NN}}(n)$ is a bijection, it is sufficient to show that $\varphi_B:\mathrm{NC}^{\mathrm{NN}}(n)\to\mathfrak{B}(n)$ is a bijection. Let us find the inverse map of φ_B.

Let $(\sigma,x)\in\mathfrak{B}(n)$. Then we construct σ' and X as follows.

If $x=\emptyset$, then $\sigma'=\sigma$ and $X=\emptyset$.

If x is an edge (a,b), then let E be the set of edges (i,j) of σ with $i\leqslant a<b\leqslant j$. Then σ' is the partition obtained from σ by removing the edges in E, and X is the set of blocks of σ' which contain an endpoint of an edge in E. Here the endpoints of an edge (i,j) are the integers i and j.

If x is a block B, then let E be the set of edges (i,j) of σ with $i<\min(B)\leqslant\max(B)<j$. Then σ' is the partition obtained from σ by removing the edges in E, and X is the set of blocks of σ' which are equal to B or contain an endpoint of an edge in E.

It is easy to see that the map $(\sigma,x)\mapsto(\sigma',X)$ is the inverse of φ_B. The 'moreover' statement is clear from the construction of ϕ_B^{NC} and φ_B. □

Since $\mathfrak{B}(n)$ is the same as $\mathrm{NC}(n)\times[n+1]$, Theorem 6.1 gives a bijective proof of $\#\mathrm{NC}_B(n)=\binom{2n}{n}$.

Remark 6.1. For $\pi\in\mathrm{NC}_B(n)$, let $Abs(\pi)$ be the partition in $\mathrm{NC}(n)$ such that B is a block of $Abs(\pi)$ if and only if $B=\{|i|:\ i\in B'\}$ for some $B'\in\pi$. Biane et al. [5, Theorem in Section 14] proved that the map $\pi\mapsto Abs(\pi)$ is an $(n+1)$-to-1 map from $\mathrm{NC}_B(n)$ to $\mathrm{NC}(n)$, thus proved $\#\mathrm{NC}_B(n)=\binom{2n}{n}$ bijectively. In fact, they proved that $\mathrm{NC}_B(n)$ is in bijection with the set of pairs (σ,x) where $\sigma\in\mathrm{NC}(n)$ and x is a block of either σ or the Kreweras complement $\mathrm{Kr}(\sigma)$. The Kreweras complement has the property that the sum of the number of blocks of σ and the number of blocks of $\mathrm{Kr}(\sigma)$ is equal to $n+1$. It is easy to check that if $\varphi_B\circ\phi_B^{\mathrm{NC}}(\pi)=(\sigma,x)$, then $\sigma=Abs(\pi)$.

We denote by $\mathfrak{D}(n)$ the set of pairs (σ,x) such that $\sigma\in\mathrm{NC}(n-1)$ and x is either $\emptyset$, an edge of σ, a block of σ or an integer in $[\pm(n-1)]$. We can also easily see that $\mathfrak{D}(n)$ is essentially the same as $\mathrm{NC}(n-1)\times[3n-2]$.

We define a map $\varphi_D:\mathrm{NC}^{\mathrm{NN}}_{\{0,\pm1\}}(n-1)\to\mathfrak{D}(n)$ as follows. Let $(\sigma,X,\epsilon)\in\mathrm{NC}^{\mathrm{NN}}_{\{0,\pm1\}}(n-1)$ and $X=\{A_1,A_2,\dots,A_k\}_<$. Then $\varphi_D(\sigma,X,\epsilon)$ is defined to be (σ',x), where σ' is the partition obtained from σ by unioning A_i and A_{k+1-i} for $i=1,2,\dots,\lfloor k/2\rfloor$, and

$$x=\begin{cases}\emptyset, & \text{if } \epsilon=0 \text{ and } k=0,\\ \big(\max(A_t),\min(A_{t+1})\big), & \text{if } \epsilon=0,\ k=2t\neq 0,\\ A_{t+1}, & \text{if } \epsilon=0 \text{ and } k=2t+1,\\ \epsilon\cdot\max(A_{\lfloor(k+1)/2\rfloor}) & \text{if } \epsilon\neq 0.\end{cases}$$

Theorem 6.2. *The map $\psi_D=\varphi_D\circ\phi_D^{\mathrm{NC}}$ is a bijection between $\mathrm{NC}_D(n)$ and $\mathfrak{D}(n)$. Moreover, if $\psi_D(\pi)=(\sigma,x)$, then*

$$\mathrm{type}(\pi)=\begin{cases}\mathrm{type}(\sigma)\uplus\{1\}, & \text{if } x=\emptyset \text{ or } x \text{ is an edge},\\ \mathrm{type}\big(\sigma\setminus\{x\}\big), & \text{if } x \text{ is a block},\\ \mathrm{type}\big(\sigma\setminus\{B\}\big)\uplus\big\{|B|+1\big\}, & \text{if } x\in\big[\pm(n-1)\big] \text{ and } B \text{ is the block}\\ & \text{of } \sigma \text{ containing } |x|.\end{cases}$$

Proof. The proof is similar to that of Theorem 6.1, hence we omit it. □

Since $\mathfrak{D}(n)$ is the same as $\mathrm{NC}(n-1)\times[3n-2]$, Theorem 6.2 gives a bijective proof of $\#\mathrm{NC}_D(n)=\frac{3n-2}{n}\binom{2(n-1)}{n-1}$.

For an integer partition $\lambda=\{1^{m_1},2^{m_2},\dots\}$, let $m_\lambda=m_1!m_2!\cdots$.

Kreweras proved the following formula for the number of $\pi\in\mathrm{NC}(n)$ with given block sizes.

Theorem 6.3. *(See [13].) Let λ be an integer partition with $|\lambda| = n$ and length ℓ. Then the number of $\pi \in \mathrm{NC}(n)$ with $\mathrm{type}(\pi) = \lambda$ is equal to*

$$\frac{n!}{m_\lambda (n-\ell+1)!}.$$

As an application of Theorems 6.1 and 6.2, we can give another proof of the following type B and type D analogs of Theorem 6.3.

Theorem 6.4. *(See [2].) Let λ be an integer partition with $|\lambda| \leqslant n$ and length ℓ. Then the number of $\pi \in \mathrm{NC}_B(n)$ with $\mathrm{type}(\pi) = \lambda$ is equal to*

$$\frac{n!}{m_\lambda (n-\ell)!}.$$

Proof. Let $|\lambda| = n-k$ and $\psi_B(\pi) = (\sigma, x) \in \mathfrak{B}(n)$.

If $k = 0$, then π does not have a zero block and x is not a block. Since σ has ℓ blocks and $n-\ell$ edges, there are $(n-\ell+1) \cdot \frac{n!}{m_\lambda (n-\ell+1)!} = \frac{n!}{m_\lambda (n-\ell)!}$ choices of $(\sigma, x) \in \mathfrak{B}(n)$.

If $k \neq 0$, then π has a zero block of size $2k$. Thus x is a block of size k in σ. Let $\lambda = \{1^{m_1}, 2^{m_2}, \ldots\}$ and $\lambda' = \mathrm{type}(\sigma)$. Note that $\lambda' = \lambda \uplus \{k\}$ and $m_{\lambda'} = m_\lambda \cdot \frac{(m_k+1)!}{m_k!} = m_\lambda (m_k+1)$. Thus, there are $\frac{n!}{m_{\lambda'}(n-\ell)!}$ choices for $\sigma \in \mathrm{NC}(n)$ and for each σ there are (m_k+1) choices for x. Thus we get the desired formula. $\square$

Theorem 6.5. *(See [3].) Let $\lambda = \{1^{m_1}, 2^{m_2}, \ldots\}$ be an integer partition with $|\lambda| \leqslant n$ and length ℓ. Then the number of $\pi \in \mathrm{NC}_D(n)$ with $\mathrm{type}(\pi) = \lambda$ is equal to*

$$\begin{cases} \dfrac{(n-1)!}{m_\lambda (n-\ell-1)!}, & \text{if } |\lambda| \leqslant n-2, \\ \big(m_1 + 2(n-\ell)\big)\dfrac{(n-1)!}{m_\lambda (n-\ell)!}, & \text{if } |\lambda| = n. \end{cases}$$

Note that if $\mathrm{type}(\pi) = \lambda$ for $\pi \in \mathrm{NC}_D(n)$, then $|\lambda|$ cannot be $n-1$.

Proof of Theorem 6.5. Let $|\lambda| = n-k$ and $\psi_D(\pi) = (\sigma, x)$.

If $k \geqslant 2$, then x is a block of size k and we can use the same argument in the proof of Theorem 6.4.

Assume $k = 0$. Then x is either $\emptyset$, an edge of σ or an integer in $[\pm(n-1)]$.

If $x = \emptyset$, then $\mathrm{type}(\sigma) = \lambda \setminus \{1\} = \{1^{m_1-1}, 2^{m_2}, \ldots\}$.

If x is an edge, then the type of σ is $\lambda \setminus \{1\}$. Since σ has $\ell-1$ blocks, there are $n-\ell$ choices of x.

Let $\lambda' = \lambda \setminus \{1\}$. Then there are $\frac{(n-1)!}{m_{\lambda'}((n-1)-(\ell-1)+1)!}$ choices of σ and $n-\ell+1$ choices of x. Thus there are

$$\frac{(n-1)!}{m_{\lambda'}(n-\ell)!} = m_1 \cdot \frac{(n-1)!}{m_\lambda (n-\ell)!} \tag{1}$$

possibilities when x is either $\emptyset$ or an edge.

Now assume that x is an integer in $[\pm(n-1)]$. If $|x|$ is contained in a block of size i, then the corresponding block in σ is of size $i+1$. Thus

$$\mathrm{type}(\sigma) = \lambda^{(i)} = \big\{1^{m_1}, \ldots, (i-1)^{m_{i-1}}, i^{m_i+1}, (i+1)^{m_{i+1}-1}, (i+2)^{m_{i+2}}, \ldots\big\}.$$

Note that $m_{\lambda^{(i)}} = m_\lambda \cdot \frac{1+m_i}{m_{i+1}}$. Thus there are $\frac{(n-1)!}{m_{\lambda^{(i)}}(n-1-\ell+1)!}$ choices of σ. For each σ, there are $1+m_i$ choices for the block containing x, and $2i$ choices for x. Thus in this case the number of possible (σ, x)'s is equal to

$$\sum_{i \geqslant 1} 2i(1+m_1)\frac{(n-1)!}{m_{\lambda^{(i)}}(n-\ell)!} = \frac{2(n-1)!}{m_\lambda (n-\ell)!}\sum_{i \geqslant 1}(1+m_i) \cdot \frac{i \cdot m_{i+1}}{1+m_i}. \tag{2}$$

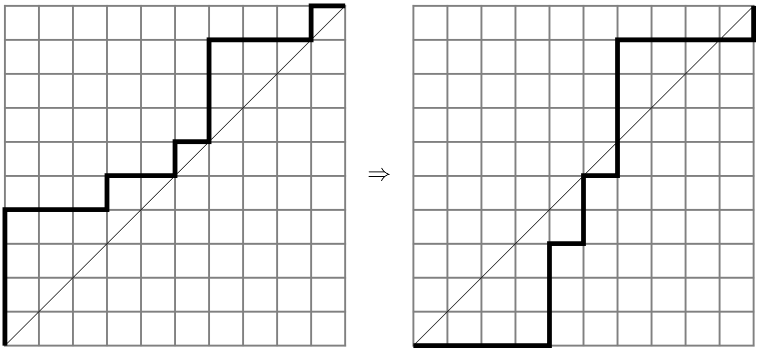


Fig. 11. A lattice path is obtained from a Dyck path by reflecting several subpaths.

Since

$$\begin{aligned}\sum_{i\geqslant 1} i\cdot m_{i+1} &= \sum_{i\geqslant 0} i\cdot m_{i+1} = \sum_{i\geqslant 0}(i+1)m_{i+1} - \sum_{i\geqslant 0} m_{i+1}\\ &= \sum_{i\geqslant 1} i\cdot m_i - \sum_{i\geqslant 1} m_i = n-\ell,\end{aligned}$$

(2) is equal to $(n-\ell)\cdot\frac{2(n-1)!}{m_\lambda(n-\ell)!}$. The sum of (1) and (2) gives the desired formula. □

7. Lattice paths

Let LP(n) denote the set of lattice paths from $(0,0)$ to (n,n) consisting of up step $(0,1)$ and east step $(1,0)$. A *Dyck path* of length $2n$ is a lattice path in LP(n) which never goes below the line $y=x$.

It is well known that NC(n) is in bijection with the set of *Dyck path* of length $2n$: the Dyck path corresponding to $\sigma\in \mathrm{NC}(n)$ is determined as follows. The $(2i-1)$th step and the $(2i)$th step are, respectively, $(0,1)$ and $(0,1)$ if i is the minimum of a non-singleton block of σ; $(1,0)$ and $(1,0)$ if i is the maximum of a non-singleton block of σ; $(0,1)$ and $(1,0)$ if $\{i\}$ is a block of σ; $(1,0)$ and $(0,1)$ otherwise.

Now let us find a bijection between $\mathrm{NC}_B(n)$ and LP(n). Since $\mathrm{NC}_B(n)$ is in bijection with $\mathrm{NC}^{\mathrm{NN}}(n)$, we will use $\mathrm{NC}^{\mathrm{NN}}(n)$ instead of $\mathrm{NC}_B(n)$.

Let $(\sigma, X)\in \mathrm{NC}^{\mathrm{NN}}(n)$. Suppose P is the Dyck path corresponding to σ. Consider a block $B\in X$ with $\min(B)=i$ and $\max(B)=j$. Since B is nonnested, the $(2i-1)$th step starts at $(i-1,i-1)$ and the $(2j)$th step ends at (j,j). Then we reflect the subpath of P consisting of the rth steps for all $r\in[2i-1,2j]$ across the line $y=x$. Let $g(\sigma,X)$ be the lattice path obtained by this reflection for each $B\in X$.

Example 7.1. Let $\sigma=\{\{1,4,5\},\{2,3\},\{6\},\{7,9\},\{8\},\{10\}\}$ and $X=\{\{1,4,5\},\{6\},\{10\}\}$. Then $(\sigma,X)\in \mathrm{NC}^{\mathrm{NN}}(10)$. The lattice path $g(\sigma,X)$ is obtained from the Dyck path corresponding to σ by reflecting the subpaths corresponding to the nonnested blocks in X. See Fig. 11.

It is easy to see that the map g is a bijection.

Proposition 7.1. *The map* $g:\mathrm{NC}^{\mathrm{NN}}(n)\to \mathrm{LP}(n)$ *is a bijection.*

Thus we get $\#\mathrm{NC}_B(n)=\#\mathrm{NC}^{\mathrm{NN}}(n)=\binom{2n}{n}$. Note that we did not use the number of Dyck paths. Since $\#\mathrm{NC}_B(n)=\#\mathfrak{B}(n)=(n+1)\cdot\#\mathrm{NC}(n)$, we get another combinatorial proof of the fact that the number of Dyck paths of length $2n$ is equal to the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Remark 7.1. Reiner [17, Proposition 17] also found a bijection between $\mathrm{NC}_B(n)$ and $\mathrm{LP}(n)$ which is different from ours. Ferrari [10, Proposition 2.5] considered the set $\widetilde{\mathrm{NC}}(n)$ of 'component-bicolored' noncrossing partitions of $[n]$ and found a bijection between this set and $\mathrm{LP}(n)$. In fact, $\widetilde{\mathrm{NC}}(n)$ is essentially the same as $\mathrm{NC}^{\mathrm{NN}}(n)$ and our bijection g is identical with Ferrari's bijection.

We can also find a bijection between $\mathrm{NC}_D(n)$ and a subset of $\mathrm{LP}(n)$. To do this, we need another interpretation for $\mathrm{NC}_D(n)$.

We denote by $\overline{\mathrm{NC}}^{\mathrm{NN}}(n)$ the set of elements $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$ such that if X has a block A containing n, then $|A| \geqslant 2$.

For $(\sigma, X, \epsilon) \in \mathrm{NC}^{\mathrm{NN}}_{\{0,\pm 1\}}(n-1)$ with $X = \{A_1, A_2, \dots, A_k\}_<$, we define $\kappa(\sigma, X, \epsilon)$ to be the pair (σ', X'), where σ' and X' are defined as follows:

- If $\epsilon = 0$, then let σ' be the partition obtained from σ by adding the singleton $\{n\}$ and let $X' = X$.
- If $\epsilon = 1$, then let σ' be the partition obtained from σ by adding n to the block A_k and let $X' = X$.
- If $\epsilon = -1$, then let σ' be the partition obtained from σ by adding n to the block A_k and let $X' = X \setminus \{A_k\}$.

One can easily check that this is a bijection.

Proposition 7.2. *The map* $\kappa : \mathrm{NC}^{\mathrm{NN}}_{\{0,\pm 1\}}(n-1) \to \overline{\mathrm{NC}}^{\mathrm{NN}}(n)$ *is a bijection.*

Let $\overline{\mathrm{LP}}(n)$ denote the set of lattice paths in $\mathrm{LP}(n)$ which do not touch $(n-1, n-1)$ and $(n, n-1)$ simultaneously. Note that the cardinality of $\overline{\mathrm{LP}}(n)$ is equal to $\binom{2n}{n} - \binom{2n-2}{n-1}$. It is easy to see that $g(\sigma, X) \in \overline{\mathrm{LP}}(n)$ for each $(\sigma, X) \in \overline{\mathrm{NC}}^{\mathrm{NN}}(n)$, and the map $g : \overline{\mathrm{NC}}^{\mathrm{NN}}(n) \to \overline{\mathrm{LP}}(n)$ is a bijection.

Proposition 7.3. *The map* $g : \overline{\mathrm{NC}}^{\mathrm{NN}}(n) \to \overline{\mathrm{LP}}(n)$ *is a bijection.*

Thus we get a combinatorial proof of $\#\,\mathrm{NC}_D(n) = \#\overline{\mathrm{NC}}^{\mathrm{NN}}(n) = \binom{2n}{n} - \binom{2n-2}{n-1}$.

8. Catalan tableaux of classical types

A *Ferrers diagram* is a left-justified arrangement of square cells with possibly empty rows and columns. The *length* of a Ferrers diagram is the sum of the number of rows and the number of columns. If a Ferrers diagram is of length n, then we label the steps in the border of the Ferrers diagram with $1, 2, \dots, n$ from north-west to south-east. We label a row (resp. column) with i if the row (resp. column) contains the south (resp. east) step labeled with i. The (i, j)*-entry* is the cell in the row labeled with i and in the column labeled with j. See Fig. 12.

For a Ferrers diagram F, a *permutation tableau* of shape F is a $0,1$-filling of the cells in F satisfying the following conditions:

1. each column has at least one 1,
2. there is no 0 which has a 1 above it in the same column and a 1 to the left of it in the same row.

The *length* of a permutation tableau is defined to be the length of its shape. A *Catalan tableau* is a permutation tableau which has exactly one 1 in each column. Let $\mathrm{CT}(n)$ denote the set of Catalan tableaux of length n. There is a simple bijection between $\mathrm{CT}(n)$ and $\mathrm{NC}(n)$ due to Burstein [7, Theorem 3.1]. His bijection can be described in the following way which is similar to that in the proof of Proposition 6 in [9].

Let $\sigma \in \mathrm{NC}(n)$. We first make the Ferrers diagram F as follows. The ith step of the border of F is south if i is the smallest integer in the block containing i; and west otherwise. We fill the (i, j)-entry with 1 if and only if i and j are in the same block whose smallest integer is i. One can easily

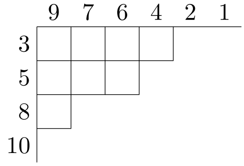


Fig. 12. A Ferrers diagram with labeled rows and columns.

	9	7	6	4	2	1
−9	0					
−7	0	0				
−6	0	0	0			
−4	1	1	0	1		
−2	0	0	0	0	0	
−1	0	0	0	0	1	1
3	0	0	0	0		
5	0	0	1			
8	0					
10						

Fig. 13. The Catalan tableau $f(\pi, X)$ of type B_{10} for $\pi = \{\{1,2\},\{3\},\{4,7,9\},\{5,6\},\{8\},\{10\}\}$ and $X = \{\{1,2\},\{4,7,9\}\}$.

check that this is a bijection. For more information of Catalan tableaux and permutation tableaux, see [20,21].

Lam and Williams [14] defined permutation tableaux of type B_n. See [15] for the 'alternative tableaux' version. The definition of permutation tableaux of type B_n in [14] can be written as follows.

Let F be a Ferrers diagram with k columns including empty columns. The *shifted* Ferrers diagram $\overline{F}$ of F is the diagram obtained from F by adding k rows of size $1, 2, \dots, k$ above it in increasing order. The rightmost cell of an added row is called *diagonal*. We label the added rows as follows. If the diagonal of an added row is in the column labeled with i, then the row is labeled with $-i$. For example, see Fig. 13; at this moment, ignore the 0's and 1's.

A *permutation tableau of type* B_n is a 0,1-filling of the cells in the shifted Ferrers diagram $\overline{F}$ for a Ferrers diagram F of length n satisfying the following conditions:

1. each column has at least one 1,
2. there is no 0 which has a 1 above it in the same column and a 1 to the left of it in the same row,
3. if a 0 is in a diagonal, then it does not have a 1 to the left of it in the same row.

A *Catalan tableau of type* B_n is a permutation tableau of type B_n such that each column has exactly one 1. A *Catalan tableau of type* D_n is a Catalan tableau of type B_n with the following additional condition: if the last row is not empty, then the left most column does not have 1 in the topmost cell. Let $\mathrm{CT}_B(n)$ and $\mathrm{CT}_D(n)$ denote the set of Catalan tableaux of type B_n and type D_n respectively.

Now we will find a bijection between $\mathrm{NC}^{\mathrm{NN}}(n)$ and $\mathrm{CT}_B(n)$.

Let $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$. Suppose F is the Ferrers diagram of length n such that the ith step of the border of F is south if i is the smallest integer in a block of σ which is not in X; and west otherwise. Let T be the 0,1-filling of the shifted Ferrers diagram $\overline{F}$ obtained as follows. For each i which is the smallest integer in a block in X, fill the $(-i, i)$-entry with 1. For each pair (i, j) of distinct integers

$$\begin{tikzcd}
 & \mathrm{CT}_B(n) \arrow[d, dash, "f"'] & \mathrm{NC}^{\mathrm{NA}}(n) \arrow[r, dash, "\overline{\rho}"] & \mathrm{NN}^{\mathrm{NA}}(n) \arrow[r, dash, "\phi_C^{\mathrm{NN}}"] & \mathrm{NN}_C(n) \\
\mathrm{NC}_B(n) \arrow[r, dash, "\phi_B^{\mathrm{NC}}"] & \mathrm{NC}^{\mathrm{NN}}(n) \arrow[ur, dash, "\overline{\xi}"] \arrow[r, dash, "\iota_B"] \arrow[dr, dash, "\varphi_B"] \arrow[d, dash, "g"'] & \mathrm{NC}^{\mathrm{NN}}(n) \arrow[r, dash, "\overline{\xi}"] & \mathrm{NC}^{\mathrm{NA}}(n) \arrow[r, dash, "\overline{\rho}"] & \mathrm{NN}^{\mathrm{NA}}(n) \arrow[d, dash, "\phi_B^{\mathrm{NN}}"] \\
 & \mathrm{LP}(n) & \mathfrak{B}(n) \arrow[r, dash, "="] & \mathrm{NC}(n) \times [n+1] & \mathrm{NN}_B(n)
\end{tikzcd}$$

Fig. 14. Bijections from $\mathrm{NC}_B(n)$.

such that i and j are in the same block B and $i = \min(B)$, fill the $(-i, j)$-entry with 1 if B is in X; and fill the (i, j)-entry with 1 otherwise. Fill the remaining entries with 0's. We define $f(\sigma, X)$ to be T. For example, see Fig. 13.

Theorem 8.1. *The map f is a bijection between $\mathrm{NC}^{\mathrm{NN}}(n)$ and $\mathrm{CT}_B(n)$.*

Proof. First, we will show that $T = f(\sigma, X) \in \mathrm{CT}_B(n)$. By the construction, each column of T contains exactly one 1, and the row of T labeled with $-i$ has a 1 if and only if the diagonal entry in the row is filled with 1. To prove $T \in \mathrm{CT}_B(n)$, it only remains to show that there is no 0 which has a 1 above it in the same column and a 1 to the left of it in the same row. Since each column has only one 1, this condition is equivalent to the following: there is no quadruple (i, j, i', j') with $i < i'$, $j < j'$ and $|i'| < j$ such that both the (i, j)-entry and the (i', j')-entry are filled with 1, where i and i' can be negative. Note that we also have $|i| \leqslant j$ and $|i'| \leqslant j'$ because there are the (i, j)-entry and the (i', j')-entry.

Suppose that we have such a quadruple (i, j, i', j'). Then we have either $|i| < |i'| < j < j'$ or $|i'| < |i| \leqslant j < j'$. Let B and B' be the blocks of σ with $|i|, j \in B$ and $|i'|, j' \in B'$. If $|i| < |i'| < j < j'$, then we must have $B = B'$ since $\sigma \in \mathrm{NC}(n)$. Then $|i| = \min(B) = \min(B') = |i'|$, which is a contradiction. If $|i'| < |i| \leqslant j < j'$, then $i < 0$. Thus B is in X, which implies that B is nonnested. However this is a contradiction because $|i'| < |i| \leqslant j < j'$ and $\sigma \in \mathrm{NC}(n)$, B cannot be nonnested.

Now we define the inverse map of f. Let $T \in \mathrm{CT}_B(n)$. Define σ to be the partition of $[n]$ such that i and j are in the same block B with $\min(B) = i$ if and only if $i < j$ and either the (i, j)-entry or the $(-i, j)$-entry of T is filled with 1. Define X to be the set of blocks B of σ such that the row of T labeled with $-\min(B)$ contains a 1. It is easy to see that the map $T \mapsto (\sigma, X)$ is the inverse of f. $\square$

Remark 8.1. Burstein's bijection between $\mathrm{CT}(n)$ and $\mathrm{NC}(n)$ in [7] is a restriction of the 'zigzag' map for permutation tableaux in [20]. We will not go into the details but our map f can also be expressed as a restriction of a type B analog of the 'zigzag' map.

If we restrict f to $\overline{\mathrm{NC}}^{\mathrm{NN}}(n)$, we get the following theorem.

Theorem 8.2. *The map $f : \overline{\mathrm{NC}}^{\mathrm{NN}}(n) \to \mathrm{CT}_D(n)$ is a bijection.*

9. Concluding remarks

Figs. 14 and 15 illustrate the objects and the bijections between them in this paper. We have two interpretations $\mathrm{NC}^{\mathrm{NN}}(n)$ and $\mathfrak{B}(n)$ for $\mathrm{NC}_B(n)$. Since both of them are closely related to $\mathrm{NC}(n)$, they may be useful to prove type B analogs of interesting properties of $\mathrm{NC}(n)$. In the author's sequel paper [12], the interpretation $\mathfrak{B}(n)$ is used to study the poset structure of $\mathrm{NC}_B(n)$ and $\mathrm{NC}_D(n)$.

$$\begin{tikzcd}
 & \mathrm{NC}^{\mathrm{NN}}_{\{0,\pm1\}}(n-1) \arrow[r, dash, "\overline{\xi}"] \arrow[d, dash, "\iota_D"'] & \mathrm{NC}^{\mathrm{NA}}_{\{0,\pm1\}}(n-1) \arrow[r, dash, "\overline{\rho}"] & \mathrm{NN}^{\mathrm{NA}}_{\{0,\pm1\}}(n-1) \arrow[d, dash, "\phi_D^{\mathrm{NN}}"] \\
\mathrm{NC}_D(n) \arrow[r, dash, "\phi_D^{\mathrm{NC}}"] & \mathrm{NC}^{\mathrm{NN}}_{\{0,\pm1\}}(n-1) \arrow[r, dash, "\varphi_D"] \arrow[d, dash, "\kappa"'] & \mathfrak{D}(n) \arrow[dr, dash, "="] & \mathrm{NN}_D(n) \\
\mathrm{CT}_D(n) \arrow[r, dash, "f"] & \overline{\mathrm{NC}}^{\mathrm{NN}}(n) \arrow[r, dash, "g"] & \overline{\mathrm{LP}}(n) & \mathrm{NC}(n-1)\times[3n-2]
\end{tikzcd}$$

Fig. 15. Bijections from $\mathrm{NC}_D(n)$.

Since we have a bijection between $\mathrm{NC}_B(n)$ and $\mathfrak{B}(n) = \mathrm{NC}(n) \times [n+1]$, one can ask the following question.

Question 9.1. Is there a natural bijection between $\mathrm{NN}_B(n)$ and $\mathrm{NN}(n) \times [n+1]$?

Acknowledgments

The author would like to thank Philippe Nadeau and Lauren Williams for their helpful comments and discussions.

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