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Colouring Steiner Quadruple Systems

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A *Steiner quadruple system of order* v (briefly SQS(v)) is a pair $(X, \mathscr{B})$, where $|X| = v$ and $\mathscr{B}$ is a collection of 4-subsets of X, called *blocks*, such that each 3-subset of X is contained in a unique block of $\mathscr{B}$. A SQS(v) exists iff $v \equiv 2, 4 \pmod 6$ or $v = 0, 1$ (the *admissible* integers). The *chromatic number* of $(X, \mathscr{B})$ is the smallest m for which there is a map $\varphi: X \to \mathbf{Z}_m$ such that $|\varphi(\beta)| \geqslant 2$ for all $\beta \in \mathscr{B}$. In this paper it is shown that for each $m \geqslant 6$ there exists v_m such that for all admissible $v \geqslant v_m$ there exists an m-chromatic SQS(v). For $m = 4, 5$ the same statement is proved for admissible v with the restriction that $v \not\equiv 2 \pmod{12}$.

1. INTRODUCTION

Should the reader wish a more comprehensive introduction to the topic of colourings of block designs there is an excellent survey article on the subject by A. Rosa and C. J. Colbourn [4].

Let $X^{[k]} = \{y : y \subseteq X \text{ and } |y| = k\}$; for a finite set $K \subset \mathbf{N}$ the notation $S(t, K, v)$ denotes a pair $(X, \mathscr{B})$, where $|X| = v$, $\mathscr{B} \subseteq \bigcup_{k \in K} X^{[k]}$, and for all $T \in X^{[t]}$ there is a unique $\beta \in \mathscr{B}$ such that $T \subseteq \beta$. The elements of $\mathscr{B}$ are called *blocks*. A *Steiner quadruple system of order* v (a SQS(v)) is then a $S(3, \{4\}, v)$ and a *Steiner triple system* (briefly STS(v)) is any $S(2, \{3\}, v)$. These systems are special *hypergraphs*, that is, pairs $(X, \mathscr{B})$ with $\mathscr{B} \subset \mathscr{P}(X)$, where $\mathscr{P}(X)$ is the power set of X. A SQS(v) exists if and only if $v \equiv 2$ or $4 \pmod 6$, $v > 0$, or $v = 0, 1$ [8] (the *admissible* integers), and a STS(v) exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v > 0$ (see [7]).

An *m-colouring* of a hypergraph $(X, \mathscr{B})$ is any mapping $\varphi: X \to C$, where $|C| = m$, the elements of C are called *colours*. The colouring φ is *proper* if $|\varphi(\beta)| \geqslant 2$ for all $\beta \in \mathscr{B}$, where $\varphi(\beta) = \{\varphi(x) : x \in \beta\}$. The system $(X, \mathscr{B})$

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is *m-chromatic* if it has a proper m-colouring but no proper $(m-1)$-colouring.

Regarding chromatic numbers of quadruple systems the problem of constructing 2-chromatic systems has received the most attention. Indeed, 2-chromatic systems are known to exist for $v \equiv 4, 8 \pmod{12}$ [5], $v = 2 \cdot 5^a 13^b 17^c$, $a+b+c>0$ [13] and $v=22$ [12]. There are not many statements about higher chromatic numbers in the literature. By contrast, for Steiner triple systems it is known [2] that there are no non-trivial 2-chromatic systems and that for each $m \geqslant 3$ there exists v_m such that for each $v \geqslant v_m$ with $v \equiv 1, 3 \pmod 6$ there exists an m-chromatic STS(v). In this paper a similar result for quadruple systems is proved.

THEOREM. *For each* $m \geqslant 6$ *and admissible integer* $v \geqslant v_m = \frac{128}{9}(4m^2+10m+6)^4 - 10$ *there exists an m-chromatic* SQS(v). *For* $m=4$ *or* 5 *the same statement is true for admissible* $v \not\equiv 2 \pmod{12}$ *and* $v_m = \frac{128}{9}(4m^2+2m+1)^4 - 10$.

A key ingredient in the proof is an infinite class of quadruple systems whose chromatic numbers tend to infinity with the order of the system. As will be demonstrated later, such a class is furnished by the affine spaces, $AG(2, n)$. In brief if $\mathscr{B}$ is the collection of all 2-dimensional affine planes in $\mathbf{F}_2^n$ then $(\mathbf{F}_n^2, \mathscr{B})$ is a SQS(2^n). It is worth pointing out that the existence of this nice class of quadruple systems makes it possible to avoid probabilistic methods in the proof of the above theorem (for comparison, see [2]).

2. Notation and Terminology

A SQS(v), $(X, \mathscr{B})$, contains the SQS(w), $(Y, \mathscr{C})$, as a subdesign if $Y \subset X$ and $\mathscr{C} \subset \mathscr{B}$. A *G-design of order v* is a $S(3, \{4, 6\}, v)$, where $6 \mid v$ and $\mathscr{B}$ contains precisely $v/6$ disjoint blocks of size 6; it is abbreviated GD(v). The notion of a subdesign for G-designs is defined similarly. The admissible integers are denoted A_S, while the admissible integers for a GD(v) are $A_G = \{v \geqslant 0 : v \equiv 0 \pmod 6\}$. The proposition that a SQS(v) exists is denoted $S(v)$, while the proposition that a GD(v) exists is denoted $G(v)$ (see [11] for an existence proof).

The notation $X = A_1 \mid A_2 \mid \cdots \mid A_n$ means that X is partitioned by the A_i. If P is a partition (colouring) of X and P contains precisely a_i subsets of size g_i, $1 \leqslant i \leqslant r$, then P has *type* $g_1^{a_1} g_2^{a_2} \cdots g_r^{a_r}$. Note that if $\varphi : X \to C$ is a colouring then $\{\varphi^{-1}(c) : c \in C\}$ is a partition of X, the partition into *colour classes*. In the sequel φ is identified with the partition of X that it induces. If the sizes of the colour classes differ by at most 1 then the colouring is *equitable*. An m-chromatic system that admits a proper, equitable m-colouring is said to be *equitably* m-chromatic. A subset $S \subset X$ is *strongly*

φ-coloured if φ assigns a different colour to each point of S, equivalently the type of φ on S is $1^{|S|}$.

The pair $(X, \mathscr{B})$ is called a *transverse quadruple system of type* n^r, or for short a TRQS(n^r), if there is a partition of X into n-sets $G_1, G_2, ..., G_r$, called *groups*, such that (1) $|\beta \cap G_i| \leqslant 1$ for each i and each $\beta \in \mathscr{B}$, and (2) for each $T \in X^{[3]}$ with $|T \cap G_i| \leqslant 1$ for each i there is a unique $\beta \in \mathscr{B}$ such that $T \subseteq \beta$.

The notations $\mathrm{SQS}(v)_m$ and $\mathrm{SQS}(v)_m^*$ denote an m-chromatic SQS(v) and an equitably m-chromatic SQS(v), respectively. The notation $\mathrm{SQS}(v)_{m^+}$ ($\mathrm{SQS}(v)_{m^-}$) denotes a SQS(v) with chromatic number at least (at most) m. Similarly the propositions $S(v)_m$, $S(v)_m^*$, $S(v)_{m^\pm}$, $G(v)_m$, $G(v)_{m^\pm}$, all have the obvious meanings. The proposition $S(v)_{[a,b]}$, $a<b$ means that $S(v)_m$ holds for each integer value $m \in [a, b]$.

When constructing designs on point sets of the form $A \times B$ the abbreviations $a_b = (a, b) \in A \times B$ and $A_b = A \times \{b\}$ will be made. The shorthand ${}^aB$ will be used for $\{a\} \times B$. The symbol δ_{pq} is the Kronecker delta: $\delta_{pq} = 1$ if $p = q$ and $\delta_{pq} = 0$ otherwise. As usual $\mathbf{Z}_m$ denotes the group of integers under addition modulo m.

3. Preliminaries

We now prove a series of technical lemmas which will be used in the next section to obtain certain recursive constructions.

Lemma 3.1. *Let $(X, \mathscr{B} \cup \Pi)$ be a G-design where $\mathscr{B} \subset X^{[4]}$ and Π is the parallel class of blocks of size* 6. *If $\Pi' \subset X^{[4]}$ is such that for each $\gamma \in \Pi$ there is $\gamma' \in \Pi'$ such that $\gamma' \subset \gamma$ then $\chi(X, \mathscr{B} \cup \Pi') \geqslant \chi(X, \mathscr{B} \cup \Pi)$.*

Proof. The proof is clear since any proper colouring of $(X, \mathscr{B} \cup \Pi')$ is also a proper colouring of $(X, \mathscr{B} \cup \Pi)$. ∎

Lemma 3.2. *Let $(X, \mathscr{B})$ be such that $\mathscr{B} \subset X^{[4]}$ and let $S \subset X$. If $\mathscr{B}_S = \{\beta \in \mathscr{B} : \beta \cap S = \varnothing\}$, then*

$$\chi(X, \mathscr{B}) \leqslant \chi(X \backslash S, \mathscr{B}_S) + \left\lceil \frac{|S|}{3} \right\rceil.$$

Proof. Properly colour $(X \backslash S, \mathscr{B}_S)$ with the minimum number of colours and partition S into $\lceil |S|/3 \rceil$ disjoint sets each of size at most 3. This gives a proper colouring of $(X, \mathscr{B})$ with $\chi(X \backslash S, \mathscr{B}_S) + \lceil |S|/3 \rceil$ colours. The lemma follows. ∎

The next lemma says that modifying the blocks incident with a "small" number of points in a quadruple system cannot increase the chromatic number by more than 1.

LEMMA 3.3. *Let* $(X, \mathscr{B})$ *be a* SQS(v) *and let* $S \subset X$, $|S| \leqslant 3$. *If* $\Gamma: \mathscr{B} \to X^{[4]}$ *satisfies* $\beta \cap S = \varnothing \Rightarrow \Gamma(\beta) = \beta$ *and* $\beta \cap S \neq \varnothing \Rightarrow \Gamma(\beta) \cap S \neq \varnothing$ *then* $\chi(X, \Gamma(\mathscr{B})) \leqslant \chi(X, \mathscr{B}) + 1$.

Proof. By Lemma 3.2,

$$\begin{aligned}\chi(X, \Gamma(\mathscr{B})) &\leqslant \chi(X \backslash S, \Gamma(\mathscr{B})_S) + \left\lceil \frac{|S|}{3} \right\rceil \\ &= (X \backslash S, \mathscr{B}_S) + \left\lceil \frac{|S|}{3} \right\rceil \\ &\leqslant \chi(X, \mathscr{B}) + 1. \quad \blacksquare\end{aligned}$$

LEMMA 3.4. *Let* $\mathscr{C}_s, \mathscr{C}'_s \subset \mathscr{P}(X)$, $s = 0, 1, ..., n$, *and for each* s *let* $\mathscr{B}_s = \{\bigcup_{j=0}^{s} \mathscr{C}'_j) \cup (\bigcup_{j=s+1}^{n} \mathscr{C}_j)$. *If* $\chi(X, \mathscr{B}_0) \leqslant m \leqslant \chi(X, \mathscr{B}_n)$ *and* $\chi(X, \mathscr{B}_{s+1}) \leqslant \chi(X, \mathscr{B}_s) + 1$ *holds for each* s *then* $\chi(X, \mathscr{B}_r) = m$ *for some* $0 \leqslant r \leqslant n$.

Proof. Clearly, somewhere along the way an m-chromatic system will be encountered. ∎

LEMMA 3.5. *Let* $(X, \mathscr{B})$ *be a* SQS(v) *with a* SQS(10), $(Y, \mathscr{C})$, *as a subdesign. If* $(Y, \mathscr{C}')$ *is another* SQS(10) *then* $\chi(X, \mathscr{B} - \mathscr{C} \cup \mathscr{C}') \leqslant \chi(X, \mathscr{B}) + 1$.

Proof. Let φ properly colour $(X, \mathscr{B})$ with the minimum number of colours. Since a SQS(10) has no independent set of size 6 [5] the type of φ on Y takes one of the forms α, $\alpha \cdot 4$, $\alpha \cdot 4^2$, $1 \cdot 4 \cdot 5$, or 5^2, where $\alpha = g_1^{l_1} g_2^{l_2} \cdots g_s^{l_s}$ is a type with each g_i less than 4. Any subset of size 4 or 5 in Y can contain at most one block of $\mathscr{C}'$. The lemma is now clear because in the worst case of all cases one new colour suffices to eliminate all the monochromatic blocks in $\mathscr{C}'$ without introducing any new monochromatic blocks. ∎

LEMMA 3.6. *Let* $(X, \mathscr{B})$ *be a* SQS(v) *and let* $Y \subset X$. *If* $(Y, \mathscr{C})$ *and* $(Y, \mathscr{C}')$ *are two* TRQS(n^4) *designs with* $n \leqslant 3$ *and* $\mathscr{C} \subset \mathscr{B}$, *then* $\chi(X, \mathscr{B} - \mathscr{C} \cup \mathscr{C}') \leqslant \chi(X, \mathscr{B}) + 1$.

Proof. Let G'_i, $i = 0, 1, 2, 3$, be the groups of $(Y, \mathscr{C}')$ and let φ be a proper colouring of $(X, \mathscr{B})$ with the minimum number of colours. If $(X, \mathscr{B} - \mathscr{C} \cup \mathscr{C}')$ has no φ-monochromatic blocks there is nothing to prove; otherwise paint G'_0 with a new colour to get a proper colouring with $\chi(X, \mathscr{B}) + 1$ colours. ∎

4. Constructions

In the first theorem the range of chromatic numbers in $[2, m]$ is realized by blending together the two doubling constructions of [5].

Theorem 4.1. $S(v)_m \Rightarrow S(2v)_{[2,m]}$.

Proof. Let $(X, \mathscr{B})$ be a $\mathrm{SQS}(v)_m$ and let $X' = X \times \mathbf{Z}_2$. For each $\beta = [x, y, z\, w] \in \mathscr{B}$ and each $\varepsilon = 0, 1$ let $\mathscr{C}_\beta^\varepsilon = \{[x_i, y_j, z_k, w_l] : i + j + k + l \equiv \varepsilon \pmod 2\}$, so that $\mathscr{C}_\beta^\varepsilon$ is the block set of a $\mathrm{TRQS}(2^4)$ with groups ${}^a\mathbf{Z}_2$, $a \in \beta$. Let $\mathscr{C}_0^0 = \mathscr{C}_0^1 = \{[x_0, y_0, x_1, y_1] : \{x, y\} \in X^{[2]}\}$. Let $\beta_1, \beta_2, ..., \beta_{q(v)}$ be the blocks in $\mathscr{B}$ and write $\mathscr{C}_s^\varepsilon$ for $\mathscr{C}_{\beta_s}^\varepsilon$, $s = 1, 2, ..., q(v)$. Now, in the notation of Lemma 3.4 (read $\mathscr{C}_s^1$ for $\mathscr{C}_s'$) each $(X, \mathscr{B}_s)$ is a $\mathrm{SQS}(2v)$ and $\chi(X, \mathscr{B}_0) = 2$ while $\chi(X, \mathscr{B}_{q(v)}) \geqslant m$, since $(X, \mathscr{B}_{q(v)})$ contains two copies of the m-chromatic system $(X, \mathscr{B})$. The result now follows from Lemma 3.4 and Lemma 3.6. ▮

Theorem 4.2. $S(v)_m \Rightarrow S(3v-2)_{m^+}$.

Proof. This is an immediate consequence of the well-known tripling construction given in [8] (see also [6]). One simply carries out the construction in such a way that the resulting $\mathrm{SQS}(3v-2)$ contains a copy of the original $\mathrm{SQS}(v)_m$. ▮

Theorem 4.3. $S(v)_m, \Rightarrow S(3v-2)_{[3,m]}$, $m \geqslant 4$.

Proof. First construct a $\mathrm{SQS}(3v-2)_3$ from the $\mathrm{SQS}(v)_m$ by appropriately modifying the construction given in [8]; full details are in [10]. A brief summary of the construction goes as follows. Start with a $\mathrm{SQS}(v)_m$, $(\{A\} \cup X, \mathscr{B})$, and let $X' = \{\infty\} \cup (X \times \mathbf{Z}_3)$ have proper 3-colouring $X' = \{\infty\} \cup X_0 \mid X_1 \mid X_2$. For each $[A, x, y, z] \in \mathscr{B}$ a $\mathrm{SQS}(10)$ is constructed on the point set $\{\infty\} \cup (\{x, y, z\} \times \mathbf{Z}_3)$ and for each $\beta = [x, y, z, w] \in \mathscr{B}$ not containing the point A a certain $\mathrm{TRQS}(3^4)$ design is constructed on the point set $\beta \times \mathbf{Z}_3$ with groups ${}^a\mathbf{Z}_3$, $a \in \beta$.

On the other hand, one may triple $(\{A\} \cup X, \mathscr{B})$ as in Theorem 4.2 to get a $\mathrm{SQS}(3v-2)_m$. It now follows from Lemmata 3.5, 3.6, and 3.4 that by replacing $\mathrm{SQS}(10)$'s and $\mathrm{TRQS}(3^4)$'s one at a time that a sequence of $\mathrm{SQS}(3v-2)$ is produced, among which are systems with chromatic numbers from 3 to m. ▮

Theorem 4.4. $G(v)_m \Rightarrow S(3v-2)_{m^+}$.

Proof. The tripling construction that will be used here is described in [9, Proposition 8]. Let $(\{A\} \cup X, \mathscr{B})$ be a $\mathrm{GD}(v)_m$; for each $\beta \in \mathscr{B}$ construct a block set, $\mathscr{E}_\beta$, as follows:

(i) $|\beta|=4$ and $A\in\beta$. Write $\beta=[A,x,y,z]$ and let $\mathscr{E}_\beta$ be the block set of a SQS(10) constructed on $\{\infty\}\cup(\{x,y,z\}\times\mathbf{Z}_3)$ so that $[\infty,x_0,y_0,z_0]$, $[\infty,x_0,x_1,x_2]$, $[\infty,y_0,y_1,y_2]$, and $[\infty,z_0,z_1,z_2]$ are blocks of the design.

(ii) $|\beta|=4$ and $A\notin\beta$. Write $\beta=[x,y,z,w]$ and let $\mathscr{E}_\beta=\{[x_i,y_j,z_k,w_l]:i+j+k+l\equiv 0 \pmod 3)\}$, so that $(\beta\times\mathbf{Z}_3,\mathscr{E}_b)$ is a TRQS(3^4) design with groups ${}^a\mathbf{Z}_3$, $a\in\beta$ and $[x_0,y_0,z_0,w_0]\in\mathscr{E}_\beta$.

(iii) $|\beta|=6$ and $A\in\beta$. Write $\beta=[A,x,y,z,t,u]$ and let $(\mathbf{Z}_4\times\mathbf{Z}_4,\mathscr{D})$ be the 3-chromatic SQS(16) constructed in [10]. Construct the block set $\mathscr{E}_\beta$ on $\{\infty\}\cup(\{x,y,z,t,u\}\times\mathbf{Z}_3)$ by making the identification

$$\begin{matrix} & 2_3 & \mathbf{3_1} & 1_2 & 1_0 & 2_1 & & & x_0 & y_0 & z_0 & t_0 & u_0 \\ 0_0 & 0_3 & \mathbf{0_1} & 1_3 & 3_0 & 1_1 & \leftrightarrow & \infty & x_1 & y_1 & z_1 & t_1 & u_1 \\ & 2_2 & 3_3 & 0_2 & 2_0 & 3_2 & & & x_2 & y_2 & z_2 & t_2 & u_2 \end{matrix}$$

and copying the blocks from $\mathscr{D}$. Note that $[\infty,x_0,y_0,t_0]\in\mathscr{E}_\beta$ and also that $[\infty,a_0,a_1,a_2]\in\mathscr{E}_\beta$, for each $a\in\beta\backslash\{A\}$. What is important here is the observation that if the elements $\mathbf{0_1}$ and $\mathbf{3_1}$ (in boldface type) are interchanged on the left-hand side then the identification used in the construction $G(v)\to S(3v-2)_3$ is obtained (see [10]).

(iv) $|\beta|=6$ and $A\notin\beta$. Write $\beta=[x,y,z,t,u,v]$ and construct a TRQS(3^6) design $(\beta\times\mathbf{Z}_3,\mathscr{E}_\beta)$ with groups ${}^a\mathbf{Z}_3$, $a\in\beta$, by copying the blocks of the TRQS(3^6) constructed in [11] via the map $\vartheta_\beta:I_6\times\mathbf{Z}_3\to\beta\times\mathbf{Z}_3$, where

$$\begin{aligned} \vartheta_\beta(1,\delta)&=x_\delta, & \vartheta_\beta(4,\delta)&=t_{\delta-1},\\ \vartheta_\beta(2,\delta)&=y_{\delta+1}, & \vartheta_\beta(5,\delta)&=u_\delta,\\ \vartheta_\beta(3,\delta)&=z_\delta, & \vartheta_\beta(6,\delta)&=v_{\delta+1}. \end{aligned}$$

Note that $[y_0,z_0,t_0,u_0]\in\mathscr{E}_\beta$, since $[(2,2),(3,0),(4,1),(5,0)]$ is a block of the original design on $I_6\times\mathbf{Z}_3$. It is important to observe that if ϑ_β is modified by setting $\vartheta_\beta(4,\delta)=t_{\delta+1}$, instead of $\vartheta(4,\delta)=t_{\delta-1}$, then the identification map used in the construction $G(v)\to S(3v-2)_3$ is obtained (again, see [10]).

Now set $\mathscr{B}'=\bigcup_{\beta\in\mathscr{B}}\mathscr{E}_\beta$, then $(X',\mathscr{B}')$ is a SQS($3v-2$) (see [9]). For each block $[A,x,y,z]$ or $[x,y,z,w]\in\mathscr{B}$ the copies $[\infty,x_0,y_0,z_0]$ and $[x_0,y_0,z_0,w_0]\in\{\infty\}\cup X_0$ are retained, while for each block $\beta=[x,y,z,w,t,u]$ (resp. $[A,x,y,z,t,u]$) there is $\gamma\in\mathscr{B}'$ with $|\gamma|=4$ and $\gamma\subset\beta_0$ (resp. $\gamma\subset[\infty,x_0,y_0,z_0,t_0]$). Thus, if $\mathscr{B}''=\{\beta\in\mathscr{B}':\beta\subset(X_0\cup\{\infty\})\}$ then Lemma 3.1 gives $\chi(X_0\cup\{\infty\},\mathscr{B}'')\geqslant\chi(X\cup\{A\},\mathscr{B})$ and, hence, $\chi(X',\mathscr{B}')\geqslant m$. ∎

THEOREM 4.5. $G(v)_m \Rightarrow S(3v-2)_{[3,m]}$.

Proof. It is enough to check that the SQS(16) and TRQS(3^6) used in the construction $G(v)_m \Rightarrow S(3v-2)_3$ may be replaced by their analogues in Theorem 4.4 without increasing the chromatic number by more than 1 each time, since Lemmata 3.5 and 3.6 ensure this for a SQS(10) or a TRQS(3^4). For each kind of replacement Lemma 3.3 may be applied. For example, interchanging 0_1 and 3_1 in the identification 0_0, 2_3, 3_1, 1_2, 1_0, 2_1, 0_3, 0_1, 1_3, 3_0, 1_1, 2_2, 3_3, 0_2, 2_0, $3_2 \leftrightarrow \infty$, $x_0, y_0, ..., u_2$ for a single block $[A, x, y, z, t, u]$ has the effect of applying a mapping Γ, where Γ satisfies the hypotheses of Lemma 3.3 for $S=\{y_0, y_1\}$. Similarly, if a single TRQS(3^6) is being replaced, one takes $S=\{t_0, t_1, t_3\}$ in the application of Lemma 3.3. ▮

THEOREM 4.6. $S(v)_m$ *or* $G(v)_m \rightarrow S(6v-10)_{6^-}$.

Proof. The hextupling construction described in [6] will be used. Let $(\{A, B\} \cup X, \mathscr{B})$ be a SQS(v) if v is admissable and a GD(v) if $6|v$, with A and B in the same block of size 6 in the latter case. Let $X'=\{\infty_0, \infty_1\} \cup (\mathbf{Z}_6 \times X)$. On the set $X'=\{\infty_0, \infty_1\} \cup (\mathbf{Z}_6 \times \mathbf{Z}_3)$ construct the two block sets $DFA(2)$ and $DFB(2)$ as follows:

$DFA(2)$,

$$[a_i, (a+2)_i, (a+3b+1)_{i+1}, (a+3b+1)_{i+2}] : a \in \mathbf{Z}_6, i \in \mathbf{Z}_3, b \in \{0, 1\}$$

$$[a_i, (a+2)_i, (a+3)_{i+k}, (a+5)_{i-k}] : a \in \mathbf{Z}_6, i \in \mathbf{Z}_3, k \in \{1, 2\}$$

$$[a_i, (a+2)_i, a_{i+1}, (a+2)_{i+1}] : a \in \mathbf{Z}_6, i \in \mathbf{Z}_3$$

$$[\infty_j, a_0, b_1, c_2] : a+b+c \equiv 3j \pmod 6, a, b, c \in \mathbf{Z}_6, j \in \{0, 1\}$$

$$[a_i, (a+3)_i, (a+3+b)_{i+1}, (a+3+b)_{i+1}] : a \in \mathbf{Z}_6, i \in \mathbf{Z}_3, b \in \{0, 1, 2\};$$

$DFB(2)$,

$$[a_i, (a+1)_i, b_{i+1}, c_{i+2}] : a+b+c \equiv 2i \pmod 6, i \in \mathbf{Z}_3, a, b, c \in \mathbf{Z}_6.$$

Let $\varphi: X' \rightarrow \mathbf{Z}_6$ be given by $\varphi(\infty_j)=1-j$, $j \in \{0, 1\}$, $\varphi(x_i)=i$, $x_i \in X \times \mathbf{Z}_6$. Construct a set of blocks, $\mathscr{E}_\beta$, for each $\beta \in \mathscr{B}$ as follows (as usual small letters denote elements different from A or B):

(i) $\beta=[A, B, x, y]$. Let $\mathscr{E}_\beta$ be the set of blocks of a SQS(14) constructed on the point set $\{\infty_0, \infty_1\} \cup \{x, y\} \times \mathbf{Z}_6$ so that $[\infty_0, \infty_1, x_0, y_0] \in \mathscr{E}_\beta$ and all blocks are properly φ-coloured. This is possible since any SQS(14) is a SQS$(14)_3^*$ [10].

(ii) $\beta=[A, B, x, y, z, w]$. Let $\mathscr{E}_\beta$ be the block set of a SQS(26) constructed on the point set $\{\infty_0, \infty_1\} \cup \{x, y, z, w\} \times \mathbf{Z}_6$ so that $[\infty_0, \infty_1, x_0, y_0] \in \mathscr{E}_\beta$ and all blocks in $\mathscr{E}_\beta$ are properly φ-coloured. One may use for this construction a $\mathrm{SQS}(26)_3^*$ (see [10]) or even the $\mathrm{SQS}(26)_2$ of [3] as both of these systems admit a proper, equitable 6-colouring.

(iii) $\beta=[A, x, y, z]$. Take for $\mathscr{E}_\beta$ the set of blocks obtained by making the idenfication $\infty_0, \infty_1, x_i, y_i, z_i \leftrightarrow \infty_0, \infty_1, 0_i, 1_i, 2_i$, $i \in \mathbf{Z}_6$, and copying the blocks from $DFA(2)$.

(iv) $\beta=[B, x, y, z]$. Form $\mathscr{E}_\beta$ by sequentially identifying x_i, y_i, z_i with $0_i, 1_i, 2_i$, $i \in \mathbf{Z}_6$, and copy the blocks from $DFB(2)$.

(v) $\beta = [x, y, z, w]$. Let $\mathscr{E}_\beta = \{[x_i, y_j, z_k, w_l] : i-j+k-l \equiv 1 \pmod 6\}$ so that $(\beta \times \mathbf{Z}_6, \mathscr{E}_\beta)$ is a $\mathrm{TRQS}(6^4)$ with groups ${}^a\mathbf{Z}_6$, $a \in \beta$.

(vi) $\beta=[x, y, z, t, u, v]$. Let $(Y, \mathscr{C})$ be any $\mathrm{TRQS}(3^6)$ design with groups G_i, $1 \leqslant i \leqslant 6$. On $Y \times \mathbf{Z}_2$ construct a 2-chromatic $\mathrm{TRQS}(6^6)$ with groups $G_i \times \mathbf{Z}_2$ by taking $[x_i, y_j, z_k, w_l]$ to be a block whenever $[x, y, z, w] \in \mathscr{C}$ and $i+j+k+l \equiv 1 \pmod 2$. Construct a copy, $(\beta \times \mathbf{Z}_6, \mathscr{E}_\beta)$, of this $\mathrm{TRQS}(6^6)$ on $\beta \times \mathbf{Z}_6$ so that the color classes are $\beta \times 2\mathbf{Z}_6$ and $\beta \times (1+2\mathbf{Z}_6)$. (This transverse quadruple system certainly admits φ as a proper coloring.)

Setting $\mathscr{B}' = \bigcup_{\beta \in \mathscr{B}} \mathscr{E}_\beta$ gives a SQS$(6v-10)$, $(X', \mathscr{B}')$, which admits φ as a proper 6-colouring. ∎

THEOREM 4.7. *$S(v)_m$ or $G(v)_m \to S(6v-10)_{(m-1)^+}$.*

Proof. Keep the notation of Theorem 4.6, as well as cases (i)–(iv), but modify the transverse quadruple systems in (v) and (vi) as follows:

(v)′ For each $\mathscr{E}_\beta$ in (v) choose a block $[x_0, y_0, z_0, w_1] \in \mathscr{E}_\beta$ and then interchange w_0 and w_1 throughout, thus obtaining a new $\mathrm{TRQS}(6^4)$, $(\beta \times \mathbf{Z}_6, \mathscr{E}'_\beta)$, with $[x_0, y_0, z_0, w_0] \in \mathscr{E}'_\beta$.

(vi)′ For each $\mathscr{E}_\beta$ in (vi) choose a block $[x_0, y_0, z_0, t_l] \in \mathscr{E}_\beta$, then $l \neq 0$ and $\mathscr{E}'_\beta$ is obtained by interchanging t_0 and t_l in each block of $\mathscr{E}_\beta$ containing either one of these points. Note that $(\beta \times \mathbf{Z}_6, \mathscr{E}'_\beta)$ is a $\mathrm{TRQS}(6^6)$ design with $[x_0, y_0, z_0, t_0] \in \mathscr{E}'_\beta$.

Now form $\mathscr{B}''$ by replacing $\mathscr{E}_\beta$ by $\mathscr{E}'_\beta$ in (v) and (vi) of Theorem 4.6, so that $(X', \mathscr{B}'')$ is a new SQS$(6v-10)$. Let $\mathscr{F} = \{[\infty_1, x_0, y_0, z_0] : [B, x, y, z] \in \mathscr{B}, x, y, z \neq A\}$, then Lemma 3.1 implies that $\chi(X', \mathscr{B}'' \cup \mathscr{F}) \geqslant m$ (examine $\{\infty_0, \infty_1\} \cup X_0$,) so Lemma 3.2 gives $\chi(X'_{\infty_1}, (\mathscr{B}'' \cup \mathscr{F})_{\infty_1}) \geqslant m-1$. However, $(\mathscr{B}'' \cup \mathscr{F})_{\infty_1} = \mathscr{B}''_{\infty_1}$, and clearly $\chi(X', \mathscr{B}'') \geqslant \chi(X'_{\infty_1}, \mathscr{B}''_{\infty_1})$, so $\chi(X', \mathscr{B}'') \geqslant m-1$, as desired. ∎

THEOREM 4.8. *$S(v)_m$ or $G(v)_m \to S(6v-10)_{[6, m-1]}$.*

Proof. The result follows from Theorems 4.6 and 4.7 and Lemmata 3.3 and 3.4. ∎

5. Proof of the Main Theorem

The following lemma gives a class of quadruple systems whose chromatic numbers tend to ∞ with the order of the system.

Lemma 5.1. $\chi(AG(2, n)) \geqslant 2^{n+1}(1+(2^{n+3}-7)^{1/2})^{-1}$.

Proof. Let $X=\mathbf{F}_2^n$, $\mathscr{B}=\{[x, y, z, w] \in X^{[4]} : x+y+z+w=0\}$, so that $(X, \mathscr{B})$ is a SQS(2^n). Let $S \subset X$ be an independent set with $|S|=s$ and suppose that $S \ni 0$ (if not translate S by one of its elements). Since S is independent and contains 0 the mapping $(S\backslash\{0\})^{[2]} \rightarrow X$ given by $\{x, y\} \rightarrow x+y$ has its range in $X\backslash S$ and is 1–1, hence $\binom{s-1}{2} \leqslant 2^n - s$. This gives the bound $s \leqslant \frac{1}{2}(1+(2^{n+3}-7)^{1/2})$; the lemma follows.

The next lemma shows how to embed a quadruple system in a G-design.

Lemma 5.2. $S(v)_m \rightarrow G(6v)_{m^+}$.

Proof. Let $(X, \mathscr{B})$ be an SQS$(v)_m$, $X' = X \times \mathbf{Z}_6$, and let $F_1, F_2, ..., F_5$ be a 1-factorization of the complete graph on $\mathbf{Z}_6$. Take for $\mathscr{B}'$ the union of the systems

$$\Pi = \{{}^x\mathbf{Z}_6 : x \in X\},$$

$$\mathscr{B}_1 = \bigcup_{[x, y, z, w] \in \mathscr{B}} \{[x_i, y_j, z_k, w_l] : i+j+k+l \equiv 0 \pmod 6\},$$

$$\mathscr{B}_2 = \bigcup_{x \neq y} \{[x_i, x_j, y_k, y_l] : (i, j), (k, l) \in F_s, s=0, 1, ..., 5\}.$$

Then $(X', \mathscr{B}')$ is a GD$(6v)$ with Π a parallel class of blocks of size 6. The new design contains two copies of the original SQS(v), one on each of the sets X_0 and X_3, and so it has chromatic number at least m. ▮

As in [6] if $w \in A_S$ let s_w be the smallest integer with the property that if $v \in A_S$ and $v \geqslant s_w$ then there exists a SQS(v) with a subsystem of order w. Similarly, if $w \in A_G$ let g_w be minimal with the property that if $v \in A_G$ and $v \geqslant g_w$ then there is a GD(v) with a GD(w) as a subsystem. The finiteness of the numbers s_w, g_w for each w is proved in [6].

Theorem 5.3 (Granville and Hartman [6]). (a) *For all* $w \in A_S$ *the inequality* $s_w \leqslant \frac{64}{27}(w+1)^4$ *holds*;

(b) *for all* $w \in A_G$ *the inequality* $g_w \leqslant \max\{8654 + w/6,\ (1.21/2)(w/6)^2 + 46(w/6)\}$ *holds*. ▮

Now the main theorem can be proved.

THEOREM 5.4. *For each* $m \geqslant 6$ *and admissible integer* $v \geqslant v_m = \frac{128}{9}(4m^2+10m+6)^4 - 10$ *there exists an m-chromatic* SQS(v). *For* $m=4$ *or* 5 *the same statement is true for admissible* $v \not\equiv 2 \pmod{12}$ *and* $v \geqslant \frac{128}{9}(4m^2+2m+1)^4 - 10$.

Proof. Fix $m \geqslant 6$ and let v_0 be such that for each $v \in (A_S \cup A_G) \cap [v_0, \infty)$ there exists a $\text{SQS}(v)_{(m+1)^+}$ or a $G(v)_{(m+1)^+}$ for $v \in A_S$ or $v \in A_G$ respectively. Put $v_m = 6v_0 - 10$, the following cases show that for each $v \in A_S \cap [v_m, \infty)$ there exists a $\text{SQS}(v)_m$:

(a) If $v \equiv 4, 8 \pmod{12}$ put $w = v/2$, then w is in $A_S \cap [v_0, \infty)$, so there exists a $\text{SQS}(w)_{m^+}$, and hence a $\text{SQS}(v)_m$ exists by Theorem 4.1.

(b) If $v \equiv 10 \pmod{12}$ put $w = (v+2)/3$, then w is in $(A_S \cup A_G) \cap [v_0, \infty)$. If w is in A_S apply Theorem 4.3 starting with a $\text{SQS}(w)_{m^+}$ to get a $\text{SQS}(v)_m$; if $w \in A_G$ apply Theorem 4.5 to a $\text{GD}(w)_{m^+}$ to get a $\text{SQS}(v)_m$.

(c) If $v \equiv 2 \pmod{12}$ put $w = (v+10)/6$, then w is in $(A_S \cup A_G) \cap [v_0, \infty)$. Apply Theorem 4.8 to a $\text{SQS}(w)_{(m+1)^+}$ or a $G(w)_{(m+1)^+}$ to obtain a $\text{SQS}(v)_m$.

Note that only the last case requires a $\text{SQS}(w)_{(m+1)^+}$, $(\text{GD}(w)_{(m+1)^+})$; the other cases require only a $\text{SQS}(v)_{m^+}$ $(\text{GD}(w)_{m^+})$. For $m=4$ and $m=5$ the above reasoning carries over verbatim except that Theorem 4.8 cannot be applied, so case (c) is omitted and, hence, the gap at $v \equiv 2 \pmod{12}$ in the statement of the theorem and the smaller value of v_m.

It remains to produce v_0. Fix $m \geqslant 4$, the inequality $2^{n+1}(1+(2^{n+3}-7)^{1/2})^{-1} \geqslant m$ holds if $2^n \geqslant \frac{1}{2}[m(2m+1) + m((2m+1)^2 - 8)^{1/2}]$, so Lemma 5.1 implies that $\chi(AG(2,n)) \geqslant m$ whenever $2^n \geqslant m(2m+1)$. Let n_0 be minimal with $2^{n_0} \geqslant m(2m+1)$, then $2^{n_0} < 2m(2m+1)$, so it follows from Theorem 5.3, part (a), that there exists a $\text{SQS}(v)_{m^+}$ whenever $v \geqslant \frac{64}{27}(4m^2+2m+1)^4$, $v \in A_S$. Also, from Lemma 5.2 and Theorem 5.3, part (b), it follows that a $\text{GD}(v)_{m^+}$ exists for $v \geqslant 2.42m^2(2m+1)^2 + 552m(2m+1)$, $v \in A_G$. Combining these estimates shows that a $\text{SQS}(v)_{m^+}$ exists whenever $v \geqslant \frac{64}{27}(4m^2+2m+1)^4$, $v \in A_S \cup A_G$. Replacing m by $m+1$ now gives the required value for v_0. ∎

6. OPEN PROBLEMS

For $\chi = 2$ the congruence classes $\pm 2 \pmod{12}$ are open for the construction of $\text{SQS}(v)_2$ except for the values $v = 2 \times 5^a 13 b^1 7^c$, $a+b+c > 0$ [13] and $v = 22$ [12] quoted in the introduction. For $\chi = 3$ the class 2 (mod 12) is open for the construction of $\text{SQS}(v)_3$, except for the values $v = 6 \times 5^a 13^b 17^c - 2$, $a+b+c > 0$, and $v = 14, 62$ [10]. For both $\chi = 4, 5$ the class 2 (mod 12) is open.

For $v \in A_S$ let Spec*$(v) = \{m : S(v)_m\}$, the *chromatic spectrum* of v, and let $\bar{\chi}(v) = \max$ Spec*(v). Is it true that Spec*$(v) = \{2, 3, ..., \bar{\chi}(v)\}$ for admissible $v \neq 14$? The exception $v = 14$ occurs because Spec*$(14) = \{3\}$; a similar conjecture has already been made for Steiner triple systems [4].

One would also like to know the order of $\bar{\chi}(v)$. A straightforward application of the Lovasz local lemma [1] gives $\bar{\chi}(v) \leqslant (e/3)^{1/3} (2v^2 - 15v + 31)^{1/3}$ while examination of the affine planes $AG(2, n)$ suggests $\bar{\chi}(v)$ is at least $2v(1 + (8v - 7)^{1/2})^{-1} \approx (v/2)^{1/2}$.

Finally, given v and m, one would like to construct (if possible) a $\mathrm{SQS}(v)_m$ in polynomial time. The methods given here fall short of this, since they require showing that a given SQS(v) is not m-chromatic prior to concluding that it is $(m + 1)$-chromatic.

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