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On the behavior of the Lorenz equation backward in time

C. Foias[a,b] and M.S. Jolly[b,*]

[a] *Department of Mathematics, Texas A&M University, College Station, TX 77843, USA*
[b] *Department of Mathematics, Indiana University, Bloomington, IN 47405, USA*

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We dedicate this paper to George R. Sell, on the occasion of his 65th birthday, with admiration, friendship, and appreciation for the inspiring role he played several times for both of us.

Abstract

The sets of solutions to the Lorenz equations that exist backward in time and are bounded at an exponential rate determined by the eigenvalues of the linear part of the equation are examined. The set associated with the middle eigenvalue is shown to project surjectively onto a plane, thereby providing a lower estimate for its dimension. Specific bounds are also found for a cone containing this set.

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0. Introduction

Among the more surprising properties of the 2-D periodic Navier–Stokes equations (NSE) is the existence of rich invariant sets formed by trajectories with given exponential growth backward in time [CFKM]. For a dissipative evolutionary equation the mere existence of solutions for all negative time *off* the global attractor $\mathscr{A}$ is significant. This is especially true for a partial differential equation such as the NSE. To be specific, each solution with backward exponential growth has an eigenvalue of the Stokes operator (i.e. the linear part of the NSE) as its exponential

*Corresponding author.
E-mail address: msjolly@indiana.edu (M.S. Jolly).

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rate. Moreover, the set M_Λ of solutions that exist for all time, and have exponential growth rate not exceeding Λ, projects *onto* the linear space spanned by all eigenvectors of the Stokes operator corresponding to eigenvalues $\leqslant \Lambda$. Though reminiscent of local invariant manifolds for steady states (or for higher dimensional objects), these sets may not be manifolds.

The Lorenz system has been the subject of many other investigations. It was derived as a three-mode truncation of partial differential equations modeling the Bénard heat convection problem [Sa]. Historically speaking, Lorenz, with his famous numerical experiments [L], breathed life into Poincare's notion of chaos just as high speed computers were becoming a common research tool in the 1960s. As a testimony to the challenge this simply stated system poses, only recently has its chaotic behavior been established, with a computer-assisted proof at that [MM]. It continues to provide a convenient testing ground for many numerical and analytical tools in dynamical systems, including the computation [Si,DV], and estimation [E,EFT] of Lyapunov exponents, the computation of invariant manifolds [JJK,KO], and exploration of global bifurcations [GS]. More complete reviews of work on the Lorenz system can be found in [GH,S].

The Lorenz system can also serve as a low-dimensional paradigm for the 2-D Navier–Stokes equations. In [FJKT] we study several global properties of the Lorenz system which are analogous to those of the NSE. In particular, we introduced and examined three invariant sets for the Lorenz system M_1, M_b, M_σ (see (1.2)) also defined in terms of the eigenvalues 1, b, σ of the linear part of the system (when written in a form similar to the NSE (see (2.1)). While it is easy to see that $M_\sigma = \mathbb{R}^3$, the nature of M_1 and M_b is not transparent. In particular, the results in [FJKT] do not say anything about the dimensions of $M_\sigma \backslash M_b$ and M_b. In this paper, by adapting to the Lorenz system the methods in [CFKM], we establish that M_b has Hausdorff dimension $\geqslant 2$ and is contained in the exterior of a cone, i.e. the two sides of the cone are in $M_\sigma \backslash M_b$. Moreover, in perfect analogy with one central result in [CFKM], M_b projects *onto* the xy-plane (see Section 5).

The main ingredients in the treatment of sets of bounded backward growth rates for PDEs are a pair of orthogonality relations of the form $(B(u,v),v) = 0$, and $(B(u,u),Au) = 0$, where A and B are respectively, the linear and bilinear operators in the equation. While the first of these relations holds in the case of the Lorenz equations using the Euclidean scalar product, the second requires the deployment of an auxiliary linear operator A_0. This adds a wrinkle to the story as the only choices possible for A_0 have only two distinct eigenvalues. The effect is that while there are conceivably three backward growth rates, the approach here distinguishes between only two of them. A priori the two extremal cases $M_b = M_1$ and $M_1 = \mathscr{A}$ are not yet ruled out. Like many features of this deceptive system, the full consequence of $A \neq A_0$ remains to be determined.

There are several motivations for the study of the backward behavior of dissipative evolutionary equations (for which the Lorenz system is an intriguing example). Most of the research on these equations is dedicated to the study of their forward time behavior, especially their global attractors [H,T]. However, in certain approaches [FT,FJ,FJK,FJL] to the localization of global attractors those invariant sets formed by solutions with slow backward in time exponential growth interfere with the detection of the attractor. Using the methods developed in those papers it is difficult to differentiate between the points on those slow growth sets which are near the attractor and points on the attractor itself. Second, in the study of the global in time solutions of dissipative equations these invariant sets are the analogue of stable invariant manifolds near a singular point when that point is infinity. Moreover, as one can check for the Lorenz system, local invariant manifold theory is not applicable due to a lack of smoothness introduced by moving infinity to the origin. Therefore the existence of various examples which display rich sets of backward time exponential growth (see also [D,V1,V2]) suggest that there may exist a wider theory of invariant sets yet to be discovered. Third, many nonlinear dissipative systems should be expected to have no solutions off the global attractor which exist for all times, e.g. [KM] for the Kuramoto–Sivashinsky equation and [Ti] for the viscous Burgers equation. As yet the only criterion for distinguishing between these two opposing behaviors seems to be the existence of a second orthogonality relation. Finally, in what concerns fluid dynamics, for a given driving force, restricting the study to the flows on or near the attractor establishes a priori bounds for the Reynolds numbers for the flows. The solutions with exponential growth are the most mathematically amenable flows with huge Reynolds numbers.

The proofs in this ODE case are naturally elementary. They demonstrate, however, how to generate a second orthogonality property by finding an alternate linear operator (here A_0). This opens the road for the study of other evolutionary equations (including infinite dimensional ones) with linear part being merely sectorial and not necessarily self-adjoint (e.g. the NSE for 2-D flows driven by moving boundaries).

1. Preliminaries

The Lorenz system

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y,\\ \dot{y} &= -y - xz,\\ \dot{z} &= -bz + xy - br,\end{aligned} \tag{1.1}$$

where $b>0$, $\sigma>0$, and $r>1$, generates a dissipative dynamical system with a nonempty compact global attractor (see e.g. [EFT] for estimates on its Hausdorff dimension). The attractor contains exactly three steady states: $u_1 = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, -1)$, $u_2 = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, -1)$, and $u_3 = (0, 0, -r)$. While (1.1) is not the classical form of the Lorenz system, it is one which more closely resembles that of the Navier–Stokes equations. In particular, offsetting the steady state u_3, which is often situated at the origin has the effect of introducing a force (see (2.1)).

We consider the common parameter settings $\sigma = 10$, $b = 8/3$ and $r = 28$. Much of the behavior backward in time can be described in terms of

$$\begin{aligned}
M_1^0 &= \left\{ u_0 \in \mathbb{R}^3 : \limsup_{t \to -\infty} \frac{\log |S(t)u_0|_2}{|t|} < 1 \right\}, \\
M_1 &= \left\{ u_0 \in \mathbb{R}^3 : \limsup_{t \to -\infty} \frac{\log |S(t)u_0|_2}{|t|} \leqslant 1 \right\}, \\
M_b &= \left\{ u_0 \in \mathbb{R}^3 : \limsup_{t \to -\infty} \frac{\log |S(t)u_0|_2}{|t|} \leqslant b \right\}, \\
M_\sigma &= \left\{ u_0 \in \mathbb{R}^3 : \limsup_{t \to -\infty} \frac{\log |S(t)u_0|_2}{|t|} \leqslant \sigma \right\},
\end{aligned} \tag{1.2}$$

where $S(t)$ is the solution operator of (1.1), $u = (x, y, z)$ and $|\cdot|_2$ denotes the usual Euclidean norm on $\mathbb{R}^3$, namely $|u|_2 = (x^2 + y^2 + z^2)^{1/2}$. Denote by P_x, P_y, and P_z the coordinate projections in $\mathbb{R}^3$. The following was proved in [FJKT].

Theorem 1.1. *The following hold.*

(i) $M_\sigma = \mathbb{R}^3$

(ii) $M_\sigma \backslash M_b = \left\{ u_0 \in \mathbb{R}^3 : \lim_{t \to -\infty} \frac{\log |P_x S(t)u_0|_2}{|t|} = \sigma \right\},$

(iii) $M_\sigma \backslash M_b = \left\{ u_0 \in \mathbb{R}^3 : \lim_{t \to -\infty} \frac{\log |S(t)u_0|_2}{|t|} = \sigma \right\},$

(iv) $M_\sigma \backslash \mathscr{A} = \left\{ u_0 \in \mathbb{R}^3 : \liminf_{t \to -\infty} \frac{\log |S(t)u_0|_2}{|t|} \geqslant 1 \right\},$

(v) $u_0 \in M_b \backslash M_1$ *implies*

$$\limsup_{t \to -\infty} \frac{\log |u(t)|_2}{|t|} = \limsup_{t \to -\infty} \frac{\log \sqrt{|P_y S(t)u_0|_2^2 + |P_z S(t)u_0|_2^2}}{|t|},$$

(vi) $M_1^0 = \mathscr{A}$ *and* $M_1 \backslash \mathscr{A} = \left\{ u_0 \in \mathbb{R}^3 : \lim_{t \to -\infty} \frac{\log |S(t)u_0|_2}{|t|} = 1 \right\}.$

By the invariance of the z-axis under the flow $S(t)$, we have immediately that M_b contains the z-axis. We will make repeated use of the following lemma, whose proof requires only a slight modification of that for Lemma 2.5 in [FJKT].

Lemma 1.2. (i) *Let* $\varphi, \psi \in C^1((-\infty, 0])$ *satisfy*

$$\frac{d\varphi}{dt} + \alpha\varphi = \psi,$$

and

$$\psi(t) = \mathcal{O}(e^{-\gamma t}) \quad as \quad t \to -\infty, \tag{1.3}$$

where $\alpha > 0$ *and* $\gamma \in [0, \alpha)$. *Then either*

$$0 < \liminf_{t \to -\infty} e^{\alpha t} |\varphi(t)| \leqslant \limsup_{t \to -\infty} e^{\alpha t} |\varphi(t)| < \infty \tag{1.4}$$

or

$$\varphi(t) = \mathcal{O}(e^{-\gamma t}) \quad as \quad t \to -\infty. \tag{1.5}$$

(ii) *Moreover*, *if instead of* (1.3) *we have*

$$\lim_{t \to -\infty} \frac{\log|\psi(t)|}{|t|} = \gamma, \tag{1.6}$$

then either (1.4) *holds*, *or*

$$\lim_{t \to -\infty} \frac{\log|\varphi(t)|}{|t|} = \gamma. \tag{1.7}$$

The proof of (i) is almost identical to that of Lemma 2.5 in [FJKT], while that of (ii) requires an easy supplement. We will not use (ii) in this paper, but mention it as an aside for possible future use.

2. The framework

We write the Lorenz system in the standard form

$$\frac{du}{dt} + Au + B(u, u) = f \tag{2.1}$$

with

$$u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A = \begin{pmatrix} \sigma & -\sigma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ 0 \\ -br \end{pmatrix},$$

$$B\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ x\tilde{z} \\ -x\tilde{y} \end{pmatrix}. \tag{2.2}$$

This is the same form as for the Navier–Stokes equations in [CFKM], with the role of the infinite-dimensional phase space now being played by $\mathbb{R}^3$. Note that B enjoys the orthogonality property

$$(B(u,v),v)_2 = 0 \quad \text{for all } u,v \in \mathbb{R}^3. \tag{2.3}$$

Here $(u,\tilde{u})_2$ is the usual Euclidean scalar product on $\mathbb{R}^3$, i.e. $(u,\tilde{u})_2 = x\tilde{x} + y\tilde{y} + z\tilde{z}$. In [CFKM] what allowed the classification of the solutions with backward exponential growth was the positivity of A and the existence of a second orthogonality, namely

$$(B(u,u),Au)_2 = 0 \quad \text{for all } u \in \mathbb{R}^3. \tag{2.4}$$

While (2.4) does not hold for the A and B in (2.2), it is remarkable that there exist another scalar product $(\cdot,\cdot)_\beta$ which makes $(Au,u)_\beta \geqslant 0$, and another positive, self-adjoint operator A_0 such that

$$(B(u,v),v)_\beta = 0, \quad \text{and } (B(u,u),A_0u)_\beta = 0 \quad \text{for all } u,v \in \mathbb{R}^3. \tag{2.5}$$

For $\beta > 0$ we define the scalar product

$$(u,\tilde{u})_\beta = \frac{\beta}{2} x\tilde{x} + y\tilde{y} + z\tilde{z}$$

with corresponding norm denoted $|u|_\beta = (u,u)_\beta^{1/2}$. It can be immediately seen that the first relation in (2.5) holds. Let $T = [t_{ij}]_{i,j=1}^3$. We denote the adjoint of T with respect to $(\cdot,\cdot)_\beta$ as $T^{*(\beta)} = [t_{ij}^{*(\beta)}]_{i,j=1}^3$ and use the defining condition

$$(T^{*(\beta)}u,\tilde{u})_\beta = (u,T\tilde{u})_\beta \tag{2.6}$$

to express $T^{*(\beta)}$ in terms of T. A straightforward computation shows that

$$T^{*(\beta)} = \begin{pmatrix} t_{11} & 2t_{21}/\beta & 2t_{31}/\beta \\ \beta t_{12}/2 & t_{22} & t_{32} \\ \beta t_{13}/2 & t_{23} & t_{33} \end{pmatrix}. \tag{2.7}$$

In particular, we have

$$A^{*(\beta)} = \begin{pmatrix} \sigma & 0 & 0 \\ -\beta\sigma/2 & 1 & 0 \\ 0 & 0 & b \end{pmatrix}.$$

Taking this scalar product of (2.1) with u, and using the orthogonality relation $(B(u,u),u)_\beta = 0$, we obtain

$$\begin{aligned}\frac{d}{dt}|u|_\beta^2 &= 2(f,u)_\beta - (Au,u)_\beta - (u,Au)_\beta \\ &= 2(f,u)_\beta - ((A + A^{*(\beta)})u,u)_\beta \\ &= 2(f,u)_\beta - 2(Au,u)_\beta, \end{aligned} \tag{2.8}$$

since

$$\left(\frac{1}{2}(A + A^{*(\beta)})u, u\right)_\beta = (Au,u)_\beta = (A^{*(\beta)}u,u)_\beta. \tag{2.9}$$

It follows from Young's inequality that

$$\begin{aligned}(Au,u)_\beta &= \frac{\beta}{2}\sigma x^2 - \frac{\beta}{2}\sigma xy + y^2 + bz^2 \\ &\geqslant \left(\frac{\beta}{2}\sigma - \frac{\beta^2\sigma^2}{4}\right)x^2 + \frac{1}{2}y^2 + bz^2 \\ &\geqslant c_1\left(\frac{\beta}{2}x^2 + y^2 + z^2\right), \end{aligned} \tag{2.10}$$

where

$$c_1 = \min\left\{\frac{\sigma}{2}\left(1 - \frac{\beta\sigma}{2}\right), \frac{1}{2}, b\right\} \geqslant \min\left\{\frac{\sigma}{4}, \frac{1}{2}, b\right\} = \frac{1}{2},$$

provided β satisfies

$$0 < \beta < 1/\sigma. \tag{2.11}$$

Henceforth we assume (2.11) holds and drop the subscript on the scalar product and norm

$$(\cdot,\cdot) = (\cdot,\cdot)_\beta, \quad |\cdot| = |\cdot|_\beta, \quad A^* = A^{*(\beta)}.$$

Elementary computations show that any self-adjoint matrix satisfying the second relation in (2.5) is of the form $A_0 = \mathrm{diag}(a,d,d)$. We will take $a = \sigma$ and $d = 1$. Note that $A_0 > 0$.

3. Analogue of the Dirichlet quotient

By the orthogonality and self-adjointness properties we have

$$\begin{aligned}\frac{d}{dt}(A_0u,u) =& -(A_0Au,u)-(A_0B(u,u),u)-(A_0f,u)\\ &-(A_0u,Au)-(A_0u,B(u,u))-(A_0u,f)\\ =& -2(Au,A_0u)+2(f,A_0u).\end{aligned} \tag{3.1}$$

Using (2.8) and (3.1), with $q=\frac{u}{|u|}$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\frac{(A_0u,u)}{|u|^2} =& \frac{1}{|u|^2}\frac{1}{2}\frac{d}{dt}(A_0u,u)-\frac{1}{|u|^4}\frac{1}{2}\left(\frac{d}{dt}\right)|u|^2(A_0u,u)\\ =& \frac{1}{|u|^2}[(f,A_0u)-(Au,A_0u)]-\frac{1}{|u|^4}[(f,u)-(Au,u)](A_0u,u)\\ =& -[(Aq,A_0q)-(A_0q,q)(Aq,q)]+\left(\frac{f}{|u|},[A_0-(A_0q,q)I]q\right)\\ =& -|[A_0-(A_0q,q)I]q|^2+([A_0-A]q,A_0q)\\ & -(A_0,q,q)([A_0-A]q,q)+\left(\frac{f}{|u|},[A_0-(A_0q,q)I]q\right).\end{aligned} \tag{3.2}$$

Let

$$\lambda_0=(A_0q,q)=\frac{(A_0u,u)}{|u|^2}$$

be the analogue of the Dirichlet quotient in [CFKM]. With

$$p=[A_0-(A_0q,q)I]q,$$

the derivation in (3.2) reads simply as

$$\frac{1}{2}\frac{d}{dt}\lambda_0=-|p|^2+(p,[A_0-A]q)+\left(\frac{f}{|u|},p\right). \tag{3.3}$$

Now denote the components of q as $q=(\xi,\eta,\zeta)$, and let $\omega=(0,\eta,\zeta)$ so that

$$\frac{\beta}{2}\xi^2+|\omega|^2=1,\quad \frac{\beta}{2}\sigma\xi^2+|\omega|^2=\lambda_0,$$

and hence

$$\frac{\beta}{2}\xi^2=\frac{\lambda_0-1}{\sigma-1}\quad\text{and}\quad |\omega|^2=\frac{\sigma-\lambda_0}{\sigma-1}. \tag{3.4}$$

We write out

$$p = \begin{pmatrix} \sigma\xi \\ \eta \\ \zeta \end{pmatrix} - \lambda_0 \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} (\sigma-\lambda_0)\xi \\ (1-\lambda_0)\eta \\ (1-\lambda_0)\zeta \end{pmatrix}, \quad p_0 \overset{\text{def}}{=} [A_0 - A]q = \begin{pmatrix} \sigma\eta \\ 0 \\ (1-b)\zeta \end{pmatrix},$$

and calculate

$$(p, p_0) = \frac{\beta}{2}(\sigma-\lambda_0)\sigma\xi\eta + (\lambda_0-1)(b-1)\zeta^2.$$

From (3.4) we have

$$\begin{aligned} |p|^2 &= \frac{\beta}{2}(\sigma-\lambda_0)^2\xi^2 + (1-\lambda_0)^2|\omega|^2 \\ &= \frac{(\sigma-\lambda_0)^2(\lambda_0-1)}{\sigma-1} + \frac{(\lambda_0-1)^2(\sigma-\lambda_0)}{\sigma-1} \\ &= (\sigma-\lambda_0)(\lambda_0-1) \end{aligned} \tag{3.5}$$

and

$$|\xi| = \left(\frac{\beta}{2}\right)^{-1/2} \frac{(\lambda_0-1)^{1/2}}{(\sigma-1)^{1/2}}. \tag{3.6}$$

Applying (3.5), along with

$$\zeta^2 = |\omega|^2 - \eta^2 = \frac{(\sigma-\lambda_0)}{(\sigma-1)} - \eta^2$$

we rewrite the first two terms in (3.3) as

$$\begin{aligned} -|p|^2 + (p, [A_0 - A]q) = &-(\sigma-\lambda_0)(\lambda_0-1) + \frac{\beta}{2}(\sigma-\lambda_0)\sigma\xi\eta + (\lambda_0-1)(b-1)\zeta^2 \\ \leqslant &-(\sigma-\lambda_0)(\lambda_0-1) + \frac{\beta}{2}(\sigma-\lambda_0)\sigma|\xi\eta| - (\lambda_0-1)(b-1)\eta^2 \\ &+ (\lambda_0-1)(b-1)\frac{(\sigma-\lambda_0)}{(\sigma-1)}. \end{aligned} \tag{3.7}$$

Then use (3.6) and Young's inequality to obtain

$$\begin{aligned} \frac{\beta}{2}(\sigma-\lambda_0)\sigma|\xi\eta| &= \left(\frac{\beta}{2}\right)^{1/2}(\sigma-\lambda_0)\frac{(\lambda_0-1)^{1/2}}{(\sigma-1)^{1/2}}\sigma|\eta| \\ &\leqslant \frac{\beta}{8}\frac{(\sigma-\lambda_0)^2\sigma^2}{(\sigma-1)(b-1)} + (\lambda_0-1)(b-1)\eta^2. \end{aligned} \tag{3.8}$$

Insert (3.8) into (3.7) and use $\lambda_0 \geqslant 1$ to find that

$$\begin{aligned} -|p|^2 + (p, [A_0 - A]q) \leqslant & -(\sigma - \lambda_0)(\lambda_0 - 1) + \frac{\beta}{8} \frac{(\sigma - \lambda_0)^2 \sigma^2}{(\sigma - 1)(b-1)} \\ & + (\lambda_0 - 1)(b-1) \frac{(\sigma - \lambda_0)}{(\sigma - 1)} \\ \leqslant & -(\sigma - \lambda_0)(\lambda_0 - 1) \frac{(\sigma - b)}{(\sigma - 1)} + \frac{\beta \sigma^2}{8} \frac{(\sigma - 1)}{(b-1)}. \end{aligned}$$

Thus by (3.5) and the fact that $b - 1 < \sigma - b$ we have

$$\begin{aligned} \frac{1}{2}\frac{d\lambda_0}{dt} \leqslant & -(\sigma - \lambda_0)(\lambda_0 - 1) \frac{(\sigma - b)}{(\sigma - 1)} + \frac{\beta \sigma^2}{8} \frac{(\sigma - 1)}{(b-1)} + \frac{br}{|u|}|p| \\ = & -(\sigma - \lambda_0)(\lambda_0 - 1) \frac{(\sigma - b)}{(\sigma - 1)} + \frac{\beta \sigma^2}{8} \frac{(\sigma - 1)}{(b-1)} + \frac{br}{|u|}(\sigma - \lambda_0)^{1/2}(\lambda_0 - 1)^{1/2} \\ \leqslant & -\frac{1}{2}(\sigma - \lambda_0)(\lambda_0 - 1) \frac{(\sigma - b)}{(\sigma - 1)} + \left[\frac{\beta \sigma^2}{8} + \frac{b^2 r^2}{2|u|^2}\right] \frac{(\sigma - 1)}{(b-1)}. \end{aligned} \tag{3.9}$$

Now assume that for some t_0, the Dirichlet quotient satisfies

$$\lambda_0(t_0) = \frac{1 + \sigma}{2} \tag{3.10}$$

so that

$$\left.\frac{1}{2}\frac{d\lambda_0}{dt}\right|_{t=t_0} \leqslant -\frac{1}{8}(\sigma - 1)(\sigma - b) + \left(\frac{\sigma - 1}{b-1}\right)\left[\frac{\beta \sigma^2}{4} + \frac{b^2 r^2}{2|u|^2}\right]. \tag{3.11}$$

Henceforth we fix $\beta \leqslant \beta_0$ where

$$\left(\frac{\sigma - 1}{b-1}\right)\frac{\beta_0 \sigma^2}{4} = \frac{1}{16}(\sigma - 1)(\sigma - b) \quad \text{i.e. } \beta_0 = \frac{11}{360}.$$

For this choice of β, (3.11) can be written as

$$\left.\frac{1}{2}\frac{d\lambda_0}{dt}\right|_{t=t_0} \leqslant -\frac{1}{16}(\sigma - 1)(\sigma - b) + \left(\frac{\sigma - 1}{b-1}\right)\frac{b^2 r^2}{2|u|^2}. \tag{3.12}$$

From (2.10) we have

$$\frac{1}{2}\frac{d}{dt}|u|^2 \leqslant |f||u| - c_1|u|^2 \leqslant -\frac{c_1}{2}|u|^2 + \frac{|f|^2}{2c_1},$$

and hence

$$|u(t)|^2 \leqslant e^{c_1(t-t_0)}|u(t_0)|^2 + \frac{|f|^2}{c_1^2}[1 - e^{c_1(t-t_0)}].$$

Thus any ball of radius ρ with $\rho > |f|/c_1$ is absorbing, and more importantly for our consideration, if

$$|u(t_0)| > \mathscr{R}_1 \stackrel{\text{def}}{=} \frac{|f|}{c_1}, \tag{3.13}$$

then

$$\left.\frac{d}{dt}|u|^2\right|_{t=t_0} \leqslant -c_1\left(|u(t_0)|^2 - \frac{|f|^2}{c_1^2}\right) < 0. \tag{3.14}$$

If

$$\left(\frac{\sigma-1}{b-1}\right)\frac{b^2r^2}{2|u(t_0)|^2} < \frac{1}{16}(\sigma-1)(\sigma-b),$$

or equivalently, if

$$|u(t_0)| > \mathscr{R}_2 \stackrel{\text{def}}{=} \left[\frac{8b^2r^2}{(b-1)(\sigma-b)}\right]^{1/2}, \tag{3.15}$$

then

$$\left.\frac{d}{dt}\frac{(A_0u(t), u(t))}{|u(t)|^2}\right|_{t=t_0} < 0.$$

We have proved the following.

Lemma 3.1. *If for some t_0 we have (with $\mathscr{R}_1$, $\mathscr{R}_2$ defined in (3.13) and (3.15) above)*

$$|u(t_0)| > \mathscr{R} \stackrel{\text{def}}{=} \max\{\mathscr{R}_1, \mathscr{R}_2\}, \quad \text{and} \quad \frac{(A_0u(t_0), u(t_0))}{|u(t_0)|^2} = \frac{\sigma+1}{2}, \tag{3.16}$$

then

$$\left.\frac{d}{dt}\frac{(A_0u(t), u(t))}{|u(t)|^2}\right|_{t=t_0} < 0, \quad \text{and} \quad \left.\frac{d}{dt}|u(t)|^2\right|_{t=t_0} < 0.$$

The following basic result will be used in Sections 4 and 5.

Lemma 3.2. *If for some* t_0 *we have*

$$|u(t_0)| > \mathscr{R} \quad \textit{and} \quad \frac{(A_0u(t_0), u(t_0))}{|u(t_0)|^2} \geqslant \frac{\sigma+1}{2}, \tag{3.17}$$

then for all $t < t_0$ *we have*

$$|u(t)| > \mathscr{R} \quad \textit{and} \quad \frac{(A_0u(t), u(t))}{|u(t)|^2} > \frac{\sigma+1}{2}. \tag{3.18}$$

Proof. If

$$\frac{(A_0u(t_0), u(t_0))}{|u(t_0)|^2} = \frac{\sigma+1}{2},$$

then by Lemma 3.1 for small enough $\varepsilon > 0$ we have for $t \in (t_0 - \varepsilon, t_0)$ that

$$|u(t)| > \mathscr{R} \quad \text{and} \quad \frac{(A_0u(t), u(t))}{|u(t)|^2} > \frac{\sigma+1}{2}.$$

By (3.14) we have that $|u(t)| > \mathscr{R}$ for all $t < t_0$. That the solution cannot enter, for any $t < t_0$ the cone

$$\frac{(A_0u, u)}{|u|^2} \leqslant \frac{\sigma+1}{2}$$

follows again from Lemma 3.1. □

4. The Dirichlet quotients and the set M_b

In order to exhibit the link between the behavior of the Dirichlet quotients and the set M_b we need two preliminary facts. Let $w = (0, y, z)$, and $(x(t), y(t), z(t)) = S(t)u_0$.

Proposition 4.1. *For any* $u_0 \in \mathbf{R}^3$ *we have*

$$\limsup_{t \to -\infty} \frac{\log|w(t)|}{|t|} \leqslant b. \tag{4.1}$$

Proof. Multiplying the second equation in (1.1) by y, the third by z, and applying Young's inequality we find that for any $\varepsilon > 0$

$$\frac{1}{2}\frac{d}{dt}|w|^2 = -y^2 - bz^2 - brz \geqslant -y^2 - bz^2 - \varepsilon z^2 - \frac{b^2r^2}{4\varepsilon} \geqslant -(b+\varepsilon)|w|^2 - \frac{b^2r^2}{4\varepsilon}.$$

By Gronwall's inequality

$$|w(t)|^2 \leqslant e^{(b+\varepsilon)(t_0-t)}\left[|w(t_0)|^2 + \frac{b^2r^2}{2\varepsilon(b+\varepsilon)}\right] - \frac{b^2r^2}{4\varepsilon(b+\varepsilon)} \leqslant e^{(b+\varepsilon)(t_0-t)}\left[|w(t_0)|^2 + \frac{b^2r^2}{2\varepsilon(b+\varepsilon)}\right]$$

from which follows (4.1). □

Proposition 4.2. *If* $\{u(t)\}_{t\in\mathbb{R}} \notin \mathscr{A}$, *and*

$$|w(t)|^2 \leqslant Cx^2(t) \quad \textit{for all} \quad t<0 \tag{4.2}$$

for some $C>0$, *then* $u_0 \notin M_b$.

Proof. Multiplying the first equation in (1.1) we have

$$\frac{d}{dt}\left(z+r-\frac{x^2}{2\sigma}\right) = -b(z+r)+x^2 = -2\sigma\left(z+r-\frac{x^2}{2\sigma}\right)+(2\sigma-b)(z+r). \tag{4.3}$$

If $u=(x,y,z)\in M_b$ then

$$c = \limsup_{t\to-\infty} \frac{\log|u(t)|}{|t|} \leqslant b < 2\sigma. \tag{4.4}$$

For any γ, $c<\gamma<\alpha=2\sigma$ we have by Lemma 1.2(i) that either there exists positive constants $\underline{c}_2$, $\bar{c}_2$, $\underline{c}_3$, $\bar{c}_3$ such that

$$\underline{c}_2 \leqslant \left|z+r-\frac{x^2}{2\sigma}\right| e^{2\sigma t} \leqslant \bar{c}_2 \quad \text{for all } t<0,$$

which by (4.2) implies

$$\underline{c}_3 \leqslant |x|e^{\sigma t} \leqslant \bar{c}_3, \tag{4.5}$$

or

$$\left|z(t)+r-\frac{x^2(t)}{2\sigma}\right| = \mathcal{O}(e^{-\gamma t}) \quad \text{as } t\to-\infty. \tag{4.6}$$

Yet if (4.5) holds, then by Theorem 1.1(ii) we have $u\in M_\sigma \backslash M_b$, a contradiction. It follows that

$$x(t)^2 = \mathcal{O}(e^{-\gamma t}) \quad \text{as } t\to-\infty,$$

and by the condition relating $|w|^2$ to x^2 we obtain

$$|u(t)| = \mathcal{O}(e^{-\gamma t/2}) \quad \text{as} \quad t\to-\infty.$$

By the definition of c we must have $c \leqslant \gamma/2$. Letting $\gamma \searrow c$ we obtain $c = 0$. From Theorem 1.1 it follows that $u(t) \in \mathscr{A}$ (for all $t \in \mathbb{R}$). □

Now if

$$(A_0 u, u) \geqslant \frac{1+\sigma}{2} |u|^2 \tag{4.7}$$

i.e.

$$\sigma \frac{\beta}{2} x^2 + y^2 + z^2 \geqslant \frac{1+\sigma}{2} \frac{\beta}{2} x^2 + \frac{1+\sigma}{2} (y^2 + z^2),$$

then simplifying, we have

$$\beta x^2 \geqslant 2|w|^2. \tag{4.8}$$

Therefore by Proposition 4.2 and the Lemma 3.2, we readily infer the following:

Theorem 4.3. *If* (3.17) *holds, then* $u(t_0) \notin M_b$.

Corollary 4.4. *We have*

$$M_b \subset \Gamma \stackrel{\text{def}}{=} \{u : |u| \leqslant \mathscr{R}\} \cup \left\{ u : (A_0 u, u) < \frac{1+\sigma}{2} |u|^2 \right\}. \tag{4.9}$$

Remark 4.5. (i) The best result in (4.9) holds for $\beta = \beta_0$.

(ii) Note that by (4.9)

$$\{u = (x, y, z) : |u| \geqslant \mathscr{R}_1,\ \beta_0 x^2 \geqslant 2(y^2 + z^2)\} \subset M_\sigma \backslash M_b,$$

hence $\dim(M_\sigma \backslash M_b) = 3$.

Corollary 4.6. *If a trajectory* $\{u(t)\}_{-\infty \leqslant t \leqslant \infty}$ *lies in* Γ, *then* $u(t) \in M_b$ *for all* $t \in \mathbb{R}$.

Proof. Indeed, in this case, we have either $u(t) \in \mathscr{A}$ for all $t \in \mathbb{R}$ and then $u(t) \in M_b$ for all $t \in \mathbb{R}$, or there exists a t_0 such that $|u(t)| \leqslant \mathscr{R}$ for $t \geqslant t_0$ and $|u(t)| > \mathscr{R}$ for $t < t_0$, and consequently

$$(A_0 u(t), u(t)) \leqslant \frac{\sigma+1}{2} |u(t)|^2 \quad \text{for} \quad t \leqslant t_0. \tag{4.10}$$

Proceeding as in the proof from (4.7) to (4.8), we deduce that (4.10) implies

$$\beta x(t)^2 \leqslant 2|w(t)|^2 \quad \text{for} \quad t \leqslant t_0. \tag{4.11}$$

Using Proposition 4.1(i), we readily obtain that

$$|u(t)|_2^2 \leqslant \left(\frac{2}{\beta}+1\right)|w(t)|^2 \leqslant C_\varepsilon^2\left(\frac{2}{\beta}+1\right)e^{-2bt} \quad \text{for all} \quad t \leqslant t_\varepsilon \leqslant t_0, \ \varepsilon > 0.$$

This shows that $u(0) \in M_b$, and consequently $u(t) \in M_b$ for all $t \in \mathbb{R}$. □

5. Geometric properties of M_b

By Young's inequality and (2.11)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u|^2 &= (f,u) - (Au,u)\\ &\geqslant -|f||u| - \frac{\beta}{2}\sigma x^2 - \frac{(\beta\sigma)^2}{8}x^2 - \frac{3}{2}y^2 - bz^2\\ &\geqslant -|f||u| - \frac{\beta}{2}\frac{3\sigma}{2}x^2 - \frac{3}{2}y^2 - bz^2\\ &\geqslant -|f||u| - \frac{3\sigma}{2}|u|^2\\ &\geqslant -\frac{|f|^2}{2\sigma} - 2\sigma|u|^2.\end{aligned}$$

Thus

$$\frac{1}{2}\frac{d}{dt}\left(|u|^2 + \frac{|f|}{4\sigma^2}\right) \geqslant -2\sigma\left(|u|^2 + \frac{|f|}{4\sigma^2}\right)$$

which provides the following estimate.

Lemma 5.1. *If*

$$|u(t_0)| \geqslant \max\left\{\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3 \stackrel{\text{def}}{=} \frac{|f|}{2\sigma}\right\}, \tag{5.1}$$

then

$$|u(t)| \geqslant \frac{1}{2}e^{-2\sigma(t-t_0)}|u(t_0)|, \quad \text{for all} \quad t \geqslant t_0.$$

Lemma 5.1 along with Lemma 3.2 are the equivalent of Lemma 3.2 in [CFKM], while the fact that $u \in \Gamma$, $|u| \geqslant \mathscr{R} \Rightarrow \beta x^2 \leqslant |w|^2$ is the equivalent of Lemma 3.4 in [CFKM]. With these results settled, the analogue of Lemma 3.9 in [CFKM] can be established by reproducing almost verbatim the proof in [CFKM]. Therefore we state the following central result and send the reader to [CFKM] for the details of its proof. We mention only that the well-known Brouwer surjectivity theorem is another basic ingredient in that proof.

Theorem 5.2. *Denote by $P_{y,z}$ the projector defined by $u=(x,y,z)\mapsto w=(0,y,z)$. Then*

$$P_{y,z}M_b = P_{y,z}\mathbb{R}^3.$$

This has the following obvious consequence.

Corollary 5.3. *The Hausdorff dimension of $M_b\backslash\mathscr{A}$, denoted $d_{\mathrm{H}}(M_b\backslash\mathscr{A})$, satisfies $d_{\mathrm{H}}(M_b\backslash\mathscr{A})\geqslant 2$.*

For the definition of the Hausdorff dimension, see [F].

Conjecture 5.4. $d_{\mathrm{H}}(M_b)<3$.

Theorem 5.5. *Take*

$$0<\varepsilon<\varepsilon_0=\max\left\{\frac{4b^2r^2}{\beta\sigma\mathscr{R}^2},1\right\}.$$

Then for all $u=(x,y,z)\in M_b$ satisfying

$$|u|\geqslant\mathscr{R}_\varepsilon\overset{\text{def}}{=}\left(\frac{4b^2r^2}{\varepsilon\beta\sigma^2}\right)^{1/2}, \tag{5.2}$$

we have

$$x^2\leqslant(\rho_\varepsilon(\beta))^2|w|^2, \tag{5.3}$$

where

$$|w|^2=y^2+z^2,\quad \rho_\varepsilon(\beta)=\frac{2}{\beta}\cdot\frac{\lambda_\varepsilon-1}{\sigma-\lambda_\varepsilon},\quad \textit{and}$$

$$\lambda_\varepsilon=\frac{1}{2}\left\{(\sigma+1)-\left[(\sigma-1)^2-\frac{\beta\sigma^2(1+\varepsilon)(\sigma-1)^2}{(b-1)(\sigma-b)}-4\varepsilon\left(\frac{\sigma-1}{\sigma-b}\right)\right]^{1/2}\right\}.$$

Moreover

$$\limsup_{|w|\to\infty,u\in M_b}\frac{|x|}{|w|}\leqslant\sqrt{\frac{45}{11}}. \tag{5.4}$$

Proof. Using (5.2) in (3.9) we have

$$\frac{1}{2}\frac{d\lambda_0}{dt} \leqslant -\frac{1}{2}(\sigma-\lambda_0)(\lambda_0-1)\frac{(\sigma-b)}{(\sigma-1)} + (1+\varepsilon)\frac{\beta\sigma^2}{8}\frac{(\sigma-1)}{(b-1)}.$$

Suppose that for some t_1, $\lambda_0 = \lambda_0(t_1)$ satisfies

$$-\frac{1}{2}(\sigma-\lambda_0)(\lambda_0-1)\left(\frac{\sigma-b}{\sigma-1}\right) + (1+\varepsilon)\frac{\beta\sigma^2}{8}\left(\frac{\sigma-1}{b-1}\right) \leqslant -\frac{\varepsilon}{2}. \tag{5.5}$$

This condition is equivalent to

$$\lambda_\varepsilon \leqslant \lambda_0 \leqslant \frac{\sigma+1}{2},$$

where the last inequality follows from (4.9) and the fact that $\mathscr{R}_\varepsilon > \mathscr{R}$. Then

$$\frac{d}{dt}\lambda_0 \leqslant -\varepsilon \quad \text{for all} \quad t \leqslant t_1,$$

and hence

$$\lambda_0(t) \in [\lambda_\varepsilon, (\sigma+1)/2], \quad \forall t \leqslant t_1.$$

But then

$$\varepsilon(t_1-t_0) \leqslant \lambda_0(t_1) + \varepsilon(t_1-t_0) \leqslant \lambda_0(t_0) \leqslant \frac{1+\sigma}{2}$$

which is impossible for $t_0 < t_1$ chosen so that

$$t_1 - t_0 > \frac{\sigma-1}{2\varepsilon}.$$

Thus, we have

$$(A_0 u, u) \leqslant \lambda_\varepsilon |u|^2, \quad \text{for} \quad |u| > \mathscr{R}_\varepsilon,$$

i.e.

$$\sigma\frac{\beta}{2}x^2 + |w|^2 \leqslant \lambda_\varepsilon \frac{\beta}{2}x^2 + \lambda_\varepsilon |w|^2. \tag{5.6}$$

Solving for x^2 yields (5.3). Holding $\beta > 0$ fixed and taking $\varepsilon \to 0$ we have

$$\lambda_\varepsilon \to \lambda_0^- = \frac{1}{2}\left\{(\sigma+1) - \left[(\sigma-1)^2 - \frac{\beta\sigma^2(\sigma-1)^2}{(b-1)(\sigma-b)}\right]^{1/2}\right\},$$

and hence

$$\rho_\varepsilon(\beta) \to \rho_0(\beta) = \frac{2}{\beta}\frac{\lambda_0^- - 1}{\sigma - \lambda_0^-}$$

$$= \frac{2}{\beta}\frac{(\sigma-1) - \left[(\sigma-1)^2 - \frac{\beta\sigma^2(\sigma-1)^2}{(b-1)(\sigma-b)}\right]^{1/2}}{(\sigma-1) + \left[(\sigma-1)^2 - \frac{\beta\sigma^2(\sigma-1)^2}{(b-1)(\sigma-b)}\right]^{1/2}}$$

$$= \frac{2}{\beta}\frac{\frac{\beta\sigma^2(\sigma-1)^2}{(b-1)(\sigma-b)}}{\left\{(\sigma-1) + \left[(\sigma-1)^2 - \frac{\beta\sigma^2(\sigma-1)^2}{(b-1)(\sigma-b)}\right]^{1/2}\right\}^2}.$$

Taking $\beta \to 0$ we obtain

$$\limsup_{|w|\to\infty, u\in M_b} \frac{|x|}{|w|} \leqslant \frac{\sigma^2}{2(b-1)(\sigma-b)},$$

which works out to (5.4). □

Remark 5.6. By Proposition 4.2 we have *along trajectories* in $M_b \backslash \mathscr{A}$ that

$$\liminf_{t\to-\infty} \frac{|x(t)|}{|w(t)|} = 0, \quad \liminf_{t\to-\infty} |w(t)| = \infty.$$

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