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Nonnegative Solutions of Algebraic Riccati Equations

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ABSTRACT

Nonnegative Hermitian solutions of various types of continuous and discrete algebraic Riccati equations are studied. The Hamiltonian is considered with respect to two different indefinite scalar products. For the set of nonnegative solutions the order structure and the topology of the set and the stability of solutions is treated. For general Hermitian solutions a method to compute the inertia is given. Although most attention is payed to the classical types arising from LQ optimal control theory, the case where the quadratic term has an indefinite coefficient is studied as well. © Elsevier Science Inc., 1997

1. INTRODUCTION AND PRELIMINARIES

In this paper we consider Riccati equations of the type

$$XBJB^*X - XA - A^*X - C^*C = 0 \tag{1.1}$$

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where A and X are $n \times n$ matrices, B is an $n \times m$ matrix, C is an $k \times n$ matrix, and finally, J is an invertible $m \times m$ Hermitian matrix. We are particularly interested in Hermitian nonnegative solutions of (1.1), which play a role in many applications. We mention the theory of H^{∞}-control, where the solution of the so-called standard problem is given in terms of nonnegative solutions of two Riccati equations of the type (1.1) (see, e.g., [7]), inner-outer factorization of rational matrix-valued functions, where in particular also the case $C = 0$ is of interest (see, e.g., [5, 14]), and finally, the theory of LQ-optimal control, where the case $J = I$ is of prime importance. The latter case has been extensively studied; we refer to [12, 15, 20, 27] and the references given there. The discrete-time counterpart of the theory of nonnegative solutions of algebraic Riccati equations is less developed.

Our approach relies heavily on notions and results from the theory of indefinite inner product spaces. In this paper the standard Hilbert-space inner product on $\mathbb{C}^n$ (or on $\mathbb{C}^{2n}$) will be denoted by $\langle x, y \rangle$, for vectors x, $y \in \mathbb{C}^n$ ($x, y \in \mathbb{C}^{2n}$). If $G = G^*$ is an $n \times n$ invertible matrix, then the number $\langle Gx, y \rangle$ is defined to be the G-inner product of the vectors x, $y \in \mathbb{C}^n$. A vector x is called G-neutral (G-negative, G-positive, G-nonnegative, G-nonpositive) if $\langle Gx, x \rangle = 0$ (< 0, > 0, ≥ 0, ≤ 0). In general the G-inner product is indefinite in the sense that such G-neutral and G-negative vectors exist. A subspace $\mathscr{M}$ is called G-neutral (G-nonnegative, G-nonpositive) if all vectors $x \in \mathscr{M}$ are G-neutral (G-nonnegative, G-nonpositive). It is a general fact (see [10, Theorem I.1.3]) that the maximal possible dimension of a G-nonnegative (G-nonpositive) subspace equals the number of positive eigenvalues of G (the number of negative eigenvalues of G, respectively). The maximal possible dimension of a G-neutral subspace equals the minimal of these two number of positive and negative eigenvalues of G; see [10, Theorem I.1.5]. A G-nonnegative subspace of maximal possible dimension is called a maximal G-nonnegative subspace, and analogously maximal G-nonpositive subspaces and maximal G-neutral subspaces are defined. If $\{x_1, \dots, x_r\}$ is a basis of some subspace $\mathscr{M}$, then the matrix

$$\begin{pmatrix} \langle Gx_1, x_1 \rangle & \cdots & \langle Gx_r, x_1 \rangle \\ \vdots & & \vdots \\ \langle Gx_1, x_r \rangle & \cdots & \langle Gx_r, x_r \rangle \end{pmatrix}$$

is called the Gram matrix of G with respect to the basis $\{x_1, \dots, x_r\}$ of $\mathscr{M}$. For more background information in this area, see [2], [10], or [13].

In connection with the Riccati equation (1.1), we shall consider the associated Hamiltonian

$$H = \begin{pmatrix} A & -BJB^* \\ -C^*C & -A^* \end{pmatrix}. \tag{1.2}$$

Introduce also the matrices

$$J_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \text{and} \quad J_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Observe that

$$J_1 H = -H^* J_1 \tag{1.3}$$

i.e., the matrix iH is self-adjoint in the indefinite inner product defined on $\mathbb{C}^{2n}$ by the invertible Hermitian matrix iJ_1. Also observe that

$$\tfrac{1}{2}(J_2 H + H^* J_2) = \begin{pmatrix} -C^*C & 0 \\ 0 & -BJB^* \end{pmatrix}. \tag{1.4}$$

For the particular case $J = I$ this says that iH is *dissipative* in the indefinite inner product defined on $\mathbb{C}^{2n}$ by the invertible Hermitian matrix $-J_2$, i.e., $(1/2i)\,[J_2(iH) - (iH)^* J_2] \leq 0$. For matrices which are self-adjoint or dissipative in an indefinite inner product there exist subspaces that are at the same time invariant with respect to the self-adjoint or dissipative matrix, and maximal nonnegative, maximal nonpositive, or neutral with respect to the indefinite inner product (see, e.g., [13]). For a constructive approach and a parametrization of all such subspaces, see [10], [19], and [21] for the self-adjoint case, and [25] for the dissipative case.

That these invariant subspaces are of importance is seen by the following observations. Let X be an $n \times n$ matrix, and form the subspace

$$\mathcal{M} = \operatorname{Im}\begin{pmatrix} I \\ X \end{pmatrix} \tag{1.5}$$

of $\mathbb{C}^{2n}$. Then X is a solution of (1.1) (not necessarily Hermitian) if and only if the subspace $\mathcal{M}$ is H-invariant. Furthermore, X is Hermitian if and only if the subspace $\mathcal{M}$ is J_1-neutral, and as $\dim \mathcal{M} = n$, this can be rephrased as $J_1\mathcal{M} = \mathcal{M}^{\perp}$. Replacing, if necessary, the matrix A by $A + \alpha I$ for $\alpha \in i\mathbb{R}$ large enough, we may assume that H is invertible. Then the statement $J_1\mathcal{M} = \mathcal{M}^{\perp}$ is equivalent to $(J_1 H)\mathcal{M} = \mathcal{M}^{\perp}$. In [17] the latter property is used to give a parametrization of all Hermitian solutions for the case $J = I$ and (A, B) controllable. (However, there the term C^*C is replaced by an arbitrary Hermitian matrix.) It is proved there that every n-dimensional J_1-neutral H-invariant subspace $\mathcal{M}$ necessarily is of the form (1.5) for some Hermitian solution X of the Riccati equation.

Finally, the matrix X is in addition also nonnegative if and only if the subspace $\mathscr{M}$ is J_2-nonnegative. Indeed, for x in $\mathbb{C}^n$ we have

$$\left\langle J_2 \begin{pmatrix} I \\ X \end{pmatrix} x, \begin{pmatrix} I \\ X \end{pmatrix} x \right\rangle = 2\langle Xx, x\rangle. \tag{1.6}$$

It is this simple observation which will play a major role in our analysis. With our choice of J_1 and J_2 any maximal J_2-nonnegative (J_2-nonpositive, J_1-neutral) subspace has dimension n.

The above considerations are summarized in the following proposition.

PROPOSITION 1.1. *If X is a nonnegative Hermitian solution of* (1.1), *then the subspace $\mathscr{M}$ in* (1.5) *is H-invariant, satisfies $J_1\mathscr{M} = \mathscr{M}^{\perp}$ (i.e., $\mathscr{M}$ is maximal J_1-neutral), and is maximal J_2-nonnegative. Conversely, if X is an $n \times n$ matrix and the subspace $\mathscr{M}$ from* (1.5) *is H-invariant, J_1-neutral, and J_2-nonnegative, then X is a nonnegative Hermitian solution of* (1.1).

In conclusion, if we are interested in nonnegative solutions of Riccati equation, then we are interested in n-dimensional subspaces of $\mathbb{C}^{2n}$ that are H-invariant, J_1-neutral, and J_2-nonnegative.

In the note, the open left and right complex half planes will be denoted by $\mathbb{C}_l$ and $\mathbb{C}_r$, respectively. For any $n \times n$ matrix D and any set $S \subset \mathbb{C}$ the notation

$$\mathscr{R}(D, S) = \operatorname{span}\{\operatorname{Ker}(D - \lambda)^n \mid \lambda \in \sigma(D) \cup S\}$$

is used for the corresponding spectral subspace. A pair of matrices (A, B), where A is $n \times n$ and B is $n \times m$, is called controllable if

$$\operatorname{Im}(B \quad AB \quad \cdots \quad A^{n-1}B) = \mathbb{C}^n.$$

For a pair of matrices (C, A), where C is $k \times n$ and A is $n \times n$, the subspace $\mathscr{V}$ is the maximal A-invariant subspace contained in $\operatorname{Ker} C$. The subspace $\mathscr{V}$ contains the subspaces $\mathscr{V}_> = \mathscr{V} \cap \mathscr{R}(A, \mathbb{C}_r)$, $\mathscr{V}_0 = \mathscr{V} \cap \mathscr{R}(A, i\mathbb{R})$, $\mathscr{V}_< = \mathscr{V} \cap \mathscr{R}(A, \mathbb{C}_l)$, and $\mathscr{V}_\leq = \mathscr{V} \cap \mathscr{R}(A, C_l \cup i\mathbb{R})$. If $\mathscr{V} = \{0\}$, then the pair (C, A) is called observable. Consequently, the pair (C, A) is observable if and only if

$$\operatorname{Ker}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = \{0\}.$$

Throughout the paper the projection $P : \mathbb{C}^{2n} \to \mathbb{C}^n$ given by $P = (I\ \ 0)$ will be used together with the corresponding canonical embedding of $\mathbb{C}^n$ into $\mathbb{C}^{2n}$ given by $P^* = (I\ \ 0)^T$.

The paper consists, besides this introduction, of seven sections. In Section 2 the general case is considered. Here, only the Hermitian solution which is *stabilizing* for the pair $(A, -BJB^*)$ is studied, i.e., the solution $X = X^*$ for which $A - BJB^*X$ has all its eigenvalues in the open left half plane. The question of nonnegativity of the stabilizing solution is reduced to the same question for a Riccati equation in fewer dimensions. The third section deals with the case $C = 0$. Here again, only the stabilizing solution is considered. This particular case allows a very explicit criterion for nonnegativity of the stabilizing solution in terms of certain Jordan chains of the matrix A. The rest of the paper treats the case $J = I$. It is assumed that the pair (A, B) is controllable. The full set of nonnegative solutions is considered. In Section 4 a parametrization of the set is given and relations with parametrizations that exist in the literature are discussed. In Section 5 applications are given to topics like isolatedness, order structure, and stability. In Section 6 the inertia of Hermitian solutions of (1.1) with $J = I$ is studied. Finally, in Sections 7 and 8, the discrete-time counterpart of the results of the Sections 4, 5, and 6 will be discussed.

Almost all of the results about algebraic Riccati equations in the existing literature that we refer to in this paper can be found in the recent book [16].

2. THE STABILIZING SOLUTION

Consider the Riccati equation

$$XBJB^*X - XA - A^*X - C^*C = 0. \tag{2.1}$$

We shall consider throughout this section the Hermitian solution which is stabilizing for the pair $(A, -BJB^*)$, if it exists, and we shall denote this solution by X. So $A - BJB^*X$ is a matrix which has all its eigenvalues in $\mathbb{C}_l$. Let H and $\mathscr{M}$ be as in (1.2) and (1.5). As $H|_{\mathscr{M}}$ is similar to $A - BJB^*X$, it follows that

$$\sigma(H|_{\mathscr{M}}) \subset \mathbb{C}_l. \tag{2.2}$$

The question whether X is nonnegative or not is reduced to a smaller Riccati equation where the corresponding stabilizing solution is invertible.

In this and the following section the orthogonal projection along $\mathscr{V}_<$ onto $\mathscr{V}_<^\perp$ is denoted by π. That is, $\pi = (0 \;\; I)$ with respect to $\mathbb{C}^n = \mathscr{V}_< \oplus \mathscr{V}_<^\perp$, and $\pi^* = (0 \;\; I)^T$ is the corresponding canonical embedding of $\mathscr{V}_<^\perp$ into $\mathscr{C}^n$.

THEOREM 2.1. *The equation* (2.1) *has a Hermitian solution X that is stabilizing for the pair $(A, -BJB^*)$ if and only if the equation*

$$\pi BJB^*\pi^* + \pi A\pi^* S + S\pi A^*\pi^* - S\pi C^*C\pi^* S = 0 \tag{2.3}$$

has an invertible Hermitian solution S acting on the space $\mathscr{V}_<^\perp$ which is stabilizing for the pair $(-\pi A^*\pi, \pi C^*C\pi^*)$. In that case

$$X = \begin{pmatrix} 0 & 0 \\ 0 & -S^{-1} \end{pmatrix} \tag{2.4}$$

with respect to the decomposition $\mathbb{C}^n = \mathscr{V}_< \oplus \mathscr{V}_<^\perp$. Moreover, the stabilizing solution X is nonnegative if and only if S is negative definite.

Proof. Assume that X is a Hermitian stabilizing solution of (2.1). Due to (2.2), the n-dimensional subspace $\mathscr{M}$ is contained in $\mathscr{R}(H, \mathbb{C}_l)$. From (1.3) it follows that H and $-H^*$ are similar; hence $\sigma(H)$ is symmetric with respect to $i\mathbb{R}$. Hence $\mathscr{M} = \mathscr{R}(H, \mathbb{C}_l)$. Observe that the subspace $P^*\mathscr{V}_<$ is a subspace of $\mathbb{R}(H, \mathbb{C}_l)$. Hence $P^*\mathscr{V}_< \subset \mathscr{M}$ and

$$\mathscr{V}_< \subset \operatorname{Ker} X. \tag{2.5}$$

Choose a subspace $\mathscr{K}$ such that $\mathscr{M} = P^*\mathscr{V}_< \dot{+} \mathscr{K}$. As $\mathscr{M}$ is J_1-neutral, we have for all $x \in \mathscr{V}_<$ and $k \in \mathscr{K}$ that $\langle J_1 P^*x, k\rangle = 0$, so $(0 \;\; I)\mathscr{K} \subset \mathscr{V}_<^\perp$.

We now show that $\operatorname{Ker} X \subset \mathscr{V}_<$. Indeed, assume $Xx = 0$. Then

$$0 = \langle (XBJB^*X - XA - A^*X - C^*C)x, x\rangle = -\langle C^*Cx, x\rangle,$$

and hence $x \in \operatorname{Ker} C$. But then

$$0 = (XBJB^*X - XA - A^*X - C^*C)x = -XAx.$$

Thus, $\operatorname{Ker} X$ is A-invariant and contained in $\operatorname{Ker} C$. Moreover, if $Xx = 0$ then $HP^*x = P^*Ax$. Hence $H|_{P^*\operatorname{Ker} X}$ is similar to $A|_{\operatorname{Ker} X}$. The former has all its eigenvalues in the open left half plane, because X is the stabilizing solution. Thus we obtain that $\operatorname{Ker} X \subset \mathscr{V}_<$.

Together with (2.5) this implies

$$\mathscr{V}_< = \operatorname{Ker} X. \tag{2.6}$$

It follows that $\mathscr{K}$ does not contain nonzero vectors of the form P^*x, and consequently that $\dim(0\ \ I)\mathscr{K} = \dim \mathscr{K}$. As $\dim \mathscr{M} = n$, we have $\dim \mathscr{K} = \dim \mathscr{V}_<^\perp$, so $(0\ \ I)\mathscr{K} = \mathscr{V}_<^\perp$. Hence there is an $S_{\mathscr{K}} : \mathscr{V}_<^\perp \to \mathbb{C}^n$ such that

$$\mathscr{K} = \begin{pmatrix} -S_{\mathscr{K}} \\ I \end{pmatrix} \mathscr{V}_<^\perp .$$

Let $S : \mathscr{V}_<^\perp \to \mathscr{V}_<^\perp$ be defined by $S = \pi S_{\mathscr{K}}$, and let

$$\mathscr{K}' = \begin{pmatrix} -\pi^*S \\ \pi^* \end{pmatrix} \mathscr{V}_<^\perp .$$

Then $\mathscr{K}'$ also complements $P^*\mathscr{V}_<$ in $\mathscr{M}$. Note that, from $\mathscr{K}' \subset \mathscr{M}$ and (1.5), the operator S must be invertible.

Next, we show that S is a solution of (2.3). This is a consequence of the H-invariance of $\mathscr{M}$. Indeed, for $y \in \mathscr{V}_<^\perp$ we have

$$H\begin{pmatrix} -\pi^*Sy \\ \pi^*y \end{pmatrix} = \begin{pmatrix} -A\pi^*Sy - BJB^*\pi^*y \\ C^*C\pi^*Sy - A^*\pi^*y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} -\pi^*Sz \\ \pi^*z \end{pmatrix}$$

for some $x \in \mathscr{V}_<$ and $z \in \mathscr{V}_<^\perp$. Hence

$$\begin{aligned} -\pi A\pi^*Sy - \pi BJB^*\pi^*y &= \pi x - \pi\pi^*Sz = -Sz = -S\pi\pi^*z \\ &= -S\pi C^*C\pi^*Sy + S\pi A^*\pi^*y, \end{aligned}$$

and S is a solution of (2.3).

From

$$\mathscr{M} = \operatorname{Im}\begin{pmatrix} I \\ X \end{pmatrix} = P^*\mathscr{V}_< \oplus \mathscr{K}' = P^*\mathscr{V}_< \oplus \operatorname{Im}\begin{pmatrix} -\pi^*S \\ \pi^* \end{pmatrix}\mathscr{V}_<^\perp$$

it follows directly that with respect to the decomposition $\mathbb{C}^n = \mathscr{V}_< \oplus \mathscr{V}_<^\perp$ the matrix X can be written as in (2.4).

Observe that

$$-\pi A^*\pi^* + \pi C^*C\pi^*S = S^{-1}(\pi BJB^*\pi^* + \pi A\pi^*S)$$
$$= S^{-1}(\pi A\pi^* + \pi BJB^*\pi^*S^{-1})S,$$

and that with respect to the decomposition $\mathscr{C}^n = \mathscr{V}_< \oplus \mathscr{V}_<^\perp$ the matrix $A - BJB^*X$ can be written as

$$A - BJB^*X = \begin{pmatrix} A|_{\mathscr{V}_<} & * \\ 0 & \pi A\pi^* + \pi BJB^*\pi^*S^{-1} \end{pmatrix}.$$

From this one sees that S is stabilizing for the pair $(-\pi A^*\pi^*, \pi C^*C\pi^*)$.

Conversely, if S is an invertible Hermitian stabilizing solution of (2.3), a direct computation shows that X from (2.4) is Hermitian and stabilizing and solves (2.1).

The last statement in the theorem follows immediately form (2.4). ■

It is easily seen that also the pair $(-\pi A^*\pi^*, \pi C^*)$ is stabilizable. Thus, the results of [11] are applicable to Equation (2.3).

In the special case when $\pi BJB^*\pi^* \geq 0$, necessary and sufficient conditions for the existence of a negative definite solution to (2.3) are given in, e.g., [20]. However, we can show that in this case the stabilizing solution of (2.3) is automatically negative definite. Indeed, because of the stabilizability of $(-\pi A^*\pi^*, \pi C^*)$ it follows from [26, Theorem 2.1] that the stabilizing solution S of (2.3) is nonpositive. By construction S is invertible, and so S is negative definite in case $\pi BJB^*\pi^* \geq 0$. Note that the nonnegativity of $\pi BJB^*\pi^*$ is a sufficient condition for negativity of the stabilizing solution, but by no means a necessary condition (as we shall see for a special case in the next section).

3. THE CASE $C = 0$

In the study of inner-outer factorization of stable rational matrix functions the following Riccati equation plays a role:

$$XBJB^*X - XA - A^*X = 0. \tag{3.1}$$

In particular, nonnegativity of the stabilizing solution is equivalent to existence of an inner-outer factorization of a certain rational matrix function (see [5, 14]).

Throughout this section we assume the existence of a Hermitian solution X of (3.1), which is stabilizing for the pair $(A, -BJB^*)$. The Hamiltonian corresponding to (3.1) is given by

$$H = \begin{pmatrix} A & -BJB^* \\ 0 & -A^* \end{pmatrix}.$$

Existence of a stabilizing solution implies $\sigma(H) \cap i\mathbb{R} = \varnothing$. Clearly, as $C = 0$, the latter is also equivalent to $\sigma(A) \cap i\mathbb{R} = \varnothing$.

Applying Theorem 2.1 in this situation, we see that (3.1) has a nonnegative stabilizing solution if and only if the solution S of

$$\pi A \pi^* S + S \pi A^* \pi^* = -\pi BJB^* \pi^* \tag{3.2}$$

which is stabilizing for the pair $(-\pi A^* \pi^*, 0)$ is negative definite. Assuming that the stabilizing solution X exists, it follows that all eigenvalues of $\pi A^* \pi^*$ are in the right half plane. Hence all eigenvalues of $\pi A \pi^* = (\pi A^* \pi^*)^*$ are also in the right half plane. It follows that $\sigma(\pi A \pi^*) \cap \sigma(-\pi A^* \pi) = \varnothing$. Thus (3.2) has a unique solution. We conclude that the stabilizing solution X of (3.1) is nonnegative if and only if the unique solution S of (3.2) is negative definite.

Recall from the previous section that we can write

$$\mathscr{M} = \operatorname{Im}\begin{pmatrix} I \\ X \end{pmatrix} = P^* \mathscr{V}_< \oplus \begin{pmatrix} -\pi^* S \\ \pi^* \end{pmatrix} \mathscr{V}_<^\perp .$$

Also, $S : \mathscr{V}_<^\perp \to \mathscr{V}_<^\perp$. Fix any basis in $\mathscr{V}_<$, say $x_1, \dots, x_m$, and any basis in $\mathscr{V}_<^\perp$, say $y_1, \dots, y_{n-m}$. Then

$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} x_m \\ 0 \end{pmatrix}, \begin{pmatrix} -\pi^* S y_1 \\ \pi^* y_1 \end{pmatrix}, \dots, \begin{pmatrix} -\pi^* S y_{n-m} \\ \pi^* y_{n-m} \end{pmatrix}$$

is a basis of $\mathscr{M}$. The Gram matrix of J_2 with respect to the basis of $\mathscr{M}$ is given by

$$\begin{pmatrix} 0 & 0 \\ 0 & \left(-2\langle Sy_i, y_j\rangle\right)_{i,j=1}^{n-m} \end{pmatrix}.$$

As X is nonnegative if and only if S is negative definite, we see that X is nonnegative if and only if the matrix $(\langle Sy_i, y_j\rangle)_{i,j=1}^{n-m}$ is negative definite. We shall compute this matrix for a particular choice of the basis in $\mathscr{V}_<^\perp$. In fact, we have, using [3, Theorem A.1.1, Appendix], the following result.

THEOREM 3.1. *Let $y_{ik} \in \mathscr{V}_<^\perp$, $i = 1,\dots,s_k$, $k = 1,\dots,r$, be a Jordan basis for $-\pi A^*\pi^*$. More precisely, assume*

$$-\pi A^*\pi^* y_{ik} = \lambda_k y_{ik} + y_{i-1k} \qquad (y_{0k} := 0).$$

Then

$$\langle Sy_{ik}, y_{jl}\rangle = \sum_{\tau=0}^{i-1}\sum_{\nu=0}^{j-1}(-1)^{\nu-\tau}\binom{\nu+\tau}{\nu}\frac{\langle \pi BJB^*\pi^* y_{i-\tau k}, y_{j-\nu l}\rangle}{\left(\lambda_l+\overline{\lambda_k}\right)^{\nu+\tau+1}}, \tag{3.3}$$

for $k, l = 1,\dots,r$, $i = 1,\dots,s_k$, and $j = 1,\dots,s_l$.

Here it is not assumed that the numbers $\lambda_1,\dots,\lambda_r$ are different. Observe, however, that $\overline{\lambda_k}$ is in the open left half plane, and therefore always different from $-\overline{\lambda_l}$.

Proof. Let $G = -\pi A^*\pi^*$ and let $\Gamma = \pi BJB^*\pi^*$. We want to solve

$$SG - (-G^*)S = \Gamma. \tag{3.4}$$

We have $\sigma(G) \in \mathbb{C}_l$ and $\sigma(-G^*) \in \mathbb{C}_r$. Write the vectors y_{ik} according to some orthonormal basis of $\mathscr{V}_<^\perp$. Let T be the matrix of size dim $\mathscr{V}_<^\perp$ with these vectors as columns. Equation (3.4) translates to

$$T^*STT^{-1}GT - (-T^*G^*T^{-*})T^*ST = T^*\Gamma T.$$

The matrix $T^{-1}GT$ consists of a diagonal of Jordan blocks

$$(T^{-1}GT)_{kk} = \lambda_k I_{s_k} + J_{s_k} = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix}$$

of size $s_k \times s_k$ for $k = 1, \dots, r$. Consider T^*ST and $T^*\Gamma T$ with the corresponding block structure. For $k, l = 1, \dots, r$ the klth block $(T^*ST)_{kl}$ satisfies

$$(T^*ST)_{kl}(\lambda_l I_{s_l} + J_{s_l}) - \left(-\overline{\lambda_k} I_{s_k} - J_{s_k}^T\right)(T^*ST)_{kl} = (T^*\Gamma T)_{kl}.$$

Let $D_k = \operatorname{diag}(-1, 1, \dots, (-1)^{s_k})$. Then

$$D_k(T^*ST)_{kl}(\lambda_l I_{s_l} + J_{s_l}) - \left(-\overline{\lambda_k} I_{s_k} + J_{s_k}^T\right) D_k(T^*ST)_{kl} = D_k(T^*\Gamma T)_{kl}.$$

According to [3, Theorem A.1.1, Appendix], the entries $(D_k(T^*ST)_{kl})_{ij}$ of $D_k(T^*ST)_{kl}$ satisfy

$$(D_k(T^*ST)_{kl})_{ij} = \sum_{\tau=0}^{i-1} \sum_{\nu=0}^{j-1} (D_k(T^*\Gamma T)_{kl})_{i-\tau\, j-\nu}(-1)^{\nu} \times \binom{\nu+\tau}{\nu} \left(\lambda_l + \overline{\lambda_k}\right)^{-\nu-\tau-1}.$$

Hence the entries $((T^*ST)_{kl})_{ij}$ of $(T^*ST)_{kl}$ satisfy

$$((T^*ST)_{kl})_{ij} = \sum_{\tau=0}^{i-1} \sum_{\nu=0}^{j-1} ((T^*\Gamma T)_{kl})_{i-\tau\, j-\nu}(-1)^{\nu-\tau} \times \binom{\nu+\tau}{\nu} \left(\lambda_l + \overline{\lambda_k}\right)^{-\nu-\tau-1}.$$

This translates immediately to the equality (3.3). ∎

Now it is possible to compute, in terms of the data from the matrices A and B, whether the stabilizing Hermitian solution of (3.1) will be nonnegative or not.

COROLLARY 3.2. *The Hermitian solution X of* (3.1) *that stabilizes the pair $(A, -BJB^*)$ is nonnegative if and only if the matrix*

$$\left(\sum_{\tau=0}^{i-1} \sum_{\nu=0}^{j-1} (-1)^{\nu-\tau} \binom{\nu+\tau}{\nu} \frac{\langle \pi BJB^* \pi^* y_{i-\tau k}, y_{j-\nu l} \rangle}{(\lambda_\lambda + \bar{\lambda}_k)^{\nu+\tau+1}} \right)_{ik, jl},$$

with the indices ik and jl running through the set

$$\{(1\ 1), \dots, (s_1\ 1), (1\ 2), \dots, (s_2\ 2), \dots, (1\ r), \dots, (s_r\ r)\},$$

is negative definite.

4. THE CASE $J = I$

In this section we consider the problem of existence and parametrization of nonnegative solutions of the algebraic Riccati equation

$$XBB^*X - XA - A^*X - C^*C = 0. \tag{4.1}$$

Throughout this section we shall assume that (A, B) is controllable, i.e.,

$$\operatorname{Im}(B\ AB\ \cdots\ A^{n-1}B) = \mathbb{C}^n.$$

The controllability guarantees the existence of a Hermitian solution X_+ such that $A - BB^*X_+$ has all its eigenvalues in the closed left half plane. This solution is unique, it is nonnegative, and it is the maximal Hermitian solution, i.e., $X_+ \geq X$ for any other Hermitian solution X of (4.1). See, e.g., [26, Theorem 2.1]. The uniqueness is from [17, Theorem 3]. The pair (C, A) is called observable if

$$\operatorname{Ker} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = \{0\}.$$

If (A, B) is controllable and (C, A) is observable, then there is just one nonnegative solution. See, e.g., [17, Section 2].

From Section 1 we recall the following observations. Let

$$H = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix};$$

then iH is iJ_1-self-adjoint and $-J_2$-dissipative. In such a case it is well known that $\mathscr{R}(H, \mathbb{C}_l)$ and $\mathscr{R}(H, \mathbb{C}_r)$ are J_1-neutral subspaces; see [2, Theorem II.3.3]. Recall that there is a one-one correspondence between Hermitian solutions and n-dimensional J_1-neutral H-invariant subspaces. If (A, B) is controllable, then for every H-invariant subspace $\mathscr{N}_+ \subset \mathscr{R}(H, \mathbb{C}_r)$ there exists a unique n-dimensional J_1-neutral H-invariant subspace $\mathscr{M}$ such that $\mathscr{M} \cap \mathscr{R}(H, \mathbb{C}_r) = \mathscr{N}_+$ (combine [10, Corollary II.4.7] with [19, Theorem 1]). This gives a full description of all n-dimensional H-invariant J_1-neutral subspaces. There is also a description of n-dimensional H-invariant J_2-nonnegative subspaces (see [25]). If the equation (4.1) has a Hermitian solution, then it is shown in [17, Theorem 1] that H has only even partial multiplicities at its purely imaginary eigenvalues (if any). Let $\mathscr{N}_0$ be the H-invariant subspace spanned by the vectors that are in the first half of Jordan chains of H corresponding to purely imaginary eigenvalues of H. Then any H-invariant maximal J_1-neutral subspace $\mathscr{M}$ is of the form

$$\mathscr{M} = \mathscr{N}_+ \dot{+} \mathscr{N}_0 \dot{+} \left[(J_1 \mathscr{N}_+)^{\perp} \cap \mathscr{R}(H, \mathbb{C}_l) \right], \tag{4.2}$$

where $\mathscr{N}_+$ is an arbitrary H-invariant subspace of $\mathscr{R}(H, \mathbb{C}_r)$; see [10, Theorems I.3.21, I.3.22]. As observed in [17, Theorem 1] (see also [10], Theorem II.4.8]), any such subspace $\mathscr{M}$ is of the form

$$\mathscr{M} = \operatorname{Im} \begin{pmatrix} I \\ X \end{pmatrix} \tag{4.3}$$

for a Hermitian solution X of (4.1). The maximal solution, for instance, is obtained by taking $\mathscr{N}_+ = \{0\}$. In order for $\mathscr{M}$ to be J_2-nonnegative it is necessary and sufficient that $\mathscr{N}_+$ be J_2-neutral (see [25], Theorem 2.7, Theorem 3.4]).

The main result of this section establishes a one-one correspondence between the set of all A-invariant subspaces $\mathscr{N}$ contained in $\mathscr{V}_>$ and the set of all nonnegative Hermitian solutions X of (4.1). Similar statements appear in [6], [15], [27], and [28]. In fact, the parametrizing set of subspaces in the theorem below is the same as the parametrizing set in [15, Theorem 3.3.4].

However, the formulas to describe the one-one correspondence and the proof are different. In particular, the maximal Hermitian solution X_+ is not needed in the approach below.

In [27] and [29] also the order structure of the set of nonnegative solutions was considered. In [28] the topology of the set of nonnegative Hermitian solutions was considered, and the isolated nonnegative Hermitian solutions were described. Both structures—the order structure and the topological structure of the set of nonnegative solutions—can easily be described based on Theorem 4.1. We shall do this in the next section.

THEOREM 4.1. *Assume* (A, B) *is controllable. Let* $\mathcal{N}$ *be an* A*-invariant subspace contained in* $\mathcal{V}_>$. *Put* $\mathcal{N}_+ = P^*\mathcal{N}$, *and let* $\mathcal{M}$ *be given by* (4.2). *Then* $\mathcal{M}$ *is of the form* (4.3) *for a nonnegative Hermitian solution* X *of* (4.1). *Conversely, if* X *is a nonnegative Hermitian solution of* (4.1), *construct* $\mathcal{M}$ *as in* (4.3) *and put* $\mathcal{N}_+ = \mathcal{M} \cap \mathcal{R}(H, \mathbb{C}_r)$. *Then* $\mathcal{N}_+$ *is the form* $\mathcal{N}_+ = P^*\mathcal{N}$, *where* $\mathcal{N}$ *is* A*-invariant and contained in* $\mathcal{V}_>$.

First let us assume that $\mathcal{N}$ is A-invariant and contained in $\mathcal{V}_>$. Put $\mathcal{N}_+ = P^*\mathcal{N}$, and let $\mathcal{M}$ be given by (4.2). It is easily seen that $\mathcal{N}_+$ is H-invariant and a subspace of $\mathcal{R}(H, \mathbb{C}_r)$. The subspace $\mathcal{M}$ is maximal J_1-neutral by construction (see, e.g., [21]). Thus $\mathcal{M}$ is of the form (4.3) for a Hermitian solution X of (4.1), by [17, Theorem 1]. Observe that also $\mathcal{N} \subset \operatorname{Ker} X$. Note also that $\operatorname{Ker} X$ always is an A-invariant subspace which is contained in $\operatorname{Ker} C$. The latter observation will be useful later.

To see that X is nonnegative, we show that $\mathcal{M}$ is J_2-nonnegative. Let us first analyze $\mathcal{N}_0$. This subspace is H-invariant and J_1-neutral, and because of the fact that H has only even partial multiplicities at its purely imaginary eigenvalues, it follows from [25, Theorem 2.7] that $\mathcal{N}_0$ is also J_2-neutral. Also from [25, Theorem 2.7] we have that $\mathcal{N}_0$ is J_2-orthogonal both to $\mathcal{R}(H, \mathbb{C}_r)$ and to $\mathcal{R}(H, \mathbb{C}_l)$. In particular, $\mathcal{N}_0$ is J_2-orthogonal both to $\mathcal{N}_+$ and to $(J_1\mathcal{N}_+)^\perp \cap \mathcal{R}(H, \mathbb{C}_l)$ and to itself (here we use in both assertions the fact that $\mathcal{N}_+$ is of the form $\mathcal{N}_+ = P^*\mathcal{N}$). Finally, from [25, Theorem 3.4] or [13], Theorem 11.6], we have that $\mathcal{R}(H, \mathbb{C}_l)$ is J_2-nonnegative. Thus the Gram matrix of J_2 with respect to the decomposition of $\mathcal{M}$ given by (4.2) is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & G \end{pmatrix}$$

for some nonnegative Hermitian matrix G. Hence $\mathcal{M}$ is J_2-nonnegative.

Conversely, let X be a nonnegative Hermitian solution of (4.1). Let $\mathcal{M}$ be as in (4.3). Put $\mathcal{N}_+ = \mathcal{M} \cap \mathcal{R}(H, \mathbb{C}_r)$. We have to show that $\mathcal{N}_+$ is the form $\mathcal{N}_+ = P^*\mathcal{N}$, where $\mathcal{N}$ is A-invariant and contained in $\mathcal{V}_>$. Observe that $\mathcal{N}_+$ is H-invariant, and J_1-neutral. As X is nonnegative, we have that $\mathcal{M}$ is J_2-nonnegative. Since $\mathcal{R}(H, \mathbb{C}_r)$ is J_2-nonpositive (see [13, Theorem 11.6]), it follows that $\mathcal{N}_+$ is J_2-neutral. Because of these observations we have

$$\left\langle J_1 H\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle = 0 \text{ and } \left\langle J_2 H\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle = 0 \qquad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{N}_+.$$

The first of these equalities translates to

$$\langle -C^*Cx - A^*y, x\rangle + \langle -Ax + BB^*y, y\rangle = 0, \tag{4.4}$$

while the second translates to

$$\langle -C^*Cx - A^*y, x\rangle + \langle Ax - BB^*y, y\rangle = 0. \tag{4.5}$$

Addition of (4.4) and (4.5) yields $\langle A^*y, x\rangle = -\|Cx\|^2$, and hence $\langle x, A^*y\rangle = -\|Cx\|^2$, while subtraction yields $\langle Ax, y\rangle = \|B^*y\|^2$. But $\langle Ax, y\rangle = \langle x, A^*y\rangle$, so $-\|Cx\|^2 = \|B^*y\|^2$, and therefore, they are both zero: $Cx = 0$ and $B^*y = 0$. But then

$$H\lfloor_{\mathcal{N}_+} = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}\Bigg|_{\mathcal{N}_+}.$$

Thus

$$\mathcal{N}_+ = \begin{pmatrix} \mathcal{N}_1 \\ \mathcal{N}_2 \end{pmatrix},$$

where $\mathcal{N}_1$ is A-invariant and contained in $\mathcal{V}_>$, while $\mathcal{N}_2$ is $-A^*$-invariant and contained in $\operatorname{Ker} B^*$. It follows from controllability of (A, B) that $\mathcal{N}_2 = \{0\}$.

From the arguments above it is clear that different nonnegative Hermitian solutions give rise to different subspaces and vice versa. Thus the correspondence between the subspaces $\mathcal{N}$ and the nonnegative solutions X given in the statement of the theorem is really a one-one correspondence. Thus the theorem is proved. ∎

The parametrization in [27] is actually based on a parametrization of the subspaces that can occur as $\operatorname{Ker} X$ for a nonnegative Hermitian solution X of (4.1).

PROPOSITION 4.2. *Let (A, B) be controllable. Let X be an arbitrary nonnegative Hermitian solution of* (4.1). *Then* $\mathscr{V}_{\leq} \subset \operatorname{Ker} X$.

Proof. Let λ be an eigenvalue of A with $\operatorname{Re}\lambda \leq 0$. Observe that $\mathscr{V} \cap \mathscr{R}(A, \{\lambda\})$ is an A-invariant subspace. Let $x_1, \dots, x_k$ be a Jordan chain of A in $\mathscr{V} \cap \mathscr{R}(A, \{\lambda\})$, i.e., $Ax_j = \lambda x_j + x_{j-1}$, where $x_0 = 0$. We shall show that $Xx_j = 0$ for all j, by induction. For x_0 this is trivial. Assume that $Xx_i = 0$ for all $i < j$. As all vectors x_j are in $\mathscr{V}$, they are in $\operatorname{Ker} C$. Then

$$\begin{aligned} 0 &= \langle (XBB^*X - XA - A^*X - C^*C)x_j, x_j \rangle \\ &= \langle XBB^*Xx_j, x_j \rangle - \langle X(\lambda x_j + x_{j-1}), x_j \rangle - \langle Xx_j, \lambda x_j + x_{j-1} \rangle. \end{aligned}$$

By the induction hypothesis it follows that

$$0 = \langle XBB^*Xx_j, x_j \rangle - (\lambda + \bar{\lambda})\langle Xx_j, x_j \rangle.$$

As X is nonnegative, we see that if $\operatorname{Re}\lambda < 0$ then $Xx_j = 0$. In case $\operatorname{Re}\lambda = 0$ we only obtain $B^*Xx_j = 0$ But then, again using the induction hypothesis,

$$\begin{aligned} 0 &= XBB^*Xx_j - XAx_j - A^*Xx_j - C^*Cx_j \\ &= -X(\lambda x_j + x_{j-1}) - A^*Xx_j = -X\lambda x_j - A^*Xx_j. \end{aligned}$$

Thus we have $A^*Xx_j = -\lambda Xx_j$ and $B^*Xx_j = 0$. The controllability of (A, B) then shows that $Xx_j = 0$. ■

Theorem 4.1 provides a parametrization of the set of nonnegative solutions of (4.1) in terms of A-invariant subspaces contained in $\mathscr{V}_{>}$. Given such a subspace $\mathscr{N}$ we have by the above proposition and the observations of the proof of Theorem 4.1 that $\mathscr{V}_{\leq} \dotplus \mathscr{N} \subset \operatorname{Ker} X \subset \mathscr{V}$. Also, it is seen from (4.2) that in fact

$$\operatorname{Ker} X = \mathscr{V}_{\leq} \dotplus \mathscr{N}. \tag{4.6}$$

Thus we can also parametrize the set of nonnegative solutions of (4.1) by the set of subspaces which are A-invariant, are contained in Ker C, and contain $\mathscr{V}_{\leq}$. It is this parametrization which is given in [27].

5. ORDER STRUCTURE, TOPOLOGY, AND STABILITY

Introduce the sets of subspaces $\mathbf{N} = \{\mathscr{N} \subset \mathscr{C}^n \mid A\mathscr{N} \subset \mathscr{N}, \mathscr{N} \subset \mathscr{V}_{>}\}$ and $\mathbf{L} = \{\mathscr{N} \subset \mathbb{C}^{2n} \mid \mathscr{N} \subset \mathscr{R}(H, \mathbb{C}_r),\ H\mathscr{N} \subset \mathscr{N}\}$. Denote by $\mathbf{P}$ the set of nonnegative Hermitian solutions of (4.1), and by $\mathbf{H}$ the set of Hermitian solutions of (4.1). These four sets are equipped with a topology as follows: $\mathbf{P}$ and $\mathbf{H}$ inherit the topology induced by the norm; on $\mathbf{N}$ and $\mathbf{L}$ the gap topology is considered, i.e., the distance between two subspaces $\mathscr{K}$ and $\mathscr{L}$ is measured by the gap $\theta(\mathscr{K}, \mathscr{L}) = \|P_{\mathscr{K}} - P_{\mathscr{L}}\|$, where $P_{\mathscr{K}}(P_{\mathscr{L}})$ is the orthogonal projection onto $\mathscr{K}(\mathscr{L})$. Furthermore, $\mathbf{N}$ and $\mathbf{L}$ are equipped with the order structure given by inclusion, and $\mathbf{P}$ and $\mathbf{H}$ inherit the order structure of the set of Hermitian matrices, i.e., $X \leq Y$ means $Y - X$ is nonnegative. Let $\gamma : \mathbf{P} \to \mathbf{N}$ be the map given by $\gamma(X) = \mathscr{N}$, where $\mathscr{N}$ is such that

$$\operatorname{Im}\begin{pmatrix} I \\ X \end{pmatrix} \cap \mathscr{R}(H, \mathbb{C}_r) = \begin{pmatrix} \mathscr{N} \\ 0 \end{pmatrix}.$$

Also define $\rho : \mathbf{H} \to \mathbf{L}$ by

$$\rho(X) = \operatorname{Im}\begin{pmatrix} I \\ X \end{pmatrix} \cap \mathscr{R}(H, \mathbb{C}_r).$$

It is known (see [21, Theorem 2.7] and [22, Theorem 4.2] for continuity, [23, Theorem 9] for the ordering) that ρ and ρ^{-1} are continuous and order reversing. As $\gamma = P\rho|_{\mathbf{P}}$ and $\gamma^{-1} = \rho^{-1}|_{P^*\mathbf{N}}$, we see that γ and γ^{-1} are continuous and order reversing. This proves immediately the following theorem.

THEOREM 5.1. *The order structure of the set of nonnegative Hermitian solutions* $\mathbf{P}$ *and the order structure of the set* $\mathbf{N}$ *are essentially the same in the following sense: Let* $X_1, X_2 \in \mathbf{P}$ *and let* $\mathscr{N}_i = \gamma(X_i)$; *then* $\mathscr{N}_1 \subset \mathscr{N}_2$ *implies* $X_1 \geq X_2$, *and conversely.*

The next theorem describes the isolated nonnegative solutions. The equivalence of (i) and (viii) in the theorem below easily translates into the description of isolated nonnegative solutions given in [28].

THEOREM 5.2. *Assume (A, B) is controllable. Let X be a nonnegative Hermitian solution of* (4.1). *Let*

$$\mathscr{M} = \operatorname{Im}\begin{pmatrix} I \\ X \end{pmatrix},$$

and let $N = P(\mathscr{M} \cap \mathscr{R}(H, \mathbb{C}_r))$. Then the following are equivalent:

(i) *X is an isolated nonnegative Hermitian solution of* (4.1);

(ii) *$\mathscr{M}$ is isolated within the set of H-invariant subspaces that are J_1-maximal neutral, and J_2-nonnegative;*

(iii) *$\mathscr{M} \cap \mathscr{R}(H, \mathbb{C}_r)$ is isolated within the set of subspaces of $\mathscr{R}(H, \mathbb{C}_r)$ that are H-invariant and J_2-neutral;*

(iv) *$\mathscr{N}$ is isolated in* $\mathbf{N}$;

(v) *$\mathscr{N}$ is isolated as an $A|_{\mathscr{V}}$-invariant subspace, and* $\operatorname{Re} \sigma(A|_{\mathscr{N}}) > 0$;

(vi) *$\mathscr{N}$ is isolated as an $A|_{\mathscr{V}_>}$-invariant subspace;*

(vii) *for every eigenvalue λ of $A|_{\mathscr{V}_>}$ with* $\dim \operatorname{Ker}(A|_{\mathscr{V}_>} - \lambda) > 1$, *either $\mathscr{N} \cap \mathscr{R}(A, \{\lambda\}) = \{0\}$ or $\mathscr{R}(A, \{\lambda\}) \cap \mathscr{V}_> \subset \mathscr{N}$;*

(viii) *for every eigenvalue λ of $A|_{\mathscr{V}_>}$ with* $\dim \operatorname{Ker}(A|_{\mathscr{V}_>} - \lambda) > 1$, *either* $\operatorname{Ker} X \cap \mathscr{R}(A, \{\lambda\}) = \{0\}$ *or* $\mathscr{R}(A, \{\lambda\}) \cap \mathscr{V}_> \subset \operatorname{Ker} X$.

Proof. (i) $\Rightarrow$ (ii): Suppose $\mathscr{M}$ is not isolated. Then there is a sequence of H-invariant, maximal J_1-neutral and J_2-nonnegative subspaces $\mathscr{M}_k$ such that $\mathscr{M}_k \to \mathscr{M}$. Then

$$\mathscr{M}_k = \operatorname{Im}\begin{pmatrix} I \\ X_k \end{pmatrix}$$

for some nonnegative solution X_k of (4.1). For all $\varepsilon > 0$ there is a number k such that $\|P_{\mathscr{M}_k} - P_{\mathscr{M}}\| < \varepsilon$. According to Theorem 13.4.2 of [9], for all $x \in \mathbb{C}^n$ there exists a vector $y_k \in \mathbb{C}^n$ such that

$$\left\| \begin{pmatrix} y_k \\ X_k y_k \end{pmatrix} - \begin{pmatrix} x \\ Xx \end{pmatrix} \right\|^2 < \varepsilon^2.$$

Hence $\|y_k - x\| < \varepsilon$ and $\|X_k y_k - Xx\| < \varepsilon$. From $X_k \le X_+$, where X_+ denotes the maximal solution of (4.1), it follows that $\|X_k\| \le \|X_+\|$. Hence

$$\|X_k x - Xx\| \le \|X_k(x - y_k)\| + \|X_k y_k - Xx\| \le (\|X_+\| + 1)\varepsilon.$$

It follows that $X_k \to X$ and X is not isolated.

(ii) $\Rightarrow$ (iii): Suppose $\mathscr{N}_+ = \mathscr{M} \cap \mathscr{R}(H, \mathbb{C}_r)$ is not isolated. Then there is a sequence of H-invariant and J_2-neutral subspaces $\mathscr{N}_{n+} \subset \mathscr{R}(H, \mathbb{C}_r)$ such that $\mathscr{N}_{n+} \to \mathscr{N}_+$. In particular $\mathscr{N}_{n+}$ and $\mathscr{N}_+$ have the same dimensions. Observe that, as $\mathscr{N}_{n+} \subset \mathscr{R}(H, \mathbb{C}_r)$, it is also J_1-neutral. Construct the n-dimensional subspaces

$$\mathscr{M}_n = \mathscr{N}_{n+} \dot{+} \mathscr{N}_0 \dot{+} \left[(J_1 \mathscr{N}_{n+})^{\perp} \cap \mathscr{R}(H, C_l) \right].$$

According to Theorem 13.4.1 of [9] there is a subsequence of $\mathscr{M}_1, \mathscr{M}_2, \dots$, denoted by $\mathscr{M}_{n_1}, \mathscr{M}_{n_2}, \dots$, that converges to some n-dimensional subspace $\mathscr{M}'$. Since $\mathscr{M}_{n_k}$ is H-invariant for all k, it follows that $\mathscr{M}'$ is H-invariant; see [9, Corollary 15.1.2]. Since $\mathscr{M}_{n_k}$ is J_1-neutral and J_2-nonnegative, it follows that $\mathscr{M}'$ is J_1-neutral and J_2-nonnegative. Write

$$\mathscr{M}' = \left[\mathscr{M}' \cap \mathscr{R}(H, \mathbb{C}_r) \right] \dot{+} \left[\mathscr{M}' \cap \mathscr{R}(H, i\mathbb{R}) \right] \dot{+} \left[\mathscr{M}' \cap \mathscr{R}(H, \mathbb{C}_l) \right].$$

Next we will show the three inclusions

$$\begin{aligned}
&\mathscr{M}' \cap \mathscr{R}(H, \mathbb{C}_r) \subset \mathscr{N}_+,\\
&\mathscr{M}' \cap \mathscr{R}(H, i\mathbb{R}) \subset \mathscr{N}_0,\\
&\mathscr{M}' \cap \mathscr{R}(H, \mathbb{C}_l) \subset (J_1 \mathscr{N}_+)^{\perp} \cap \mathscr{R}(H, \mathbb{C}_l).
\end{aligned}$$

For any $x_+ \in \mathscr{M}' \cap \mathscr{R}(H, \mathbb{C}_r)$ there are vectors $x_{n_k} = x_{n_k+} + x_{n_k0} + x_{n_k-} \in \mathscr{M}_{n_k}$ such that $\|x_+ - x_{n_k}\| \to 0$. By continuity of the projection on $\mathscr{R}(H, \mathbb{C}_r)$ along $\mathscr{R}(H, \mathbb{C}_l \cup i\mathbb{R})$ if follows that $\|x_+ - x_{n_k+}\| \to 0$. From [9, Theorem 13.4.2] it follows that $x_+ \in \mathscr{N}_+$. Hence $\mathscr{M}' \cap \mathscr{R}(H, \mathbb{C}_r) \subset \mathscr{N}_+$. For any $x_0 \in \mathscr{M}' \cap \mathscr{R}(H, i\mathbb{R})$ there are vectors $x_{n_k} = x_{n_k+} + x_{n_k0} + x_{n_k-} \in \mathscr{M}_{n_k}$ such that $\|x_- - x_{n_k}\| \to 0$. Projection on $\mathscr{R}(H, i\mathbb{R})$ along $\mathscr{R}(H, \mathbb{C}_l \cup \mathbb{C}_r)$ gives $\|x_0 - x_{n_k0}\| \to 0$. Clearly the sequence $\mathscr{M}_{n_k} \cap \mathscr{R}(H, i\mathbb{R}) = \mathscr{N}_0$ converges to $\mathscr{N}_0$. Thus $x_0 \in \mathscr{N}_0$ and $M' \cap \mathscr{R}(H, i\mathbb{R}) \subset \mathscr{N}_0$. For any $x_- \in \mathscr{M}' \cap \mathscr{R}(H, \mathbb{C}_l)$ there are vectors $x_{n_k} = x_{n_k+} + x_{n_k0} + x_{n_k-} \in \mathscr{M}_{n_k}$ such that $\|x_- - x_{n_k}\| \to 0$. Projection on $\mathscr{R}(H, \mathbb{C}_l)$ along $\mathscr{R}(H, i\mathbb{R} \cup \mathbb{C}_r)$ gives $\|x_- - x_{n_k-}\| \to 0$. Let y be some arbitrary vector in $\mathscr{N}_+$. Choose a sequence of vectors $y_{n_k} \in \mathscr{N}_{n_k}$ that converges to y. Then

$$\langle J_1 y, x_- \rangle = \lim_{k \to \infty} \langle J_1 y_{n_k}, x_{n_k-} \rangle = 0.$$

Thus $x_- \in (J_\Gamma \mathcal{N}_+)^\perp$ and $M' \cap \mathcal{R}(H, \mathbb{C}_l) \subset (J_\Gamma \mathcal{N}_+)^\perp \cap \mathcal{R}(H, \mathbb{C}_l)$. Combining the three inclusions gives $\mathcal{M}' \subset \mathcal{M}$, but, as $\mathcal{M}'$ and $\mathcal{M}$ have the same dimension, equality holds. Hence $\mathcal{M}$ is not isolated.

(iii) $\Rightarrow$ (iv): From the proof of Theorem 4.1 it follows that $\mathcal{M} \cap \mathcal{R}(H, \mathbb{C}_r) = P^*\mathcal{N}$. Since the canonical embedding P^* is continuous, it follows that $\mathcal{N}$ is isolated if $\mathcal{M} \cap \mathcal{R}(H, \mathbb{C}_r)$ is isolated.

(iv) $\Rightarrow$ (i): Assume X is not isolated. Then there are nonnegative solutions X_n of (4.1) such that $\|X_n - X\| \to 0$. From [9, Theorem 13.5.1] it follows that $\operatorname{Ker} X_n \to \operatorname{Ker} X$. Hence $\dim \operatorname{Ker} X_n = \dim \operatorname{Ker} X$ for n large enough. Let

$$\mathcal{M}_n = \operatorname{Im}\begin{pmatrix} I \\ X_n \end{pmatrix}$$

and let $\mathcal{N}_n = P(\mathcal{M}_n \cap \mathcal{R}(H, \mathbb{C}_r))$. From (4.6) it follows that $\dim \mathcal{N}_n = \dim \mathcal{N}$ for n large enough. Let $\mathcal{N}_{n_1}, \mathcal{N}_{n_2}, \ldots,$ be a subsequence converging to some subspace $\mathcal{N}'$. Then $\dim \mathcal{N}' = \mathit{dim} \mathcal{N}_{n_k}$ for k large enough. Thus $\dim \mathcal{N}' = \dim \mathcal{N}$. For any $x \in \mathcal{N}'$ there are vectors $x_{n_k} \in \mathcal{N}_{n_k}$ converging to x. Note that $x_{n_k} \in \operatorname{Ker} X_{n_k}$ for all k. Hence $Xx = (X - X_{n_k})x + X_{n_k}(x - x_{n_k})$. This implies that $x \in \operatorname{Ker} X$. Thus $P^*x \in \mathcal{M}$. Moreover, since $P^*x_{n_k} \in \mathcal{R}(H, \mathbb{C}_r)$, we have that $P^*x \in \mathcal{R}(H, \mathbb{C}_r)$. Hence $x \in P(\mathcal{M} \cap \mathcal{R}(H, \mathbb{C}_r)) = \mathcal{N}$ and $\mathcal{N}' \in \mathcal{N}$. As $\mathcal{N}'$ and $\mathcal{N}$ have the same dimension, equality holds. From Theorem 4.1 it follows that $\mathcal{N}_{n_k} \in \mathbf{N}$. Thus $\mathcal{N}$ is not isolated.

(v) and (vi) are essentially just reformulations of (iv), while the equivalence of (vii) with (vi) is known from [4, Theorem 8.1]. The reformulation of (vii) and (viii) is a consequence of the fact that $\operatorname{Ker} X = \mathcal{N} \dot{+} \mathcal{V}_{\leq}$, as observed in (4.6). ∎

Next, we shall consider the question of stability of nonnegative Hermitian solutions under small perturbations of the coefficients of the algebraic Riccati equation (4.1). A nonnegative Hermitian solution X_0 of (4.1) is called *stably nonnegative* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|A - A_1\| + \|B - B_1\| + \|C - C_1\| < \delta$ implies that the algebraic Riccati equation

$$XB_1B_1^*X - XA_1 - A_1^*X - C_1^*C_1 = 0$$

has a nonnegative Hermitian solution X_1 such that $\|X_0 - X_1\| < \varepsilon$.

The following lemma, which may be of independent interest, will be useful in the description of stably nonnegative solutions. It shows that the set of pairs (C, A) which are not observable is an open and dense subset of the set of all pairs (C, A) of the same size.

LEMMA 5.3. *Assume (A, B) is controllable and (C, A) is not observable. For $\varepsilon > 0$ small enough there exist $A(\varepsilon)$, $C(\varepsilon)$ such that $(C(\varepsilon), A(\varepsilon))$ is observable, $(A(\varepsilon), B)$ is controllable, and $\|C - C(\varepsilon)\| + \|A - A(\varepsilon)\| < \varepsilon$.*

Proof. Let $C = (c_{ij})_{i,j=1}^{m;n}$, and let $A = (a_{ij})_{i,j=1}^{n}$. Consider the first n rows of

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}.$$

This gives an $n \times n$ matrix, which we shall denote by $\mathscr{O}$. Its determinant is a polynomial in the variables c_{ij} and a_{ij}. the pair (C, A) is certainly observable if $\det \mathscr{O} \neq 0$. Let us identify the pair of matrices (C, A) with the matrix $(C^T\ A^T)^T$ in $\mathbb{C}^{(m+n)\times n}$. It is well known that the solution of the equation $\det \mathscr{O} = 0$ is an algebraic variety in $\mathbb{C}^{(n+m)\times n}$ of lower dimension. Hence, in every neighborhood of (C_0, A_0) there is pair of matrices (C, A) with $\deg \mathscr{O} \neq 0$, i.e., an observable pair of matrices. The controllability of the pair (A, B) means that the matrix $(B\ AB\ \cdots\ A^{n-1}B)$ has full rank, and the rank does not change under small perturbations of A. ■

THEOREM 5.4. *Assume (A, B) is controllable. Then there is only one stably nonnegative solution of* (4.1), *being the maximal one.*

Proof. Let X_+ be the maximal solution of (4.1). Let

$$\mathscr{M} = \operatorname{Im}\begin{pmatrix} I \\ X_+ \end{pmatrix}.$$

The n-dimensional subspace $\mathscr{M}$ is H-invariant, J_1-neutral, and $J_2 =$ nonnegative. Note that, according to Theorem 5.1, the subspace $\mathscr{N} = P(\mathscr{M} \cap \mathscr{R}(H, \mathbb{C}_r))$ that corresponds to X_+ equals $\mathscr{N} = \{0\}$. Let $A_m \to A$, $B_m \to B$, and $C_m \to C$, and let X_m be the maximal solution of the perturbed equation

$$XB_m B_m^* X - XA_m - A_m^* X - C_m^* C_m = 0.$$

Let H_m be the corresponding Hamiltonian. Clearly H_m converges to H. According to [26, Theorem 2.1], the matrix X_m is a nonnegative solution. Let

$$\mathscr{M}_m = \operatorname{Im}\begin{pmatrix} I \\ X_m \end{pmatrix}.$$

The n-dimensional subspace $\mathscr{M}_m$ is H_m-invariant, J_1-neutral, and J_2-nonnegative. Let $\mathscr{N}_{m0}$ be the H_m-invariant subspace spanned by the vectors that are in the first half of Jordan chains of H_m corresponding to pure imaginary eigenvalues of H_m. From Theorem 5.1 it follows that $P(\mathscr{M}_m \cap \mathscr{R}(H_m, \mathbb{C}_r)) = \{0\}$. Due to (4.2) it follows that $\mathscr{M}_m = \mathscr{N}_{m0} \dot{+} \mathscr{R}(H_m, \mathbb{C}_l)$ and $\sigma(H_m|_{\mathscr{M}_m}) \subset \mathbb{C}_l \cup i\mathbb{R}$.

First we show that X_+ is a stably nonnegative solution of (4.1). As the set of subspaces of dimension n is compact, we may as well assume from the start that $\mathscr{M}_m \to \mathscr{M}'$ for some n-dimensional subspace $\mathscr{M}'$. Since $\mathscr{M}_m$ is H_m-invariant, J_1-neutral, and J_2-nonnegative, it follows that $\mathscr{M}'$ is H-invariant, J_1-neutral, and J_2-nonnegative. Hence $\mathscr{M}'$ has the form

$$\mathscr{M}' = \operatorname{Im}\begin{pmatrix} I \\ X' \end{pmatrix}$$

for some nonnegative solution X' of (4.1). Since $\sigma(H_m|_{\mathscr{M}_m}) \subset \mathbb{C}_l \cup i\mathbb{R}$ for all m, it follows that $\sigma(H|_{\mathscr{M}'}) \subset \mathbb{C}_l \cup i\mathbb{R}$; see [9, Theorem 15.1.4]. Hence $P(\mathscr{M}' \cap \mathscr{R}(H, \mathbb{C}_r)) = \{0\} = \mathscr{N}$, and from Theorem 4.1 it follows that $X' = X_+$. Hence X_+ is a stably nonnegative solution.

Conversely, to show that the maximal solution is the only stably nonnegative solution, let $(C_\varepsilon, A_\varepsilon) \to (C, A)$ be such that for each ε the pair $(C_\varepsilon, A_\varepsilon)$ is observable and (A_ε, B) is controllable (this is possible by Lemma 5.3). Then the algebraic Riccati equation

$$XBB^*X - XA_\varepsilon - A_\varepsilon^* X - C_\varepsilon^* C_\varepsilon = 0$$

has only one nonnegative Hermitian solution X_ε. Hence X_ε is the maximal solution of the perturbed equation. From the first part of this proof it follows that X_ε converges to the maximal solution of (4.1). ■

6. INERTIA OF HERMITIAN SOLUTIONS

Consider again the algebraic Riccati equation

$$XBB^*X - XA - A^*X - C^*C = 0, \tag{6.1}$$

under the usual assumption that (A, B) is controllable. Let X be a Hermitian solution, and consider

$$\mathscr{M} = \operatorname{Im}\begin{pmatrix} I \\ X \end{pmatrix}. \tag{6.2}$$

The problem we wish to solve is the following: can we characterize the inertia of X in terms of properties of the subspace $\mathscr{M}$?

To start with, the following proposition shows that $\dim \operatorname{Ker} X = \dim(\mathscr{M} \cap P^*\mathscr{V})$.

PROPOSITION 6.1. *Assume (A, B) is controllable. Let X be Hermitian solution of* (6.1), *and let $\mathscr{M}$ be as in* (6.2). *Then*

$$P^* \operatorname{Ker} X = \mathscr{M} \cap P^*\mathscr{V}.$$

Proof. Put $\mathscr{N} = P^* \operatorname{Ker} X$. Clearly $\mathscr{N} \subset \mathscr{M}$. Moreover, if $x \in \operatorname{Ker} X$, we have already seen that $x \in \mathscr{V}$. This shows that $\mathscr{N} \subset \mathscr{M} \cap P^*\mathscr{V}$. Conversely, let $x \in \mathscr{V}$ such that $P^*x \in \mathscr{M}$. Then $Xx = 0$ by the definition of $\mathscr{M}$. Consequently, $\mathscr{M} \cap P^*\mathscr{V} \subset \mathscr{N}$. This proves the proposition. ■

Recall that the Hamiltonian H corresponding to (6.2) is

$$H = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}.$$

As in Section 4, let $\mathscr{N}_0$ be the H-invariant subspace spanned by vectors that are in the first half of Jordan chains of H corresponding to pure imaginary eigenvalues of H. The subspace N_0 is J_1-neutral and J_2-neutral. By the remark after the proof of Theorem 4.1, we have that $\mathscr{N}_0 = P^*P\mathscr{N}_0$ where $P\mathscr{N}_0 \subset \mathscr{V}_0$, i.e., $P\mathscr{N}_0 \subset \mathscr{V}_0 \cap \operatorname{Ker} X$. Let us consider $\mathscr{V}_0$. Let x be an eigenvector of $A|_{\mathscr{V}_0}$. Then $Ax = \lambda_0 x$ where $\operatorname{Re} \lambda_0 = 0$, and $Cx = 0$. It follows easily that $\langle XBB^*Xx, x\rangle = 0$, i.e., $B^*Xx = 0$. Then

$$0 = (XBB^*X - XA - A^*X - C^*C)x = -X\lambda_0 x - A^*Xx.$$

Hence $A^*Xx = -\lambda_0 Xx$. As (B^*, A^*) is observable, we get that $Xx = 0$. Thus $x \in \operatorname{Ker} X$. Analogously, an induction argument with respect to the Jordan chains of $A|_{\mathscr{V}_0}$ shows that $\mathscr{V}_0 \subset \operatorname{Ker} X$. Summarizing, we have

$$P\mathscr{N}_0 \subset \mathscr{V}_0 \subset \operatorname{Ker} X.$$

In fact the first two subspaces are equal: $P\mathscr{N}_0 = \mathscr{V}_0$. Indeed, let $x \in \mathscr{V}_0$. Then $x \in \operatorname{Ker} X$ and $P^*x \in \mathscr{M}$. From $x \in \operatorname{Ker} C \cap \mathscr{R}(A, i\mathbb{R})$ it follows that

$$P^*x \in \mathscr{R}(H|_{\mathscr{M}}, i\mathbb{R}) = \mathscr{N}_0,$$

where the equality follows from (4.2). Hence $x \in P\mathscr{N}_0$, and we conclude that $P\mathscr{N}_0 = \mathscr{V}_0$ and even $\mathscr{N}_0 = P^*\mathscr{V}_0$.

Let us denote by π (ν) the number of positive (negative) eigenvalues of X, multiplicities taken into account, and let $\delta = \dim \operatorname{Ker} X$.

If we write

$$\mathscr{M} = [\mathscr{M} \cap \mathscr{R}(H, \mathbb{C}_l)] \dotplus (\mathscr{M} \cap \mathscr{N}_0) \dotplus [\mathscr{M} \cap \mathscr{R}(H, \mathbb{C}_r)],$$

and

$$\mathscr{M} \cap \mathscr{R}(H, \mathbb{C}_l) = (\mathscr{M} \cap P^*\mathscr{V}_<) \dotplus \mathscr{K}_< ,$$

$$\mathscr{M} \cap \mathscr{R}(H, \mathbb{C}_r) = (\mathscr{M} \cap P^*\mathscr{V}_>) \dotplus \mathscr{K}_>$$

for some $\mathscr{K}_<$ and $\mathscr{K}_>$, then

$$\begin{aligned}\mathscr{M} &= \operatorname{Im}\begin{pmatrix} I \\ X \end{pmatrix} \\ &= \mathscr{K}_< \dotplus (\mathscr{M} \cap P^*\mathscr{V}_<) \dotplus (\mathscr{M} \cap \mathscr{N}_0) \dotplus (\mathscr{M} \cap P^*\mathscr{V}_>) \dotplus \mathscr{K}_> \\ &= \mathscr{K}_< \dotplus P^* \operatorname{Ker} X \dotplus \mathscr{K}_> . \end{aligned} \tag{6.3}$$

It follows that $\nu + \pi = \dim \mathscr{K}_< + \dim \mathscr{K}_>$. The relation (1.6) shows that π, ν, and δ are the numbers of positive, negative, and zero squares, respectively, of the quadratic form $\langle J_2 \cdot, \cdot \rangle$ on $\mathscr{M}$. Then the equation (6.3) implies that $\nu + \pi = \dim \mathscr{K}_< + \dim \mathscr{K}_>$. From [25, Lemma 3.3] it follows that $\mathscr{R}(H, \mathbb{C}_l)$ is J_2-nonpositive and $\mathscr{R}(H, \mathbb{C}_r)$ is J_2-nonegative. Hence $\mathscr{K}_<$ is J_2-nonpositive, and $\mathscr{K}_>$ is J_2-nonnegative. This shows that $\pi \leq \dim \mathscr{K}_>$ and $\nu \leq \dim \mathscr{K}_<$. Combining these observations we see that $\pi = \dim \mathscr{K}_>$ and $\nu = \dim \mathscr{K}_<$, and we arrive at the following theorem.

THEOREM 6.2. *Assume (A, B) is controllable. Let X be a Hermitian solution of* (6.1), *and let $\mathscr{M}$ be as in* (6.2). *Then*

$$\pi = \dim[\mathscr{M} \cap \mathscr{R}(H, \mathbb{C}_l)] - \dim(\mathscr{M} \cap P^*\mathscr{V}_<),$$

$$\nu = \dim[\mathscr{M} \cap \mathscr{R}(H, \mathbb{C}_r)] - \dim(\mathscr{M} \cap P^*\mathscr{V}_>),$$

$$\delta = \dim[\mathscr{M} \cap P^*\mathscr{V}].$$

7. NONNEGATIVE SOLUTIONS FOR THE DISCRETE ALGEBRAIC RICCATI EQUATION

In this section we do the analogue of Section 4 for the discrete algebraic Riccati equation.

The equation number consideration is

$$X = A^*XA + Q - A^*XB(R + B^*XB)^{-1}B^*XA \tag{7.1}$$

where A is an $n \times n$ matrix, $Q \geq 0$ is also an $n \times n$ matrix, $R > 0$ is an $m \times m$ matrix, and B is an $n \times m$ matrix. The matrix X is to be found. The equation plays a role in the study of LQ-optimal control for discrete-time systems. We would like to have a parametrization of all nonnegative Hermitian solutions X. We shall assume throughout that A is invertible and that (A, B) is controllable.

Introduce

$$T = \begin{pmatrix} A + BR^{-1}B^*A^{*-1}Q & -BR^{-1}B^*A^{-1} \\ -A^{*-1}Q & A^{*-1} \end{pmatrix}. \tag{7.2}$$

Recall that

$$J_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \text{and} \quad J_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Straightforward computation yields that $T^*J_1T = J_1$, i.e., T is J_1-unitary. Again by straightforward computation it is checked that

$$\begin{aligned} J_2 - T^*J_2T &= 2\begin{pmatrix} Q + QA^{-1}BR^{-1}B^*A^{*-1}Q & -QA^{-1}BR^{-1}B^*A^{*-1} \\ -A^{-1}BR^{-1}B^*A^{*-1}Q & A^{-1}BR^{-1}B^*A^{-1} \end{pmatrix} \\ &= 2\left[\begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} Q \\ -I \end{pmatrix} A^{-1}BR^{-1}B^*A^{*-1}\begin{pmatrix} Q & -I \end{pmatrix}\right] \geq 0. \end{aligned} \tag{7.3}$$

Thus $T^*J_2T \leq J_2$, i.e., T is J_2-contractive.

In this and the next section we will use $\mathbb{C}_{\text{in}}$ for the set of complex numbers inside the open unit disc and $\mathbb{C}_{\text{out}}$ for the set of complex numbers outside the closed unit disc. The unit circle will be denoted by $\mathbb{T}$. In the rest of this paper $\mathscr{V}$ denotes the maximal A-invariant subspace contained in Ker Q. The subspace $\mathscr{V}$ contains the subspaces $\mathscr{V}_< = \mathscr{V} \cap \mathscr{R}(A, \mathbb{C}_{\text{in}})$, $\mathscr{V}_> = \mathscr{V} \cap \mathscr{R}(A, \mathbb{C}_{\text{out}})$, $\mathscr{V}_0 = \mathscr{V} \cap \mathscr{R}(A, \mathbb{T})$, and $\mathscr{V}_\leq = \mathscr{V} \cap \mathscr{R}(A, \mathbb{C}_{\text{in}} \cup \mathscr{T})$. The notation $\mathscr{V}$, etc., has been used for the analogous subspaces in the previous sections, and no confusion will arise. As before, $P = (I\ 0)$ denotes the orthogonal projection of $\mathbb{C}^{2n}$ onto $\mathbb{C}^n$, and $P^* = (I\ 0)^T$ is the corresponding canonical embedding.

The following propositions are known.

Proposition 7.1 ([18, Theorem 0.2; see also [24, Theorem 1.1]). *Assume (A, B) is controllable, A is invertible, $Q \geq 0$, and $R > 0$. Every n-dimensional T-invariant J_1-neutral subspace $\mathscr{M}$ is of the form*

$$\mathscr{M} = \operatorname{Im}\begin{pmatrix} I \\ X \end{pmatrix} \tag{7.4}$$

for some Hermitian solution X of (7.1). *Conversely, if X is a Hermitian solution of* (7.1), *then the n-dimensional subspace $\mathscr{M}$ constructed as in* (7.4) *is T-invariant and J_1-neutral. There exists a Hermitian solution of* (7.1) *if and only if T has only even partial multiplicities corresponding to its eigenvalues on the unit circle.*

Proposition 7.2 [26, Theorem 3.1]. *Assume (A, B) is controllable, A is invertible, $Q \geq 0$, and $R > 0$. Then the algebraic Riccati equation* (7.1) *has a nonnegative solution. In fact, there exists a nonnegative solution X_+ of* (7.1), *the maximal solution, such that $X_+ \geq X$ for any other Hermitian solution X of* (7.1).

Proposition 7.3 [13, Theorems 7.1, 11.2; 1, Proposition 1.5]. *Assume $J = J^*$ is an invertible matrix. Assume T is J-contractive, i.e., $T^*JT \leq J$. Then $\mathscr{R}(T, \mathbb{C}_{\text{out}})$ is J-nonpositive and $\mathscr{R}(T, C_{\text{in}})$ is J-nonnegative. For any Jordan chain $x_1, \ldots, x_n$ of T corresponding to an eigenvalue on the unit circle, the subspace* span$\{x_1, \ldots, x_m\}$, *where $m = [n/2]$, is J-neutral. If T is J-unitary, then both $\mathscr{R}(T, \mathbb{C}_{\text{out}})$ and $\mathscr{R}(T, \mathbb{C}_{\text{in}})$ are J-neutral.*

Assume X is a Hermitian solution of (7.1). From (1.6) it follows that X is nonnegative if and only if the corresponding subspace $\mathscr{M}$ is maximal J_2-nonnegative.

Combining Propositions 7.1 and 7.2, it follows that the matrix T has only even partial multiplicities for its eigenvalues on the unit circle. Let $\mathscr{N}_0$ denote the T-invariant subspace spanned by the vectors that are in the first halves of Jordan chains of T corresponding to eigenvalues of T on the unit circle. From Proposition 7.3 it follows that $\mathscr{N}_0$ is J_1-neutral and J_2-neutral.

THEOREM 7.4. *Assume (A, B) is controllable, A is invertible, $R > 0$, and $Q \geq 0$. Then there is a one-one correspondence between the set of all A-invariant subspaces $\mathscr{N}$ contained in $\mathscr{V}_>$ and the set of all nonnegative Hermitian solutions X of (7.1). More precisely, let $\mathscr{N}$ be such a subspace, and let $\mathscr{M}$ be given by*

$$\mathscr{M} = P^*\mathscr{N} \dot{+} \mathscr{N}_0 \dot{+} \left[(J_1 P^*\mathscr{N})^{\perp} \cap \mathscr{R}(T, \mathbb{C}_{\text{in}})\right]. \tag{7.5}$$

Then

$$\mathscr{M} = \operatorname{Im}\begin{pmatrix} I \\ X \end{pmatrix} \tag{7.6}$$

for a nonnegative Hermitian solution X of (7.1). Conversely, assume X is a nonnegative Hermitian solution of (7.1), and let $\mathscr{M}$ be as in (7.6). Then $\mathscr{M} \cap \mathscr{R}(T, \mathbb{C}_{\text{out}}) = P^*\mathscr{N}$ for some A-invariant subspace $\mathscr{N}$ contained in $\mathscr{V}_>$.

Proof. Assume $\mathscr{N}$ is A-invariant and contained in $\mathscr{V}_>$. Clearly, $P^*\mathscr{N}$ is T-invariant and both J_1-neutral and J_2-neutral. Assume $x \perp J_1 P^*\mathscr{N}$ for some $x \in \mathbb{C}^{2n}$. Using $J_1 = T^* J_1 T$ and $TP^*\mathscr{N} = P^*\mathscr{N}$, it follows that $Tx \perp J_1 P^*\mathscr{N}$. Hence $(J_1 P^*\mathscr{N})^{\perp}$ is T-invariant. We conclude that the subspace $\mathscr{M}$ is (7.5) is T-invariant, J_1-neutral, and J_2-nonnegative. Due to Proposition 7.1, there exists a nonnegative solution X of (7.1) such that (7.6) holds.

Conversely, assume X is nonnegative solution of (7.1), and let $\mathscr{M}$ be the subspace from (7.6). Since $\mathscr{M}$ is J_2-nonnegative and J_1-neutral and $\mathscr{R}(T, \mathbb{C}_{\text{out}})$ is J_2-nonpositive, it follows that the T-invariant subspace $\mathscr{M} \cap \mathscr{R}(T, \mathbb{C}_{\text{out}})$ is both J_1-neutral and J_2-neutral. Consider some arbitrary vector

$$\begin{pmatrix} x \\ Xx \end{pmatrix} \in \mathscr{M} \cap \mathscr{R}(T, \mathbb{C}_{\text{out}}).$$

From equation (1.6) and $X \geqslant 0$ it follows that $Xx = 0$. Hence $\mathscr{M} \cap R(T, \mathbb{C}_{\text{out}}) = P^*\mathscr{N}$, where $\mathscr{N} = P(\mathscr{M} \cap \mathscr{R}(T, \mathbb{C}_{\text{out}}))$. For any $x \in \mathscr{N}$ the vector

$$TP^*x = \begin{pmatrix} (A + BR^{-1}B^*A^{-*}Q)x \\ -A^{-*}Qx \end{pmatrix}$$

is also in $\mathscr{M} \cap \mathscr{R}(T, \mathbb{C}_{\text{out}}) = P^*\mathscr{N}$. Hence $-A^{-*}Qx = 0$. Therefore $Qx = 0$ and $TP^*x = P^*Ax$. It follows that $Ax \in \mathscr{N}$. Thus $\mathscr{N}$ is A-invariant and contained in $\mathscr{V}_>$. ∎

If, moreover, the pair (Q, A) is observable, then $\mathscr{V}_> = \{0\}$ and it follows directly from the above theorem that there is only one nonnegative solution of (7.1), which is well known. In that case, this solution is the maximal solution.

The following lemma gives a direct connection between a nonnegative solution X of (7.1) and the A-invariant subspace $\mathscr{N}$ contained in $V_>$ that is associated with X according to the one-one correspondence given in Theorem 7.4.

LEMMA 7.5. *Assume (A, B) is controllable, A is invertible, $R > 0$, and $Q \geq 0$. If X is a nonnegative solution of (7.1) and $\mathscr{N}$ is the subspace associated with X according to the one-one correspondence given in Theorem 7.4, then*

$$\operatorname{Ker} X = \mathscr{V}_{\leq} \dot{+} \mathscr{N}.$$

Proof. Assume X is a nonnegative solution of (7.1), and let $\mathscr{M}$ be the T-invariant J_1-neutral J_2-nonnegative subspace of (7.6). Recall that $\mathscr{N} = P(\mathscr{M} \cap \mathscr{R}(T, \mathbb{C}_{\text{out}}))$. From Theorem 7.4 it follows directly that

$$(\operatorname{Ker} X) \cap \mathscr{V}_> = \mathscr{N}. \tag{7.7}$$

Assume $x \in \operatorname{Ker} X$. Then $P^*x \in \mathscr{M}$. Clearly P^*x is J_2-neutral. Also TP^*x is in the T-invariant subspace $\mathscr{M}$. Since T is J_2-contractive, we have

$$\langle J_2 TP^*x, TP^*x \rangle \leq \langle J_2 P^*x, P^*x \rangle = 0.$$

On the other hand, $\mathscr{M}$ is J_2-nonnegative, so $\langle J_2 TP^*x, TP^*x \rangle = 0$. Hence $P^*x \in \operatorname{Ker}(J_2 - T^*J_2T)$, and from (7.3) it follows that $Qx = 0$. Hence $TP^*x = P^*Ax$ from the definition of T. Evidently $P^*Ax \in \mathscr{M}$. From (7.6) it follows that $Ax \in \operatorname{Ker} X$. Hence $\operatorname{Ker} X$ is A-invariant and $\operatorname{Ker} X \subset \mathscr{V} = \mathscr{V}_{\leq} \dot{+} \mathscr{V}_>$. From (7.7) it follows that $\operatorname{Ker} X \subset \mathscr{V}_{\leq} \dot{+} \mathscr{N}$.

Conversely, assume $x \in \mathscr{V}_{\leq} \dot{+} \mathscr{N}$. If $x \in \mathscr{N}$, then $x \in \operatorname{Ker} X$ due to (7.7). Assume $x_1, \ldots, x_k$ is a Jordan chain of A contained in $\mathscr{V}_{\leq}$. Let $x_0 = 0$. We have $Ax_j = \lambda x_j + x_{j-1}$ for some $|\lambda| \leq 1$ and $j = 1, \ldots, k$. Assume $x_{j-1} \in \operatorname{Ker} X$ for some $j \in \{1, \ldots, k\}$. The Riccati equation (7.1) gives

$$\langle Xx_j, x_j \rangle = |\lambda|^2 \langle Xx_j, x_j \rangle + |\lambda|^2 \big\langle (R + B^*XB)^{-1} B^*Xx_j, B^*Xx_j \big\rangle.$$

Here X is nonnegative and $(R + B^*XB)^{-1}$ is positive, since R is positive. If $|\lambda| < 1$, it follows directly that $Xx_j = 0$. If $|\lambda| = 1$, then we only have $B^*Xx_j = 0$. The latter, combined with (7.1), gives $Xx_j = \lambda A^* X_{xj}$. Using the controllability of (A, B), we conclude that $Xx_j = 0$. By induction it follows that $\mathscr{V}_{\leq} \subset \operatorname{Ker} X$. ■

The partial ordering of the set of nonnegative solutions is similar to the partial ordering of A-invariant subspaces contained in $\mathscr{V}_{>}$, just as it is the case in Theorem 5.1.

THEOREM 7.6. *Assume (A, B) is controllable, A is invertible, $R > 0$, and $Q \geq 0$. Let X_1 and X_2 be two nonnegative solutions of (7.1). Let $\mathscr{N}_1$ and $\mathscr{N}_2$ be the A-invariant subspaces associated with X_1 and X_2, respectively, according to the one-one correspondence of Theorem 7.4. Then $X_1 \geq X_2$ if and only if $\mathscr{N}_1 \subset \mathscr{N}_2$.*

Proof. Let

$$\mathscr{M}_1 := \operatorname{Im}\begin{pmatrix} I \\ X_1 \end{pmatrix}, \qquad \mathscr{M}_2 := \operatorname{Im}\begin{pmatrix} I \\ X_2 \end{pmatrix}.$$

Recall that $\mathscr{N}_1 = P(\mathscr{M}_1 \cap \mathscr{R}(T, \mathbb{C}_{\text{out}}))$ and $\mathscr{N}_2 = P(\mathscr{M}_2 \cap \mathscr{R}(T, \mathbb{C}_{\text{out}}))$. Assume $X_1 \geq X_2$. For any $x \in \mathscr{N}_1$ the vector $P^*x \in \mathscr{R}(T, \mathbb{C}_{\text{out}})$ and $x \in \operatorname{Ker} X_1$ according to Lemma 7.5. From $X_2 \leq X_1$ and X_2 being nonnegative it follows that $\langle X_2 x, x \rangle = 0$ and $x \in \operatorname{Ker} X_2$. Hence $P^*x \in \mathscr{M}_2 \cap \mathscr{R}(T, \mathbb{C}_{\text{out}})$ and $x \in \mathscr{N}_2$.

Conversely, assume $\mathscr{N}_1 \subset \mathscr{N}_2$. For some arbitrary $x \in \mathbb{C}^n$ consider the vector

$$\begin{pmatrix} x \\ X_2 x \end{pmatrix} \in \mathscr{M}_2 = [\mathscr{M}_2 \cap \mathscr{R}(T, \mathbb{C}_{\text{out}})] \dotplus \mathscr{N}_0 \dotplus [\mathscr{M}_2 \cap \mathscr{R}(T, \mathbb{C}_{\text{in}})].$$

Let $x_+ \in P(\mathscr{M}_2 \cap \mathscr{R}(T, \mathbb{C}_{\text{in}}))$ be the vector for which

$$\begin{pmatrix} x - x_+ \\ X_2(x - x_+) \end{pmatrix} \in [\mathscr{M}_2 \cap \mathscr{R}(T, \mathbb{C}_{\text{out}})] \dotplus \mathscr{N}_0.$$

From Theorem 7.4 we know that all vectors of $\mathscr{M}_2 \cap \mathscr{R}(T, \mathbb{C}_{\text{out}})$ have the form P^*y for some $y \in \operatorname{Ker} X_2$. Exactly analogously, the same can be proved for vectors of $\mathscr{N}_0$; see the second part of the proof of Theorem 7.4.

Hence $P^*(x - x_+) \in [\mathscr{M}_2 \cap \mathscr{R}(T, \mathbb{C}_{\text{out}})] \dot{+} \mathscr{N}_0$ and $x - x_+ \in \mathscr{N}_2 \dot{+} P\mathscr{N}_0$, and $X_2(x - x_+) = 0$. Recall that $\mathscr{M}_2 \cap \mathscr{R}(T, \mathbb{C}_{\text{in}}) = (J_1 P^* \mathscr{N}_2)^\perp \cap \mathscr{R}(T, \mathbb{C}_{\text{in}})$ and $\mathscr{M}_1 \cap \mathscr{R}(T, \mathbb{C}_{\text{in}}) = (J_1 P^* \mathscr{N}_1)^\perp \cap \mathscr{R}(T, \mathbb{C}_{\text{in}})$. From the assumption $\mathscr{N}_1 \subset \mathscr{N}_2$ it follows that

$$\begin{pmatrix} x_+ \\ X_2 x_+ \end{pmatrix} \in \mathscr{M}_2 \cap \mathscr{R}(T, C_{\text{in}}) \subset \mathscr{M}_1 \cap \mathscr{R}(T, \mathbb{C}_{\text{in}}).$$

Hence $X_2 x_+ = X_1 x_+$ and $X_1 x_+ \in \operatorname{Im} X_2 = (\operatorname{Ker} X_2)^\perp$. It follows that

$$\begin{aligned} \langle (X_1 - X_2) x, x \rangle &= \langle (X_1 - X_2)(x - x_+), x \rangle \\ &= \langle X_1(x - x_+), x \rangle \\ &= \langle X_1(x - x_+), (x - x_+) \rangle \geq 0. \end{aligned}$$

Hence $X_1 \geq X_2$. ∎

The isolated nonnegative solutions can be described as in Theorem 5.2.

THEOREM 7.7. *Assume (A, B) is controllable, A is invertible, $R > 0$ and $Q \geq 0$. Let X be a nonnegative Hermitian solution of* (7.1). *Let*

$$\mathscr{M} = \operatorname{Im} \begin{pmatrix} I \\ X \end{pmatrix},$$

and let $\mathscr{N} = P(\mathscr{M} \cap \mathscr{R}(T, \mathbb{C}_{\text{out}}))$. Then the following are equivalent:

(i) *X is an isolated nonnegative Hermitian solution of* (7.1);

(ii) *$\mathscr{M}$ is isolated within the set of T-invariant subspaces that are maximal J_1-neutral and J_2-nonnegative*;

(iii) *$\mathscr{M} \cap \mathscr{R}(T, C_{\text{out}})$ is isolated within the set of subspaces of $\mathscr{R}(T, \mathbb{C}_{\text{out}})$ that are T-invariant and J_2-neutral*;

(iv) *$\mathscr{N}$ is isolated within the set of A-invariant subspaces contained in $\mathscr{V}_>$*;

(v) *for every eigenvalue λ of $A|_{\mathscr{V}_>}$ with* $\dim \operatorname{Ker}(A|_{\mathscr{V}_>} - \lambda) > 1$, *either $\mathscr{N} \cap \mathscr{R}(A, \lambda) = \{0\}$ or $\mathscr{R}(A, \lambda) \cap \mathscr{V}_> \subset \mathscr{N}$*;

(vi) *for every eigenvalue λ of $A|_{\mathscr{V}_>}$ with* $\dim \operatorname{Ker}(A|_{\mathscr{V}_>} - \lambda) > 1$, *either* $\operatorname{Ker} X \cap \mathscr{R}(A, \lambda) = \{0\}$ *or* $\mathscr{R}(A, \lambda) \cap \mathscr{V}_> \subset \operatorname{Ker} X$.

Proof. The proof is exactly analogous to the proof of Theorem 5.2 ∎

A nonnegative Hermitian solution X_0 of (7.1) is called *stably nonnegative* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|A - A_1\| + \|B - B_1\| + \|Q - Q_1\| + \|R - R_1\| < \delta$ and $R^*_1 = R_1$, $Q_1 \geq 0$ imply that the algebraic Riccati equation

$$X = A_1^* X A_1 + Q_1 - A_1^* X B_1 (R_1 + B_1^* X B_1)^{-1} B_1^* X A_1$$

has a nonnegative Hermitian solution X_1 such that $\|X_0 - X_1\| < \varepsilon$.

THEOREM 7.8. *Assume* (A, B) *is controllable,* A *is invertible,* $Q \geq 0$, *and* $R > 0$. *Then the only stably nonnegative solution of* (7.1) *is the maximal solution.*

Proof. To prove that the maximal solution is a stably nonnegative solution, proceed analogously to the proof of Theorem 5.4. To show that the other nonnegative solutions are not stably nonnegative solutions, use the perturbation $Q_\varepsilon = Q + \varepsilon I$. Then $Q_\varepsilon > 0$ and the pair (Q_ε, A) is observable. The perturbed equation has only one nonnegative solution that converges to the maximal solution of (7.1). ∎

8. INERTIA OF HERMITIAN SOLUTIONS FOR THE DISCRETE ALGEBRAIC RICCATI EQUATION

Finally, we study the inertia of a general Hermitian solution to the discrete algebraic Riccati equation (7.1). In Proposition 7.1 the set of Hermitian solutions is characterized by the set of n-dimensional T-invariant J_1-neutral subspaces. For studying the inertia we will use a slightly different characterization.

PROPOSITION 8.1. *Assume* (A, B) *is controllable,* A *is invertible,* $Q \geq 0$, *and* $R > 0$. *Every* T*-invariant* J_1*-neutral subspace* $\mathscr{M}_>$ *contained in* $\mathscr{R}(T, \mathbb{C}_{\mathrm{out}})$ *is of the form*

$$\mathscr{M}_> = \operatorname{Im}\begin{pmatrix} I \\ X \end{pmatrix} \cap \mathscr{R}(T, \mathbb{C}_{\mathrm{out}}) \tag{8.1}$$

for some Hermitian solution X of (7.1). *Conversely, if X is a Hermitian solution of* (7.1), *then the subspace $\mathcal{M}_>$ constructed as in* (8.1) *is T-invariant and J_1-neutral.*

Proof. Starting with a T-invariant J_1-neutral subspace $M_>$ contained in $\mathcal{R}(T, \mathbb{C}_{\text{out}})$, construct

$$\mathcal{M} = \mathcal{M}_> \dot{+} \mathcal{N}_0 \dot{+} \left[(J_1 \mathcal{M}_<)^{\perp} \cap \mathcal{R}(T, \mathbb{C}_{\text{in}})\right],$$

where $\mathcal{N}_0$ is the span of the first halves of all the Jordan chains of T corresponding to unimodular eigenvalues. Then apply Proposition 7.1.

Conversely, starting with a Hermitian solution X, apply Proposition 7.1 to get the T-invariant J_1-neutral subspace $\mathcal{M}$, and let $\mathcal{M}_> = \mathcal{M} \cap \mathcal{R}(T, \mathbb{C}_{\text{out}})$. ∎

Let X be some Hermitian solution to the equation (7.1). Let $\mathcal{M}_>$ be the T-invariant J_1-neutral subspace that corresponds to X according to the above proposition. Let $\mathcal{N}_0$ be the span of the first halves of all the Jordan chains of T corresponding to unimodular eigenvalues. Let $\mathcal{M}_< = (J_1 \mathcal{M}_>)^{\perp} \cap \mathcal{R}(T, \mathbb{C}_{\text{in}})$. Then $M = M_> \dot{+} \mathcal{N}_0 \dot{+} \mathcal{M}_<$ is n-dimensional, T-invariant, and J_1-neutral, and according to Proposition 7.1

$$\mathcal{M} = Im\begin{pmatrix} I \\ X \end{pmatrix}.$$

From Proposition 7.3, recall that $\mathcal{M}_>$ is J_2-nonpositive, $\mathcal{N}_0$ is J_2-neutral, and $\mathcal{M}_<$ is J_2-nonnegative. The J_2-neutral parts $\mathcal{M}_>^0 = (J_2 \mathcal{M}_>)^{\perp} \cap \mathcal{M}_>$ and $\mathcal{M}_<^0 = (J_2 \mathcal{M}_<)^{\perp} \cap \mathcal{M}_<$ of $\mathcal{M}_>$ and $\mathcal{M}_<$ will be considered.

PROPOSITION 8.2 (From [8, Lemma 2.1]). *The subspaces $\mathcal{M}_>^0$ and $\mathcal{M}_<^0$ are T-invariant.*

Proof. From $\mathcal{M}_> \subset \mathcal{R}(T, \mathbb{C}_{\text{out}})$ it follows that $T|_{\mathcal{M}_>}$ is invertible. For any $x \in \mathcal{M}_>^0$ we have

$$0 = \left\langle J_2 x, x \right\rangle \leq \left\langle J_2 (T|_{\mathcal{M}_>})^{-1} x, (T|_{\mathcal{M}_>})^{-1} x \right\rangle \leq 0.$$

Hence $(T|_{\mathcal{M}_>})^{-1} x$ is J_2-neutral. Since $\mathcal{M}_>$ is a J_2-nonpositive subspace, it follows that $J_2 (T|_{\mathcal{M}_>})^{-1} x \perp \mathcal{M}_>$. Hence $(T|_{\mathcal{M}_>})^{-1} \mathcal{M}_>^0 \subset \mathcal{M}_>^0$. The dimension of the left-hand side cannot be smaller than the dimension of the right-hand side. Therefore $(T|_{\mathcal{M}_>})^{-1} \mathcal{M}_>^0 = M_>^0$ and $\mathcal{M}_>^0 = TM_>^0$.

For $x \in \mathscr{M}_<^0$ we have

$$0 \le \langle J_2 Tx, Tx \rangle \le \langle J_2 x, x \rangle = 0.$$

Hence Tx is J_2-neutral. Since $\mathscr{M}_<$ is J_2-nonnegative, it follows that $J_2 Tx \perp \mathscr{M}_<$. Hence $Tx \in \mathscr{M}_<^0$. ∎

The three subspaces $\mathscr{M}_>^0$, $\mathscr{N}_0$, and $\mathscr{M}_<^0$ are T-invariant, J_1-neutral, and J_2-neutral. One of the aims of this section is to show that the sum of their dimensions equals the dimension of the kernel of X.

LEMMA 8.5. *Assume (A, B) is controllable, A is invertible, $Q \ge 0$, and $R > 0$. Then for any T-invariant, J_1-neutral, J_2-neutral subspace $\mathscr{N}$ the projection on the second coordinate $(0\ \ I)\mathscr{N} = 0$, and moreover, $(I\ \ 0)\mathscr{N} \subset \mathscr{V}$.*

Proof. For any

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathscr{N}$$

it follows from (7.3) that $Qx = 0$ and $B^* A^{*-1} y = 0$. Hence

$$T^k \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^k x \\ (A^{*-1})^k y \end{pmatrix}$$

for $x = 1, 2, \ldots$. Hence $x \in \mathscr{V}$ and $B^*(A^*)^k (A^{*-1})^n y = 0$ for $k = 0, 1, \ldots, n-1$. The controllability of (A, B) gives that $(A^{*-1})^n y = 0$. Hence $y = 0$. ∎

From the above lemma if follows that $P\mathscr{M}_<^0$, $P\mathscr{N}_0$, and $P\mathscr{M}_>^0$ are contained in Ker X. On the other hand it follows that $M_<^0 \subset P^*\mathscr{V}_<$, $\mathscr{N}_0 \subset P^*\mathscr{V}_0$, and $\mathscr{M}_>^0 \subset P^*\mathscr{V}_>$. Combining, we have $\mathscr{M}_<^0 \subset P^*\mathscr{V}_< \cap \mathscr{M}$, $\mathscr{N}_0 \subset P^*\mathscr{V}_0 \cap \mathscr{M}$, and $\mathscr{M}_>^0 \subset P^*\mathscr{V}_> \cap \mathscr{M}$. The subspace $P^*\mathscr{V}$ is T-invariant and J_2-neutral, and therefore also the inverse inclusions hold, so that we have

$$\begin{aligned} \mathscr{M}_<^0 &= P^*\mathscr{V}_< \cap \mathscr{M}, \\ \mathscr{N}_0 &= P^*\mathscr{V}_0 \cap \mathscr{M}, \qquad\qquad (8.2) \\ \mathscr{M}_>^0 &= P^*\mathscr{V}_> \cap \mathscr{M}. \end{aligned}$$

For a Hermitian solution X of (7.1) let us denote by $\pi(\nu)$ the number of positive (negative) eigenvalues of X, multiplicities taken into account, and let $\delta = \dim \operatorname{Ker} X$.

THEOREM 8.4. *Assume (A, B) is controllable, A is invertible, $Q \geq 0$, and $R > 0$. Let X be a Hermitian solution to* (7.1), *and let $\mathscr{M}$ be as in* (7.4). *Then*

$$\pi = \dim[\mathscr{M} \cap \mathscr{R}(T, \mathbb{C}_{\text{in}})] - \dim(\mathscr{M} \cap P^*\mathscr{V}_<),$$

$$\nu = \dim[\mathscr{M} \cap \mathscr{R}(T, \mathbb{C}_{\text{out}})] - \dim(\mathscr{M} \cap P^*\mathscr{V}_>),$$

$$\delta = \dim(\mathscr{M} \cap P^*\mathscr{V}).$$

Proof. The vectors in the subspace $\mathscr{M}_< \ominus \mathscr{M}_<^0$ are strictly J_2-positive, and the vectors in the subspace $\mathscr{M}_> \ominus \mathscr{M}_<^0$ are strictly J_2-negative. From (1.6) it follows that the subspaces $P(\mathscr{M}_< \ominus \mathscr{M}_<^0)$ and $P(\mathscr{M}_> \ominus \mathscr{M}_>^0)$ are strictly X-positive and X-negative, respectively. Hence $\pi \geq \dim(\mathscr{M}_< \ominus \mathscr{M}_<^0)$ and $\nu \geq \dim(\mathscr{M}_> \ominus \mathscr{M}_>^0)$. From the definition of $\mathscr{M}$ it follows that the subspace $P(P^*\mathscr{V} \cap \mathscr{M})$ is contained in Ker X. Hence $\delta \geq \dim(P^*\mathscr{V} \cap \mathscr{M})$. From (8.2) it follows that the three lower bounds add up to n, which is the number of eigenvalues of X. Hence the lower bounds give the exact numbers of negative and positive eigenvalues of X and of dim Ker X. ■

COROLLARY 8.5. *Assume (A, B) is controllable, A is invertible, $Q \geq 0$, and $R > 0$. For any Hermitian solution X of* (7.1) *the inertia π, ν, δ satisfies $\pi \leq \dim \mathscr{R}(T, \mathbb{C}_{\text{in}})$, $\nu \leq \dim \mathscr{R}(T, \mathbb{C}_{\text{out}})$, and $\delta \geq \frac{1}{2} \dim \mathscr{R}(T, \mathbb{T})$.*

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