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Essential spectra of some matrix operators and application to two-group transport operators with general boundary conditions

Nedra Moalla [a,*], Mondher Damak [b], Aref Jeribi [b]

[a] *Département de Mathématiques, Institut Préparatoire aux Etudes d'Ingénieur de Sfax, Route Menzel Chaker, Km 0.5, BP 805, 3018, Sfax, Tunisie*

[b] *Département de Mathématiques, Faculté des Sciences de Sfax, Route de Soukra, Km 3.5, BP 802, 3018, Sfax, Tunisie*

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Abstract

In this article we investigate the essential spectra of a 2×2 block operator matrix on a Banach space. Furthermore, we apply the obtained results to determine the essential spectra of two-group transport operators with general boundary conditions in the Banach space $L_p([-a,a] \times [-1,1]) \times L_p([-a,a] \times [-1,1])$, $a > 0$.

Keywords: Matrix operators; Essential spectra; Fredholm perturbation; Transport equation

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1. Introduction

In this article we are concerned with the essential spectra of operators defined by a 2×2 block operator matrix

$$L_0 := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

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[*] Corresponding author.
E-mail addresses: nedra.moalla@ipeis.rnu.tn (N. Moalla), aref.jeribi@fss.rnu.tn (A. Jeribi).

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that act on the product of Banach spaces $X \times Y$. In general, the operators occurring in L_0 are unbounded. The operator A acts on the Banach space X and has the domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space Y and the intertwining operator B (respectively C) is defined on the domain $\mathcal{D}(B)$ (respectively $\mathcal{D}(C)$) and acts between these spaces. Note that, in general, L_0 is neither a closed nor a closable operator, even if its entries are closed. It's showed that under some conditions L_0 is closable (see, [1]). We shall denote L its closure.

Many problems in mathematical physics can be described by systems of mixed order linear differential equations. Important physical information is given by the localization of the essential spectra. The study of the problem of the essential spectrum of these operators was done by different authors [27,33–35]. The most general results for Douglis–Nirenberg elliptic systems were obtained by G. Grubb and G. Geymonat [13]. A successful approach has recently been developed by F.V. Atkinson, H. Langer, R. Mennicken and A.A. Shkalikov in [1,45]. M. Damak and A. Jeribi [3] have, recently, extended these results to a large class of operators. But the theoretical results of the authors cited above cannot solve some physical problems, in particular the essential spectra of two-group transport operators in L_1-spaces.

To describe the essential spectra of a class of linear two-group transport operators, with abstract boundary conditions, in the Banach space $X_p \times X_p$, $1 \leqslant p < \infty$, where

$$X_p := L_p\big([-a,a] \times [-1,1]\big), \quad a > 0,$$

we will consider the operator

$$A_H = T_H + K$$

where T_H and K are defined by

$$T_H\psi = \begin{pmatrix} -v\frac{\partial\psi_1}{\partial x} - \sigma_1(v)\psi_1 & 0 \\ 0 & -v\frac{\partial\psi_2}{\partial x} - \sigma_2(v)\psi_2 \end{pmatrix} := \begin{pmatrix} T_{H_1} & 0 \\ 0 & T_{H_2} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

and

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

where K_{ij}, $1 \leqslant i, j \leqslant 2$, are bounded linear operators defined on X_p by

$$\begin{cases} K_{ij} : X_p \to X_p, \\ \psi_j \mapsto K_{ij}\psi_j(x,v) = \int_{-1}^{1} \kappa_{ij}(x,v,v')\psi_j(x,v')\,dv'. \end{cases}$$

Each operator T_{H_j}, $j = 1, 2$, is defined by

$$\begin{cases} T_{H_j} : \mathcal{D}(T_{H_j}) \subset X_p \to X_p, \\ \psi_j \mapsto (T_{H_j}\psi_j)(x,v) = -v\frac{\partial\psi_j}{\partial x}(x,v) - \sigma_j(v)\psi_j(x,v), \\ \mathcal{D}(T_{H_j}) = \{\psi_j \in X_p \text{ such that } v\frac{\partial\psi_j}{\partial x} \in X_p \text{ and } \psi_j^i = H_j\psi_j^o\}. \end{cases}$$

The function $\psi_j(x,v)$ represents the number density of gas particles having the position x and the direction cosine of propagation v. The variable v may be thought of as the cosine of the angle between the velocity of particles and the x-direction. The function $\sigma_j(.)$ is a measurable function called the collision frequency. The boundary conditions are modelled by

$$\psi_j^i = H_j\psi_j^o, \quad j = 1, 2,$$

see Section 4 for more details.

For a self-adjoint operator in a Hilbert space, there seems to be only one reasonable way to define the essential spectrum: the set of all points of the spectrum that are not isolated eigenvalues of finite algebraic multiplicity (see, for example, [39,49,50]). When dealing with non self-adjoint closed, densely defined linear operator, T, on a Banach space X, various notions of essential spectrum appear in application of spectral theory (see, for instance, [10,14,15,26,40,41,49]) and the references therein. Motivated by the description of the essential spectra of transport operators, A. Jeribi has, recently, discussed in [16–23] the essential spectra of closed densely defined linear operators under additive perturbations.

Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X,Y)$ (respectively $\mathcal{C}(X,Y)$) the set of all bounded (respectively closed, densely defined) linear operators from X into Y and we denote by $\mathcal{K}(X,Y)$ the subspace of compact operators from X into Y. For $T \in \mathcal{C}(X,Y)$, we write $\mathcal{D}(T) \subset X$ for the domain, $N(T) \subset X$ for the null space and $R(T) \subset Y$ for the range of T. The nullity, $\alpha(T)$, of T is defined as the dimension of $N(T)$ and the deficiency, $\beta(T)$, of T is defined as the codimension of $R(T)$ in Y. Let $\sigma(T)$ (respectively $\rho(T)$) denote the spectrum (respectively the resolvent set) of T. The set of upper semi-Fredholm operators is defined by

$$\Phi_+(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } Y\}$$

and the set of lower semi-Fredholm operators is defined by

$$\Phi_-(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \beta(T) < \infty\}.$$

$\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$ denote the set of Fredholm operators from X into Y and $\Phi_\pm(X,Y) := \Phi_+(X,Y) \cup \Phi_-(X,Y)$ the set of semi-Fredholm operators from X into Y. While the number $i(T) := \alpha(T) - \beta(T)$ is called the index of T, for $T \in \Phi(X,Y)$. A complex number λ is in Φ_{+T}, Φ_{-T}, $\Phi_{\pm T}$ or Φ_T if $\lambda - T$ is in $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_\pm(X,Y)$ or $\Phi(X,Y)$, respectively. If $X = Y$ then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ and $\Phi_\pm(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi_\pm(X)$, respectively.

In this paper we are concerned with the following essential spectra:

$$\begin{aligned}
\sigma_{e1}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_+(X)\} := \mathbb{C} \setminus \Phi_{+T},\\
\sigma_{e2}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_-(X)\} := \mathbb{C} \setminus \Phi_{-T},\\
\sigma_{e3}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_\pm(X)\} := \mathbb{C} \setminus \Phi_{\pm T},\\
\sigma_{e4}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi(X)\} := \mathbb{C} \setminus \Phi_T,\\
\sigma_{e5}(T) &:= \mathbb{C} \setminus \rho_5(T),\\
\sigma_{e6}(T) &:= \mathbb{C} \setminus \rho_6(T),
\end{aligned}$$

where $\rho_5(T) := \{\lambda \in \Phi_T \text{ such that } i(\lambda - T) = 0\}$ and $\rho_6(T)$ denotes the set of those $\lambda \in \rho_5(T)$ such that all scalars near λ are in $\rho(T)$. They can be ordered as

$$\sigma_{e3}(T) := \sigma_{e1}(T) \cap \sigma_{e2}(T) \subseteq \sigma_{e4}(T) \subseteq \sigma_{e5}(T) \subseteq \sigma_{e6}(T).$$

The subsets $\sigma_{e1}(.)$ and $\sigma_{e2}(.)$ are the Gustafson and Weidmann essential spectra [15]. $\sigma_{e3}(.)$ is the Kato essential spectrum [26]. $\sigma_{e4}(.)$ is the Wolf essential spectrum [15,50]. $\sigma_{e5}(.)$ is the Schechter essential spectrum [40,43] and $\sigma_{e6}(.)$ denotes the Browder essential spectrum [15,24,37]. Note that all these sets are closed and if X is a Hilbert space and T is a self-adjoint operator on X, then all these sets coincide.

To study the Wolf essential spectrum of the operator matrix L in Banach spaces, the authors in [1,45] used the compactness condition for the operator $(\lambda - A)^{-1}$ (respectively

$C(\lambda - A)^{-1}$ and $((\lambda - A)^{-1}B)^*$). They showed that, under certain additional assumptions, $\sigma_{e4}(L) = \sigma_{e4}(\overline{D - C(\lambda_0 - A)^{-1}B})$ (respectively $\sigma_{e4}(L) = \sigma_{e4}(A) \cup \sigma_{e4}(\overline{D - C(\lambda_0 - A)^{-1}B})$), where L (respectively $\overline{D - C(\lambda_0 - A)^{-1}B}$) is the closure of L_0 (respectively $D - C(\lambda_0 - A)^{-1}B$) and λ_0 is any number in the resolvent set of A. In [3] the authors determine the essential spectra of L by assuming that $(\lambda - A)^{-1} \in \mathcal{I}(X)$ where $\mathcal{I}(X)$ is a nonzero two-sided ideal of $\mathcal{L}(X)$ contained in the set of Fredholm perturbations. But the above assumptions are not always satisfactory in the classical transport theory. In fact in L_1-spaces the operator $C(\lambda - A)^{-1} := K_{21}(\lambda - T_{H_1} - K_{11})^{-1}$ is weakly compact (see Lemma 4.3).

The aim of this paper is to extend the obtained results into a large class of operators and to investigate the six essential spectra of a matrix operators. More precisely, let $\mathcal{I}(X)$ be an arbitrary nonzero two-sided ideal of $\mathcal{L}(X)$ contained in $\mathcal{F}(X)$, where $\mathcal{F}(X)$ denotes the set of Fredholm perturbations. If for some $\mu \in \rho(A)$ the operator $C(A - \mu)^{-1}$ is in $\mathcal{I}(X)$ and $M(\mu) \in \mathcal{F}(X \times X)$, then

$$\sigma_{e4}(L) = \sigma_{e4}(A) \cup \sigma_{e4}\big(S(\mu)\big)$$

and

$$\sigma_{e5}(L) \subseteq \sigma_{e5}(A) \cup \sigma_{e5}\big(S(\mu)\big),$$

where $S(\mu)$ is the closure of $D - C(\mu - A)^{-1}B$ and $M(\mu)$ is the operator defined by

$$M(\mu) := \begin{pmatrix} 0 & \overline{(\mu - A)^{-1}B} \\ C(\mu - A)^{-1} & C(\mu - A)^{-1}\overline{(\mu - A)^{-1}B} \end{pmatrix}.$$

If in addition, Φ_A is connected then $\sigma_{e5}(L) = \sigma_{e5}(A) \cup \sigma_{e5}(S(\mu))$. Moreover, if $\mathcal{I}(X)$ satisfies some additional (reasonable) conditions, we get

$$\sigma_{ei}(L) = \sigma_{ei}(A) \cup \sigma_{ei}\big(S(\mu)\big), \quad i = 1, 2,$$

and

$$\sigma_{e3}(L) = \sigma_{e3}(A) \cup \sigma_{e3}\big(S(\mu)\big) \cup \big[\sigma_{e2}(A) \cap \sigma_{e1}\big(S(\mu)\big] \cup \big[\sigma_{e1}(A) \cap \sigma_{e2}\big(S(\mu)\big)\big]$$

(see Theorem 3.2). Our results extend and improve many known ones in the literature. In particular, the results obtained in [1,3,45] are now special cases of the ones obtained here.

Our paper is organized as follows. In the next section we recall some definitions and preliminary results. In Section 3 we investigate the essential spectra of L. The main result of this section is Theorem 3.2. Finally, in Section 4 we apply the results obtained in Section 3 to investigate the essential spectra of a two-group transport operator with general boundary conditions on L_p-spaces, $1 \leqslant p < \infty$.

2. Notations and preliminary results

In this section we recall some definitions and we give some lemmas that we will need in the sequel.

In the next proposition we will recall some well-known properties of the Fredholm-sets (see, for example, [8,43]).

Proposition 2.1.

- (i) *Φ_{+T}, Φ_{-T} and Φ_T are open.*
- (ii) *$i(\lambda - T)$ is constant on any component of Φ_T.*
- (iii) *$\alpha(\lambda - T)$ and $\beta(\lambda - T)$ are constant on any component of Φ_T except on a discrete set of points at which they have larger values.*

Definition 2.1. Let X and Y be two Banach spaces. An operator $A \in \mathcal{L}(X, Y)$ is said to be weakly compact if $A(B)$ is relatively weakly compact in Y for every bounded subset $B \subset X$.

The family of weakly compact operators from X to Y is denoted by $\mathcal{W}(X, Y)$. If $X = Y$ the family of weakly compact operators on X, $\mathcal{W}(X) := \mathcal{W}(X, X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (cf. [7,9]).

Definition 2.2. Let X be a Banach space. An operator $S \in \mathcal{L}(X)$ is called strictly singular if, for every infinite-dimensional subspace M of X, the restriction of S to M is not a homeomorphism.

Let $\mathcal{S}(X)$ denote the set of strictly singular operators on X.

The concept of strictly singular operators was introduced in the pioneering paper by Kato [25] as a generalization of the notion of compact operators. For a detailed study of the properties of strictly singular operators we refer to [9,25]. Note that $\mathcal{S}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$. If X is a Hilbert space then $\mathcal{S}(X)= \mathcal{K}(X)$. The class of weakly compact operators in L_1-spaces (respectively $\mathcal{C}(\Omega)$-spaces with Ω a compact Haussdorff space) is nothing else than the family of strictly singular operators on L_1-spaces (respectively $\mathcal{C}(\Omega)$-spaces) (see [38, Theorem 1]).

Let X be a Banach space. If N is a closed subspace of X, we denote by π_N the quotient map $X \to X/N$. The codimension of N, $\operatorname{codim}(N)$, is defined to be the dimension of the vector space X/N.

Definition 2.3. Let X be a Banach space. An operator $S \in \mathcal{L}(X)$ is said to be strictly cosingular if there exists no closed subspace N of X with $\operatorname{codim}(N) = \infty$ such that $\pi_N S : X \to X/N$ is surjective.

Let $\mathcal{CS}(X)$ denote the set of strictly cosingular operators on X. This class of operators was introduced by Pelczynski [38], it forms a closed two-sided ideal of $\mathcal{L}(X)$ (cf. [46]).

Definition 2.4. A Banach space X is said to have the Dunford–Pettis property (for short property DP) if for each Banach space Y every weakly compact operator $T : X \to Y$ takes weakly compact sets in X into norm compact sets of Y.

It is well known that any L_1-space has the property DP [6]. Also, if Ω is a compact Hausdorff space, $C(\Omega)$ has the property DP [12]. For further examples we refer to [5] or [7, pp. 494, 497, 508 and 511]. Note that the property DP is not preserved under conjugation. However, if X is a Banach space whose dual has the property DP, then X has the property DP (see, [12]). For more information we refer to the paper by J. Diestel [5] which contains a survey and exposition of the Dunford–Pettis property and related topics.

Definition 2.5. Let X be a Banach space and $R \in \mathcal{L}(X)$. R is said to be a Riesz operator if $\Phi_R = \mathbb{C} \backslash \{0\}$.

For further information on the family of Riesz operators we refer to [2,24] and the references therein.

Definition 2.6. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X, Y)$.

(i) The operator F is called Fredholm perturbation if $U + F \in \Phi(X, Y)$ whenever $U \in \Phi(X, Y)$.
(ii) F is called an upper (respectively lower) semi-Fredholm perturbation if $U + F \in \Phi_+(X, Y)$ (respectively $U + F \in \Phi_-(X, Y)$) whenever $U \in \Phi_+(X, Y)$ (respectively $U \in \Phi_-(X, Y)$).

We denote by $\mathcal{F}(X, Y)$ the set of Fredholm perturbations and by $\mathcal{F}_+(X, Y)$ (respectively $\mathcal{F}_-(X, Y)$) the set of upper semi-Fredholm (respectively lower semi-Fredholm) perturbations.

Remark 2.1. Let $\Phi^b(X, Y)$ denote the set $\Phi(X, Y) \cap \mathcal{L}(X, Y)$. If in Definition 2.6 we replace $\Phi(X, Y)$ by $\Phi^b(X, Y)$, we obtain the sets $\mathcal{F}^b(X, Y)$, $\mathcal{F}^b_+(X, Y)$ and $\mathcal{F}^b_-(X, Y)$.

The set of Fredholm perturbations, $\mathcal{F}^b(X, Y)$, was introduced and investigated in [8]. In particular, it is shown that $\mathcal{F}^b(X, Y)$ is a closed subset of $\mathcal{L}(X, Y)$ and if $X = Y$, then $\mathcal{F}^b(X) := \mathcal{F}^b(X, X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.

Remark 2.2. In [42], it is proved that $\mathcal{F}^b(X)$ is the largest ideal of $\mathcal{L}(X)$ contained in the family of Riesz operators.

We recall the following result established in [28].

Lemma 2.1. [28, Lemma 2.3] *Let X be a Banach space. Then*

$$\mathcal{F}^b(X) = \mathcal{F}(X),$$

where $\mathcal{F}(X) := \mathcal{F}(X, X)$.

An immediate consequence of Lemma 2.1 is that $\mathcal{F}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$. We can deduce from Lemma 2.2 in [28] and Theorem 3.1 in [9] the following inclusions:

$$\mathcal{K}(X) \subset \mathcal{S}(X) \subset \mathcal{F}_+(X) \subset \mathcal{F}(X),$$
$$\mathcal{K}(X) \subset \mathcal{CS}(X) \subset \mathcal{F}_-(X) \subset \mathcal{F}(X),$$

where $\mathcal{F}_-(X) := \mathcal{F}_-(X, X)$ and $\mathcal{F}_+(X) := \mathcal{F}_+(X, X)$.

Remark 2.3. It is proved in [32, Section 3] that if X is a Banach space with the property DP, then

$$\mathcal{W}(X) \subset \mathcal{F}_+(X) \cap \mathcal{F}_-(X).$$

We say that X is weakly compactly generating (w.c.g.) if the linear span of some weakly compact subset is dense in X. For more details and results see [5]. In particular, all separable and all reflexive Banach spaces are w.c.g. as well as $L_1(\Omega, d\mu)$ if (Ω, μ) is σ-finite. It is proved in [47] that if X is a w.c.g. Banach space then

$$\mathcal{F}_+(X) = \mathcal{S}(X) \quad \text{and} \quad \mathcal{F}_-(X) = \mathcal{CS}(X).$$

Remark 2.4. Let (Ω, Σ, μ) be a positive measure space and let X_p denote the spaces $L_p(\Omega, d\mu)$ with $1 \leqslant p < \infty$. Since the spaces X_p, $1 \leqslant p < \infty$, are w.c.g., then we can deduce from what precedes that

$$\mathcal{K}(X_p) \subset \mathcal{F}_+(X_p) \cap \mathcal{F}_-(X_p).$$

We say that X is subprojective, if given any closed infinite dimensional subspace M of X, there exists a closed infinite dimensional subspace N contained in M and a continuous projection from X onto N. Clearly any Hilbert space is subprojective. The spaces c_0, l_p $(1 \leqslant p < \infty)$ and L_p $(2 \leqslant p < \infty)$ are also subprojective [48].

We say that X is superprojective if every subspace V having infinite codimension in X is contained in a closed subspace W having infinite codimension in X as it exists a bounded projection from X to W. The spaces l_p $(1 < p < \infty)$ and L_p $(1 < p \leqslant 2)$ are superprojective [48].

Let X be a w.c.g. Banach space. It is proved in [44] that if X is superprojective (respectively subprojective), then $\mathcal{S}(X) \subset \mathcal{CS}(X)$ (respectively $\mathcal{CS}(X) \subset \mathcal{S}(X)$). Accordingly, we have the following result:

Proposition 2.2. *Let X be a w.c.g. Banach space, then*

(i) *If X is superprojective, then $\mathcal{S}(X) \subset \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$.*
(ii) *If X is subprojective, then $\mathcal{CS}(X) \subset \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$.*

3. Essential spectra of *L*

The purpose of this section is to discuss the essential spectra of the matrix operator L, closure of L_0, on the space $X \times X$, where X is a Banach space.

In the product space $X \times X$, we consider an operator which is formally defined by a matrix

$$L_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the operator A acts on X and has domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space X, and the intertwining operator B (respectively C) is defined on the domain $\mathcal{D}(B)$ (respectively $\mathcal{D}(C)$) and acts on X. In what follows, we will assume that the following conditions, introduced in [45], hold:

(H1) A is closed, densely defined linear operator on X with nonempty resolvent set $\rho(A)$.
(H2) The operator B is densely defined linear operator on X and for some (hence for all) $\mu \in \rho(A)$, the operator $(A - \mu)^{-1}B$ is closable. (In particular, if B is closable, then $(A - \mu)^{-1}B$ is closable.)
(H3) The operator C satisfies $\mathcal{D}(A) \subset \mathcal{D}(C)$, and for some (hence for all) $\mu \in \rho(A)$, the operator $C(A - \mu)^{-1}$ is bounded. (In particular, if C is closable, then $C(A - \mu)^{-1}$ is bounded.)
(H4) The lineal $\mathcal{D}(B) \cap \mathcal{D}(D)$ is dense in X, and for some (hence for all) $\mu \in \rho(A)$ the operator $D - C(A - \mu)^{-1}B$ is closable, we will denote by $S(\mu)$ its closure.

Remark 3.1.

- (i) It follows, from the closed graph theorem that the operator

  $$G(\mu) := \overline{(A-\mu)^{-1}B}$$

  is bounded on X.
- (ii) We emphasize that neither the domain of $S(\mu)$ nor the property of being closable depend on μ. Indeed, it follows from the Hilbert identity that

  $$S(\lambda) = S(\mu) + (\mu - \lambda)F(\lambda)G(\mu), \tag{3.1}$$

  where

  $$F(\lambda) := C(A-\lambda)^{-1}, \quad \lambda, \mu \in \rho(A).$$

  Since the operators $F(\lambda)$ and $G(\mu)$ are bounded, then the difference $S(\lambda) - S(\mu)$ is bounded. Therefore, neither the domain of $S(\mu)$ nor the property of being closable depend on μ.

We recall the following result which describes the closure of the operator L_0.

Theorem 3.1. [1] *Let conditions* (H1)–(H3) *be satisfied and the lineal* $M := \mathcal{D}(B) \cap \mathcal{D}(D)$ *be dense in* X. *Then the operator* L_0 *is closable if and only if the operator* $S(\mu)$, $\mu \in \rho(A)$, *is closable in* X. *Moreover, the closure* L *of* L_0 *is given by*

$$L = \mu - \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \mu - A & 0 \\ 0 & \mu - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} \tag{3.2}$$

or, spelled out

$$\begin{cases} L : \mathcal{D}(L) \subset X \times X \to X \times X, \\ \binom{x}{y} \to L\binom{x}{y} = \binom{A(x+G(\mu)y) - \mu G(\mu)y}{C(x+G(\mu)y) - S(\mu)y}, \\ \mathcal{D}(L) = \left\{\binom{x}{y} \in X \times X \text{ such that } x + G(\mu)y \in \mathcal{D}(A),\ y \in \mathcal{D}(S(\mu))\right\}. \end{cases}$$

Note that, in view of Remark 3.1(ii) the description of the operator L does not depend on the choice of the point $\mu \in \rho(A)$.

Unless otherwise stated in all that follows $\mathcal{I}(X)$ will denote an arbitrary nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying

(H5) $\mathcal{I}(X) \subseteq \mathcal{F}(X)$

and we will denote, for $\mu \in \rho(A)$, by $M(\mu)$ the operator

$$M(\mu) := \begin{pmatrix} 0 & G(\mu) \\ F(\mu) & F(\mu)G(\mu) \end{pmatrix}. \tag{3.3}$$

Remark 3.2. It should be observed that if $\mathcal{I}(X)$ is a nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying (H5), then

$$\mathcal{F}_0(X) \subseteq \mathcal{I}(X) \subseteq \mathcal{F}(X),$$

where $\mathcal{F}_0(X)$ stands for the ideal of finite rank operators. This follows from Lemma 2.1 and [8, Proposition 4, p. 70].

Lemma 3.1. *Let $\mathcal{I}(X)$ be any nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying* (H5). *If $F(\mu) \in \mathcal{I}(X)$ for some $\mu \in \rho(A)$, then $F(\mu) \in \mathcal{I}(X)$ for all $\mu \in \rho(A)$.*

Proof. Let $\mu_0 \in \rho(A)$, such that $F(\mu_0) \in \mathcal{I}(X)$. We have

$$F(\mu) = F(\mu_0)\big[I + (\mu - \mu_0)(\mu_0 - A)^{-1}\big]^{-1},$$

for all μ in $\rho(A)$. This implies, by the ideal propriety of $\mathcal{I}(X)$, that $F(\mu) \in \mathcal{I}(X)$. □

Lemma 3.2. *Let $\mathcal{I}(X)$ be a nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying* (H5). *If $F(\mu) \in \mathcal{I}(X)$ for some $\mu \in \rho(A)$, then*

(i) *$\sigma_{ei}(S(\mu))$, $i = 4, 5$ does not depend on μ.*
(ii) *If $\mathcal{I}(X) \subset \mathcal{F}_+(X)$, then $\sigma_{e1}(S(\mu))$ does not depend on μ.*
(iii) *If $\mathcal{I}(X) \subset \mathcal{F}_-(X)$ or $[\mathcal{I}(X)]^* \subset \mathcal{F}_+(X^*)$, then $\sigma_{e2}(S(\mu))$ does not depend on μ.*
(iv) *If $\mathcal{I}(X) \subset \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$, then $\sigma_{e3}(S(\mu))$ does not depend on μ.*

Proof. The proof of this lemma follows from Eq. (3.1) and [28, Theorem 3.1]. □

We are now in the position to express the main result of this section. In the following we will denote the complement of a subset $\Omega \subset \mathbb{C}$ by ${}^C\Omega$.

Theorem 3.2. *Let the matrix operator L_0 satisfy conditions* (H1)–(H4)*, and let $\mathcal{I}(X)$ be any nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying* (H5). *If for some $\mu \in \rho(A)$, the operator $F(\mu) \in \mathcal{I}(X)$, then*

(i) *If $M(\mu) \in \mathcal{F}(X \times X)$ for some $\mu \in \rho(A)$, then*

$$\sigma_{e4}(L) = \sigma_{e4}(A) \cup \sigma_{e4}\big(S(\mu)\big)$$

and

$$\sigma_{e5}(L) \subseteq \sigma_{e5}(A) \cup \sigma_{e5}\big(S(\mu)\big).$$

Moreover, if ${}^C\sigma_{e4}(A)$ is connected, then

$$\sigma_{e5}(L) = \sigma_{e5}(A) \cup \sigma_{e5}\big(S(\mu)\big).$$

If in addition, ${}^C\sigma_{e5}(L)$ is connected, $\rho(L) \neq \emptyset$, ${}^C\sigma_{e5}(S(\mu))$ is connected and $\rho(S(\mu)) \neq \emptyset$, then

$$\sigma_{e6}(L) = \sigma_{e6}(A) \cup \sigma_{e6}\big(S(\mu)\big).$$

(ii) *If $\mathcal{I}(X) \subseteq \mathcal{F}_+(X)$ and the operator $M(\mu) \in \mathcal{F}_+(X \times X)$ for some $\mu \in \rho(A)$, then*

$$\sigma_{e1}(L) = \sigma_{e1}(A) \cup \sigma_{e1}\big(S(\mu)\big).$$

(iii) *If $\mathcal{I}(X) \subseteq \mathcal{F}_-(X)$ and the operator $M(\mu) \in \mathcal{F}_-(X \times X)$, then*

$$\sigma_{e2}(L) = \sigma_{e2}(A) \cup \sigma_{e2}\big(S(\mu)\big).$$

(iv) *If $\mathcal{I}(X) \subseteq \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$ and the operator $M(\mu) \in \mathcal{F}_+(X \times X) \cap \mathcal{F}_-(X \times X)$ for some $\mu \in \rho(A)$, then*

$$\sigma_{e3}(L) = \sigma_{e3}(A) \cup \sigma_{e3}\big(S(\mu)\big) \cup \big[\sigma_{e2}(A) \cap \sigma_{e1}\big(S(\mu)\big)\big] \cup \big[\sigma_{e1}(A) \cap \sigma_{e2}\big(S(\mu)\big)\big].$$

Remark 3.3.

(a) If X is a w.c.g. Banach space and superprojective (respectively subprojective), then the ideal $\mathcal{I}(X) = \mathcal{S}(X)$ (respectively $\mathcal{I}(X) = \mathcal{CS}(X)$) satisfies the conditions of Theorem 3.2 (see Proposition 2.2). Also, if we take X a Banach space with the property DP and $\mathcal{I}(X) = \mathcal{W}(X)$ (see Remark 2.3) or if we consider the ideal $\mathcal{K}(X_p)$ in the L_p spaces, $1 \leqslant p \leqslant \infty$.

(b) The ideal of finite rank operators $\mathcal{F}_0(X)$ is the minimal subset of $\mathcal{L}(X)$ for which the conditions of Theorem 3.2 are valid regardless of the Banach spaces.

(c) It is noted that, in the paper [3] the authors suppose that the operator $(A - \mu)^{-1} \in \mathcal{I}(X)$ but in our case we suppose only that $C(A - \mu)^{-1} \in \mathcal{I}(X)$, which is a weaker condition, and we usually obtain the same result. So, Theorem 3.2 may be regarded as an extension of [3, Theorem 4.2] to a larger class of operators.

(d) In the papers [1,45], the authors studied only the Wolf essential spectrum. Theorem 3.2 is an extension of their results to different other essential spectra.

(e) If $F(\mu)$ and $G(\mu)$ are in $\mathcal{K}(X)$, for some $\mu \in \rho(A)$, then $M(\mu) \in \mathcal{K}(X \times X) \subset \mathcal{F}(X \times X)$.

(f) Let $X = L_1(\Omega, d\mu)$ where (Ω, Σ, μ) is a positive measure space. If $F(\mu)$ and $G(\mu)$ are in $\mathcal{W}(X)$, for some $\mu \in \rho(A)$, then $M(\mu) \in \mathcal{W}(X \times X) \subset \mathcal{F}(X \times X)$.

Proof of Theorem 3.2. (i) Let $\mu \in \rho(A)$ be such that $M(\mu) \in \mathcal{F}(X \times X)$ and set $\lambda \in \mathbb{C}$. While writing $\lambda - L = \mu - L + (\lambda - \mu)$ and using relation (3.2), we have

$$\begin{aligned} \lambda - L &= \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \lambda - A & 0 \\ 0 & \lambda - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (\lambda - \mu) M(\mu) \\ &:= U V(\lambda) W - (\lambda - \mu) M(\mu). \end{aligned} \tag{3.4}$$

Since $M(\mu) \in \mathcal{F}(X \times X)$, then $\lambda - L$ is a Fredholm operator if and only if $UV(\lambda)W$ is a Fredholm operator. Now, observe that the operators U and W are bounded and have bounded inverse, hence the operator $UV(\lambda)W$ is a Fredholm operator if and only if $V(\lambda)$ has this property if and only if $\lambda - A$ and $\lambda - S(\mu)$ are Fredholm operators on X. Therefore,

$$\sigma_{e4}(L) = \sigma_{e4}(A) \cup \sigma_{e4}\big(S(\mu)\big). \tag{3.5}$$

The use of [28, Proposition 3.1(i)] and Eq. (3.4) show that, for $\lambda \in \Phi_L$,

$$i(\lambda - L) = i(\lambda - A) + i\big(\lambda - S(\mu)\big). \tag{3.6}$$

It follows, immediately, from Eqs. (3.5) and (3.6) that $\sigma_{e5}(L) \subseteq \sigma_{e5}(A) \cup \sigma_{e5}(S(\mu))$. Suppose now that ${}^{C}\sigma_{e4}(A) = \rho_4(A)$ is connected. By assumption (H1), $\rho(A)$ is nonempty. Let $\mu_0 \in \rho(A)$, then $\mu_0 - A \in \Phi(X)$ and $i(\mu_0 - A) = 0$. Since $\rho(A) \subseteq \rho_4(A)$ and $i(\lambda - A)$ is constant on any component of Φ_A, then $i(\lambda - A) = 0$ for all $\lambda \in \rho_4(A)$. It follows, from Eqs. (3.5) and (3.6) that

$$\sigma_{e5}(L) = \sigma_{e5}(A) \cup \sigma_{e5}\big(S(\mu)\big). \tag{3.7}$$

Assume further, that ${}^{C}\sigma_{e5}(L)$ is connected. We have the set $\rho_5(L) = {}^{C}\sigma_{e5}(L)$ contains points of $\rho(L)$, which is a nonempty set. Thus, since $\alpha(\lambda - L)$ and $\beta(\lambda - L)$ are constant on any component of Φ_L except possibly on a discrete set of points at which they have large values (see Proposition 2.1), then $\rho_5(L) \subset \rho_6(L)$. This together with the inclusion $\sigma_{e5}(L) \subset \sigma_{e6}(L)$ leads to $\sigma_{e5}(L) = \sigma_{e6}(L)$. Since, ${}^C\sigma_{e4}(A)$ is connected, then it follows from what precedes that $\sigma_{e5}(A) = \sigma_{e4}(A)$. So, ${}^C\sigma_{e5}(A)$ is connected. Using the same reasoning as before, we show that $\sigma_{e5}(A) = \sigma_{e6}(A)$. The condition that ${}^C\sigma_{e5}(S(\mu))$ is connected leads to $\sigma_{e5}(S(\mu)) = \sigma_{e6}(S(\mu))$, and the result of the assertion (i) follows from Eq. (3.7).

(ii) Let $\mu \in \rho(A)$ be such that $M(\mu)$ is an upper semi-Fredholm perturbation. Then, from Eq. (3.4), we have $\lambda - L \in \Phi_+(X \times X)$ if and only if $UV(\lambda)W \in \Phi_+(X \times X)$ if and only if $\lambda - A$ and $\lambda - S(\mu)$ are in $\Phi_+(X)$, since the operators U and W are bounded and have bounded inverse. Then the result of (ii) follows.

(iii) The proof of this assertion may be checked in the same way as the proof of (ii).

(iv) This assertion is an immediate consequence of (ii) and (iii). □

Remark 3.4.

(a) If the operators A, B, C and D are everywhere defined and bounded, the hypothesis of Theorem 3.2(iii) can be replaced by $[\mathcal{I}(X)]^* \subset \mathcal{F}_+(X^*)$ and $[M(\mu)]^* \in \mathcal{F}_+(X^* \times X^*)$ for some $\mu \in \rho(A)$. Indeed, it is sufficient to write the relation (3.4) for the adjoint, thus

$$\bar{\lambda} - L^* = W^*\big[V(\lambda)\big]^*U^* - (\bar{\lambda} - \bar{\mu})\big[M(\mu)\big]^*.$$

Now, using the fact that $\alpha(\bar{\lambda} - L^*) = \beta(\lambda - L)$ and $\alpha([V(\lambda)]^*) = \beta(V(\lambda))$ (cf. [9,26]) and arguing as the proof of Theorem 3.2(ii), we derive, easily, the result.

(b) Assume that the operator L acts on the product of Banach spaces $X \times Y$. Using Proposition 2 in [8, pp. 69–70] we can verify that if $F(\mu) \in \mathcal{F}^b(X, Y)$ for some $\mu \in \rho(A)$, then $F(\mu) \in \mathcal{F}^b(X, Y)$ for all $\mu \in \rho(A)$ and $\sigma_{ei}(S(\mu))$, $i = 4, 5$ does not depend on μ. Therefore, it can be showed that the result of Theorem 3.2(i) remains valid if $F(\mu) \in \mathcal{F}^b(X, Y)$ and $M(\mu) \in \mathcal{F}(X \times Y)$.

4. Application to two-group transport operators

The aim of this section is to apply Theorem 3.2 to study the essential spectra of a class of linear two-group transport operators on L_p-spaces, $1 \leqslant p < \infty$, with abstract boundary conditions.

Let

$$X_p := L_p\big((-a, a) \times (-1, 1); dx\, dv\big), \quad a > 0,\ 1 \leqslant p < \infty.$$

We consider the following two-group transport operators with abstract boundary conditions:

$$A_H = T_H + K,$$

where

$$T_H\psi = \begin{pmatrix} -v\frac{\partial\psi_1}{\partial x} - \sigma_1(v)\psi_1 & 0 \\ 0 & -v\frac{\partial\psi_2}{\partial x} - \sigma_2(v)\psi_2 \end{pmatrix} = \begin{pmatrix} T_{H_1} & 0 \\ 0 & T_{H_2} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

and

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

with K_{ij}, $i, j = 1, 2$, are bounded linear operators defined on X_p by

$$\begin{cases} K_{ij} : X_p \to X_p, \\ u \mapsto K_{ij}u(x,v) = \int_{-1}^{1} \kappa_{ij}(x,v,v')u(x,v')\,dv', \end{cases} \tag{4.1}$$

and the kernels $\kappa_{ij} : (-a,a) \times (-1,1) \times (-1,1) \to \mathbb{R}$ are assumed to be measurable.

Each operator T_{H_j}, $j = 1,2$, is defined by

$$\begin{cases} T_{H_j} : \mathcal{D}(T_{H_j}) \subset X_p \to X_p, \\ \varphi \mapsto (T_{H_j}\varphi)(x,v) = -v\frac{\partial\varphi}{\partial x}(x,v) - \sigma_j(v)\varphi(x,v), \\ \mathcal{D}(T_{H_j}) = \{\varphi \in W \text{ such that } \varphi^i = H_j\varphi^o\}, \end{cases}$$

where W is the space defined by

$$W = \left\{\varphi \in X_p \text{ such that } v\frac{\partial\varphi}{\partial x} \in X_p\right\}$$

and $\sigma_j(.) \in L^\infty(-1,1)$. φ^o, φ^i represent the outgoing and the incoming fluxes related by the boundary operator H_j ("o" for the outgoing and "i" for the incoming) and given by

$$\begin{cases} \varphi^i(v) = \varphi(-a,v), & v \in (0,1), \\ \varphi^i(v) = \varphi(a,v), & v \in (-1,0), \\ \varphi^o(v) = \varphi(-a,v), & v \in (-1,0), \\ \varphi^o(v) = \varphi(a,v), & v \in (0,1). \end{cases}$$

We denote by X_p^o and X_p^i the following boundary spaces:

$$X_p^o := L_p\big[\{-a\} \times (-1,0); |v|\,dv\big] \times L_p\big[\{a\} \times (0,1); |v|\,dv\big] := X_{1,p}^o \times X_{2,p}^o$$

equipped with the norm

$$\begin{aligned} \|u^o, X_p^o\| &:= \big(\|u_1^o, X_{1,p}^o\|^p + \|u_2^o, X_{2,p}^o\|^p\big)^{\frac{1}{p}} \\ &= \left[\int_{-1}^{0} \big|u(-a,v)\big|^p |v|\,dv + \int_{0}^{1} \big|u(a,v)\big|^p |v|\,dv\right]^{\frac{1}{p}}, \end{aligned}$$

and

$$\begin{aligned} X_p^i &:= L_p\big[\{-a\} \times (0,1); |v|\,dv\big] \times L_p\big[\{a\} \times (-1,0); |v|\,dv\big] \\ &:= X_{1,p}^i \times X_{2,p}^i \end{aligned}$$

equipped with the norm

$$\begin{aligned} \|u^i, X_p^i\| &:= \big(\|u_1^i, X_{1,p}^i\|^p + \|u_2^i, X_{2,p}^i\|^p\big)^{\frac{1}{p}} \\ &= \left[\int_{0}^{1} \big|u(-a,v)\big|^p |v|\,dv + \int_{-1}^{0} \big|u(a,v)\big|^p |v|\,dv\right]^{\frac{1}{p}}. \end{aligned}$$

It is well known that any function u in W possesses traces on the spatial boundary $\{-a\} \times (-1,0)$ and $\{a\} \times (0,1)$ which respectively belong to the spaces X_p^o and X_p^i (see, for instance, [4] or [11]). They are denoted, respectively, by u^o and u^i.

It is clear that the operator A_H is defined on $\mathcal{D}(T_{H_1}) \times \mathcal{D}(T_{H_2})$. We will denote the operator A_H by

$$A_H := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$\begin{cases} A_{11} = T_{H_1} + K_{11}, \\ A_{12} = K_{12}, \\ A_{21} = K_{21}, \\ A_{22} = T_{H_2} + K_{22}. \end{cases}$$

Remark 4.1.

(i) It is well known that the operators T_{H_j}, $j = 1, 2$, are closed, densely defined linear operators with a nonempty resolvent set. Then the assumptions (H1)–(H4), introduced in Section 3, are satisfied for the operator A_H, since K_{ij}, $i, j = 1, 2$, are bounded.

(ii) To verify that the operator $M(\mu)$ defined by (3.3) is compact on $X_p \times X_p$, $1 < p < \infty$ (respectively weakly compact on $X_1 \times X_1$), we shall prove that the operators

$$F(\lambda) := K_{21}(\lambda - A_{11})^{-1} \quad \text{and} \quad G(\lambda) := (\lambda - A_{11})^{-1} K_{12}$$

are compact on X_p, $1 < p < \infty$ (respectively weakly compact on X_1) (see Remark 3.3).

In view of the previous remark we will determine the expression of the resolvent of the operator T_{H_1}. Let $\varphi \in X_p$, $\lambda \in \mathbb{C}$ and consider the resolvent equation for T_{H_1}

$$(\lambda - T_{H_1})\psi_1 = \varphi, \tag{4.2}$$

where the unknown ψ_1 must be in $\mathcal{D}(T_{H_1})$. Let

$$\lambda_j^* = \liminf_{|v| \to 0} \sigma_j(v), \qquad j = 1, 2,$$

and

$$\lambda_0^j := \begin{cases} -\lambda_j^*, & \text{if } \|H_j\| \leqslant 1, \\ -\lambda_j^* + \frac{1}{2a}\log(\|H_j\|), & \text{if } \|H_j\| > 1. \end{cases}$$

Therefore, for $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -\lambda_1^*$, the solution of (4.2) is formally given by

$$\psi_1(x, v) = \begin{cases} \psi_1(-a, v) e^{-\frac{(\lambda + \sigma_1(v))|a + x|}{|v|}} + \frac{1}{|v|}\int_{-a}^{x} e^{-\frac{(\lambda + \sigma_1(v))|x - x'|}{|v|}} \varphi(x', v)\, dx', & 0 < v < 1, \\ \psi_1(a, v) e^{-\frac{(\lambda + \sigma_1(v))|a - x|}{|v|}} + \frac{1}{|v|}\int_{x}^{a} e^{-\frac{(\lambda + \sigma_1(v))|x - x'|}{|v|})} \varphi(x', v)\, dx', & -1 < v < 0. \end{cases} \tag{4.3}$$

Accordingly, $\psi_1(a, v)$ and $\psi_1(-a, v)$ are given by

$$\psi_1(a, v) = \psi_1(-a, v) e^{-2a\frac{(\lambda + \sigma_1(v))}{|v|}} + \frac{1}{|v|}\int_{-a}^{a} e^{-\frac{(\lambda + \sigma_1(v))|a - x|}{|v|}} \varphi(x, v)\, dx, \quad 0 < v < 1, \tag{4.4}$$

$$\psi_1(-a,v) = \psi_1(a,v)e^{-2a\frac{(\lambda+\sigma_1(v))}{|v|}} + \frac{1}{|v|}\int_{-a}^{a} e^{-\frac{(\lambda+\sigma_1(v))|a+x|}{|v|}}\varphi(x,v)\,dx, \quad -1<v<0. \tag{4.5}$$

For the clarity of our subsequent analysis, we introduce the following bounded operators:

$$\begin{cases} M_\lambda : X_p^i \to X_p^o, \quad M_\lambda u := (M_\lambda^+ u, M_\lambda{}^- u) \quad \text{with} \\ M_\lambda^+ u(-a,v) := u(-a,v)e^{-2a\frac{(\lambda+\sigma_1(v))}{|v|}}, \quad 0<v<1, \\ M_\lambda^- u(a,v) := u(a,v)e^{-2a\frac{(\lambda+\sigma_1(v))}{|v|}}, \quad -1<v<0, \end{cases}$$

$$\begin{cases} B_\lambda : X_p^i \to X_p, \quad B_\lambda u := \chi_{(-1,0)}(v)B_\lambda^- u + \chi_{(0,1)}(v)B_\lambda{}^+ u \quad \text{with} \\ B_\lambda^+ u(x,v) := u(-a,v)e^{-\frac{(\lambda+\sigma_1(v))|a+x|}{|v|}}, \quad 0<v<1, \\ B_\lambda^- u(x,v) := u(a,v)e^{-\frac{(\lambda+\sigma_1(v))|a-x|}{|v|}}, \quad -1<v<0, \end{cases}$$

$$\begin{cases} G_\lambda : X_p \to X_p^o, \quad G_\lambda\varphi := (G_\lambda^+\varphi, G_\lambda^-\varphi) \quad \text{with} \\ G_\lambda^+\varphi(-a,v) := \frac{1}{|v|}\int_{-a}^{a} e^{-\frac{(\lambda+\sigma_1(v))|a-x|}{|v|}}\varphi(x,v)\,dx, \quad 0<v<1, \\ G_\lambda^-\varphi(a,v) := \frac{1}{|v|}\int_{-a}^{a} e^{-\frac{(\lambda+\sigma_1(v))|a+x|}{|v|}}\varphi(x,v)\,dx, \quad -1<v<0 \end{cases}$$

and finally, we consider

$$\begin{cases} C_\lambda : X_p \to X_p, \quad C_\lambda\varphi := \chi_{(-1,0)}(v)C_\lambda^-\varphi + \chi_{(0,1)}(v)C_\lambda^+\varphi \quad \text{with} \\ C_\lambda^+\varphi(x,v) := \frac{1}{|v|}\int_{-a}^{x} e^{-\frac{(\lambda+\sigma_1(v))|x-x'|}{|v|}}\varphi(x',v)\,dx', \quad 0<v<1, \\ C_\lambda^-\varphi(x,v) := \frac{1}{|v|}\int_{x}^{a} e^{-\frac{(\lambda+\sigma_1(v))|x-x'|}{|v|}}\varphi(x',v)\,dx', \quad -1<v<0, \end{cases}$$

where $\chi_{(-1,0)}(.)$ and $\chi_{(0,1)}(.)$ denote, respectively, the characteristic functions of the intervals $(-1,0)$ and $(0,1)$. The operators M_λ, B_λ, G_λ, and C_λ are bounded on their respective spaces. Their norms are bounded above, respectively by $e^{-2a(\operatorname{Re}\lambda+\lambda_1^*)}$, $(p\operatorname{Re}\lambda+\lambda_1^*)^{-1/p}$, $(\operatorname{Re}\lambda+\lambda_1^*)^{-1/q}$ and $(\operatorname{Re}\lambda+\lambda_1^*)^{-1}$, where q denotes the conjugate of p. For the details we refer to [30].

Using the operators defined above and the fact that ψ_1 must satisfy the boundary conditions, we can write Eqs. (4.4) and (4.5) in the operators form

$$\psi_1^o = M_\lambda H_1\psi_1^o + G_\lambda\varphi.$$

It follows, from the norm estimate of M_λ, that $\|M_\lambda H_1\| < 1$ for $\operatorname{Re}\lambda > \lambda_0^1$. This gives

$$\psi_1^o = \sum_{n\geqslant 0}(M_\lambda H_1)^n G_\lambda\varphi. \tag{4.6}$$

On the other hand, Eq. (4.3) can be written as

$$\psi_1 = B_\lambda H_1\psi_1^o + C_\lambda\varphi. \tag{4.7}$$

Substituting (4.6) into (4.7), we get

$$\psi_1 = \sum_{n\geqslant 0} B_\lambda H_1(M_\lambda H_1)^n G_\lambda\varphi + C_\lambda\varphi.$$

Therefore,

$$(\lambda - T_{H_1})^{-1} = \sum_{n\geqslant 0} B_\lambda H_1 (M_\lambda H_1)^n G_\lambda + C_\lambda. \tag{4.8}$$

Notice that the collision operators K_{ij}, $i, j = 1, 2$, defined in (4.1), act only on the velocity v', so x may be seen, simply, as a parameter in $[-a, a]$. Then, we will consider K_{ij} as a function

$$K_{ij}(.) : x \in [-a, a] \longrightarrow K_{ij}(x) \in \mathcal{L}\big(L_p\big([-1, 1]; dv\big)\big).$$

In the sequel, we will make the following assumptions introduced in [36]:

$$\text{(H6)}\quad \begin{cases} \text{– the function } K_{ij}(.) \text{ is measurable, i.e., if } \mathcal{O} \text{ is an open subset of} \\ \quad \mathcal{L}(L_p([-1,1]; dv)), \text{ then } \{x \in [-a,a] \text{ such that } K_{ij}(x) \in \mathcal{O}\} \text{ is measurable,} \\ \text{– there exists a compact subset } \mathcal{C} \subseteq \mathcal{L}(L_p([-1,1]; dv)) \text{ such that} \\ \quad K_{ij}(x) \in \mathcal{C} \text{ a.e. on } [-a,a], \\ \text{– } K_{ij}(x) \in \mathcal{K}(L_p([-1,1]; dv)) \text{ a.e. on } [-a,a]. \end{cases}$$

Definition 4.1. A collision operator in the form (4.1) is said to be regular if it satisfies the assumptions (H6).

We recall the following lemma established in [36].

Lemma 4.1. [36, Lemma 2.3] *A regular collision operator K can be approximated, in the uniform topology, by a sequence K_n of collision operators of the form*

$$\kappa_n(x, v, v') = \sum_{j=1}^{n} \alpha_j(x) f_j(v) g_j(v'),$$

where $\alpha_j(.) \in L^\infty(-a, a)$, $f_j(.) \in L^p(-1, 1)$ and $g_j \in L^q(-1, 1)$ (q denote the conjugate of p).

Lemma 4.2. *If $\kappa_{21}(x, v, v')/|v'|$ defines a regular operator, then $K_{21}(\lambda - T_{H_1})^{-1}$ is weakly compact on X_1.*

Proof. In view of (4.8), the operator $K_{21}(\lambda - T_{H_1})^{-1}$ is given by

$$K_{21}(\lambda - T_{H_1})^{-1} = \sum_{n\geqslant 0} K_{21} B_\lambda H_1 (M_\lambda H_1)^n G_\lambda + K_{21} C_\lambda.$$

Then, to prove the weak compactness of $K_{21}(\lambda - T_{H_1})^{-1}$, it suffices to prove the weak compactness of the operators $K_{21}B_\lambda$ and $K_{21}C_\lambda$. Observe that C_λ is nothing else but $(\lambda - T_1)^{-1}$, where T_1 is the streaming operator for the vacuum boundary conditions. According to Remark 2.4 in [36] the operator $K_{21}C_\lambda$ is weakly compact on X_1. Thus, it suffices to prove that $K_{21}B_\lambda$ is weakly compact on X_1.

Let $u \in X_1^i$, we have

$$K_{21} B_\lambda u(x, v) = \int_{-1}^{1} \kappa_{21}(x, v, v') B_\lambda u(x, v')\, dv' = \widetilde{K}_{21} \widetilde{B}_\lambda u(x, v),$$

where

$$\begin{cases} \widetilde{K}_{21}: X_1 \to X_1, \\ \psi \to \widetilde{K}_{21}u(x,v) = \int_{-1}^{1} \frac{\kappa_{21}(x,v,v')}{|v'|} u(x,v')\,dv', \end{cases}$$

and $\widetilde{B}_\lambda = |v'|B_\lambda$. Then it is sufficient to establish the weak compactness of $\widetilde{K}_{21}\widetilde{B}_\lambda$. The fact that $\widetilde{K}_{21}$ is regular and the use of Lemma 4.1 allows us to establish the result for an operator whose kernel is

$$\frac{\kappa_{21}(x,v,v')}{|v'|} = \sum_{j=1}^{n} \alpha_j(x) f_j(v) g_j(v'),$$

where $\alpha_j(.) \in L^\infty(-a,a)$, $f_j(.) \in L^1(-1,1)$ and $g_j \in L^\infty(-1,1)$. Therefore, we restrict ourselves to

$$\frac{\kappa_{21}(x,v,v')}{|v'|} = \alpha(x) f(v) g(v'),$$

where $\alpha(.) \in L^\infty(-a,a)$, $f(.) \in L^1(-1,1)$ and $g \in L^\infty(-1,1)$, since the weak compactness is stable by summation. We claim that the operator $\widetilde{K}_{21}\widetilde{B}_\lambda$ satisfies the following estimate:

$$\|\widetilde{K}_{21}\widetilde{B}_\lambda\| \leqslant 2a\|g\|_\infty \|\alpha\|_\infty \|f\|. \tag{4.9}$$

Indeed, let $u \in X_1^i$,

$$\widetilde{K}_{21}\widetilde{B}_\lambda u(x,v) = \alpha(x) f(v) \left[\int_0^1 g(v') u(-a,v') e^{-\frac{(\lambda+\sigma_1(v'))|a+x|}{|v'|}} |v'|\,dv' + \int_{-1}^{0} g(v') u(a,v') e^{-\frac{(\lambda+\sigma_1(v'))|a-x|}{|v'|}} |v'|\,dv' \right].$$

Therefore,

$$\left|\widetilde{K}_{21}\widetilde{B}_\lambda u(x,v)\right| \leqslant \|g\|_\infty \|\alpha\|_\infty \left|f(v)\right| \left[\int_0^1 \left|u(-a,v')\right| e^{-\frac{(\operatorname{Re}\lambda+\lambda_1^*)|a+x|}{|v'|}} |v'|\,dv' + \int_{-1}^{0} \left|u(a,v')\right| e^{-\frac{(\operatorname{Re}\lambda+\lambda_1^*)|a-x|}{|v'|}} |v'|\,dv' \right].$$

Thus, for $\operatorname{Re}\lambda > -\lambda_1^*$, we have

$$\left|\widetilde{K}_{21}\widetilde{B}_\lambda u(x,v)\right| \leqslant \|g\|_\infty \|\alpha\|_\infty \left|f(v)\right| \left\|u, X_1^i\right\|.$$

Then the claim is proved. The inequality (4.9) shows that the operator $\widetilde{K}_{21}\widetilde{B}_\lambda$ depends continuously (in the uniform topology) on $f(.)$. Since the set of bounded functions which vanish in neighborhood of $v=0$ is dense in $L_1(-1,1)$, $\widetilde{K}_{21}\widetilde{B}_\lambda$ is a limit, in the uniform topology, of integral operators with bounded kernels. The use of [7, Corollary 11, p. 294] make us conclude that $\widetilde{K}_{21}\widetilde{B}_\lambda$ is weakly compact on X_1^i. Now, the weak compactness of $K_{21}(\lambda - T_{H_1})$ follows. □

Remark 4.2. Lemma 4.2 is a generalization of Remark 2.4 in [36] to general boundary conditions.

Lemma 4.3. *Let* $\lambda \in \rho(T_{H_1})$ *be such that* $r_\sigma((\lambda - T_{H_1})^{-1}K_{11}) < 1$ ($r_\sigma(.)$ *the spectral radius*).

(i) *If* $\kappa_{21}(x, v, v')/|v'|$ *defines a regular operator, then the operator* $F(\lambda) = K_{21}(\lambda - A_{11})^{-1}$ *is weakly compact on* X_1.
(ii) *If* K_{21} *is regular, then the operator* $F(\lambda) = K_{21}(\lambda - A_{11})^{-1}$ *is compact on* X_p *for* $1 < p < \infty$.
(iii) *If the operator* K_{12} *is regular, then* $G(\lambda) = (\lambda - A_{11})^{-1}K_{12}$ *is compact on* X_p *for* $1 < p < \infty$ *and weakly compact on* X_1.

Proof. In [31, Proposition 3.1] it is shown that $\lim_{\operatorname{Re}\lambda\to+\infty} \|(\lambda - T_{H_1})^{-1}\| = 0$. Then there exists $\lambda \in \rho(T_{H_1})$ such that $r_\sigma((\lambda - T_{H_1})^{-1}K_{11}) < 1$. For a such λ, the equation $(\lambda - T_{H_1} - K_{11})\varphi = \psi$ may be transformed into

$$\varphi - (\lambda - T_{H_1})^{-1}K_{11}\varphi = (\lambda - T_{H_1})^{-1}\psi,$$

since $\lambda \in \rho(T_{H_1})$. The fact that $r_\sigma((\lambda - T_{H_1})^{-1}K_{11}) < 1$ implies

$$(\lambda - A_{11})^{-1} = \sum_{n\geqslant 0}\big[(\lambda - T_{H_1})^{-1}K_{11}\big]^n(\lambda - T_{H_1})^{-1}. \tag{4.10}$$

(i) The use of Lemma 4.2 implies that for all n in $\mathbb{N}$, $K_{21}[(\lambda - T_{H_1})^{-1}K_{11}]^n(\lambda - T_{H_1})^{-1}$ is weakly compact on X_1. Now, the result follows from Eq. (4.10) and the fact that $\mathcal{W}(X_1)$ is a closed two-sided ideal of $\mathcal{L}(X_1)$.

(ii) The proof of this assertion follows immediately from Eq. (4.10) and Theorem 2.2 in [30].

(iii) Equation (4.10) leads to

$$G(\lambda) = \sum_{n\geqslant 0}\big[(\lambda - T_{H_1})^{-1}K_{11}\big]^n(\lambda - T_{H_1})^{-1}K_{12}.$$

Therefore, the hypothesis on K_{12} together with Lemma 3.1 in [19] imply the compactness of $G(\lambda)$ on X_p for $1 < p < \infty$ and its weak compactness on X_1. □

Remark 4.3.

(i) Let us note that according to Theorem 1 in [38] we have

$$\mathcal{W}(X_1) = \mathcal{S}(X_1).$$

If $1 < p < \infty$, X_p is reflexive and then $\mathcal{L}(X_p) = \mathcal{W}(X_p)$. On the other hand, it follows from [8, Theorem 5.2] that $\mathcal{K}(X_p) \subset_{\neq} \mathcal{S}(X_p) \subset_{\neq} \mathcal{W}(X_p)$ with $p \neq 2$. For $p = 2$ we have $\mathcal{K}(X_p) = \mathcal{S}(X_p) = \mathcal{W}(X_p)$.

(ii) The essential spectra of the operator T_j, $j = 1, 2$ (T_j designates the streaming operator with vacuum boundary conditions, i.e., $H_j = 0$), were analyzed in detail in [29, Remark 4.1]. In particular it is shown that

$$\sigma_{ei}(T_j) = \big\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \leqslant -\lambda_j^*\big\} \quad \text{for } i = 1, \ldots, 6.$$

In view of Eq. (4.8), we have for $\operatorname{Re}\lambda > \lambda_0^j$, $j = 1, 2$,

$$(\lambda - T_{H_j})^{-1} - (\lambda - T_j)^{-1} = \sum_{n \geqslant 0} B_\lambda H_j (M_\lambda H_j)^n G_\lambda$$

(C_λ is nothing else but $(\lambda - T_j)^{-1}$). If the operators H_j, $j = 1, 2$, are strictly singular on X_p, for $1 \leqslant p < \infty$, then $(\lambda - T_{H_j})^{-1} - (\lambda - T_j)^{-1}$ are strictly singular too. Therefore, the use of Theorem 3.3 in [28] and Remark 4.3(ii) imply that

$$\sigma_{ei}(T_{H_j}) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \leqslant -\lambda_j^*\} \quad \text{for } i = 1, \ldots, 5.$$

The fact that $C\sigma_{e5}(T_{H_j})$, $j = 1, 2$, are connected and $\rho(T_{H_j}) \neq \emptyset$ imply that $\sigma_{e5}(T_{H_j}) = \sigma_{e6}(T_{H_j})$.

Remark 4.4. According to Remarks 3.3, 2.4 and Lemma 4.3, the hypothesis $M(\mu)$ in $\mathcal{K}(X_p \times X_p)$, for $1 < p < \infty$ (respectively in $\mathcal{W}(X_1 \times X_1)$) is verified. Hence, for $\mathcal{I}(X) = \mathcal{K}(X_p)$, $1 < p < \infty$ (respectively $\mathcal{I}(X) = \mathcal{W}(X_1)$), all the results of Section 3 are applicable for the operator A_H.

We are now ready to express the essential spectra of two-group transport operators with general boundary conditions.

Theorem 4.1. *If the operators $H_j \in \mathcal{S}(X_p)$, $j = 1, 2$, and the operators K_{11}, K_{22}, K_{12} are regular and if in addition $\kappa_{21}(x, v, v')$ (respectively $\kappa_{21}(x, v, v')/|v'|$) defines a regular operator on X_p, for $1 < p < \infty$ (respectively on X_1), then*

$$\sigma_{ei}(A_H) = \{\lambda \in \mathbb{C} \textit{ such that } \operatorname{Re}\lambda \leqslant -\min(\lambda_1^*, \lambda_2^*)\}, \quad \textit{for } i = 1, \ldots, 6.$$

Proof. Let $\lambda \in \rho(T_{H_1})$ such that $r_\sigma((\lambda - T_{H_1})^{-1} K_{11}) < 1$, then $\lambda \in \rho(A_{11}) \cap \rho(T_{H_1})$. From Eq. (4.10) we have

$$(\lambda - A_{11})^{-1} - (\lambda - T_{H_1})^{-1} = \sum_{n \geqslant 1} [(\lambda - T_{H_1})^{-1} K_{11}]^n (\lambda - T_{H_1})^{-1}.$$

Since K_{11} is regular, then it follows from [16, Lemma 3.1] that the operator $(\lambda - A_{11})^{-1} - (\lambda - T_{H_1})^{-1}$ is compact on X_p, for $1 < p < \infty$, and weakly compact on X_1. The use of [28, Theorem 3.3] leads to

$$\sigma_{ei}(A_{11}) = \sigma_{ei}(T_{H_1}) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \leqslant -\lambda_1^*\}, \quad i = 1, \ldots, 6. \tag{4.11}$$

Let $\mu \in \rho(A_{11})$. The operator $S(\mu)$ is given by

$$S(\mu) = A_{22} - K_{21} G(\mu).$$

By Lemma 4.3, the operator $K_{21} G(\mu)$ is compact on X_p, for $1 < p < \infty$, and weakly compact on X_1, then it follows from [28, Theorem 3.1] that $\sigma_{ei}(S(\mu)) = \sigma_{ei}(A_{22})$, $i = 1, \ldots, 6$. According to the same reasoning as the previous one, we have

$$\sigma_{ei}(S(\mu)) = \sigma_{ei}(A_{22}) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \leqslant -\lambda_2^*\}, \quad i = 1, \ldots, 6. \tag{4.12}$$

Applying Theorem 3.2 and using Eqs. (4.11) and (4.12), we get

$$\sigma_{ei}(A_H) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \leqslant -\min(\lambda_1^*, \lambda_2^*)\}, \quad \text{for } i = 1, \ldots, 6. \qquad \square$$

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