The Non-existence of Certain Geometries of Type C₃

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In this note, we investigate finite connected flat geometries of type C₃ of uniform odd order in which the lines and the planes form a projective space. We give an algebraic description of such geometries by sets of alternating bilinear forms or, equivalently, by sets of skew symmetric matrices. These sets of skew symmetric matrices will be identified in a natural way with the full point set of a projective plane. By a general result concerning two-valued functions defined on projective planes of order 5, we finally prove the non-existence of finite connected flat geometries of type C₃ of uniform order 5 in which the lines and the planes form a projective space.

1. INTRODUCTION

A fundamental theorem of Tits [7] shows that every geometry described by a Coxeter diagram arises as a quotient of a building if and only if the same is true for all residual geometries of type C₃ or H₃. This paper is concerned with a certain class of geometries of type C₃ which are not buildings.

For the definitions of most of the notation used in this paper the reader is referred to [5], [6] or [7]. In addition, we say that the geometry S of type C₃ has uniform order q, if each flag of rank two of S is contained in exactly q + 1 maximal flags. Moreover, we call S flat, iff each point of S is incident with each plane of S.

The following result is due to Ott [3]:

PROPOSITION 1.1. Each finite connected geometry of type C₃ of uniform order is either a building or flat.

While the buildings of type C₃ are completely determined by Tits [6], there do not exist general results concerning flat geometries of type C₃ of uniform order. We only have the following elementary observation. For a proof see [4].

PROPOSITION 1.2. Let S be a finite connected flat geometry of type C₃ of uniform order q. Then the planes and the lines of S form a 2-design D(S) with λ = 1. (The points of D(S) are the planes of S and the blocks of D(S) are the lines of S.) For the parameters v, b, k, r of D(S) we have v = q³ + q² + q + 1, b = (q² + q + 1)(q² + 1), k = q + 1 and r = q² + q + 1.

It has been conjectured repeatedly that the 2-design D(S) of Proposition 1.2 is always a projective space, and in fact, the 2-design which arises from the Aschbacher geometry [1], the only known flat geometry of type C₃, is a projective space, namely PG(3, 2).

The major contribution of the present paper is to show that under this additional (projective) condition, the geometry S gives rise to a structured set of alternating bilinear forms which can be interpreted in a natural way as the full point set of a projective plane. This algebraic description will be the object of the second section of this note.

Finally, in the third section, the algebraic approach developed in the second section will be used to prove the following main result.
THEOREM 1.3. There is no finite connected geometry $S$ of type $C_3$ of uniform order $q$ which satisfies the following conditions:

(i) $S$ is not a building.

(ii) The planes and the lines of $S$ form a projective space $D(S)$. (The points of $D(S)$ are the planes of $S$ and the blocks of $D(S)$ are the lines of $S$.)

There exist two results which suggest this theorem while their proofs are completely different from the techniques used in this paper. The first, due to de Clerk and Thas [2], shows that there is no finite connected geometry of type $C_3$ of uniform order $q$ for even $q$ with $q \neq 2$ that satisfies conditions (i) and (ii) of Theorem 1.3. The other, due to Sarah Rees [5], proves the same statement for $q = 3$.

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2. AN ALGEBRAIC DESCRIPTION OF CERTAIN FLAT GEOMETRIES OF TYPE $C_3$ OF UNIFORM ORDER

Let $q$ be an odd prime power and let $V(d, q)$ denote the $d$-dimensional vector space over $GF(q)$.

DEFINITION 2.1. (i) Let $\mathcal{A}(4, q)$ denote the set of all non-degenerate alternating bilinear forms on $V(4, q)$. The two forms $a, b \in \mathcal{A}(4, q)$ are called compatible iff

$$a - tb \in \mathcal{A}(4, q)$$

for each $t \in GF(q)$.

(ii) By $\mathcal{S}(4, q)$ we denote the set of all non-singular skew symmetric $4 \times 4$-matrices over $GF(q)$. Further, we call the matrices $X, Y \in \mathcal{S}(4, q)$ compatible iff

$$X - tY \in \mathcal{S}(4, q)$$

for each $t \in GF(q)$.

For each $a \in \mathcal{A}(4, q)$ let $M(a)$ denote the Gram matrix of $a$ with respect to a fixed basis of $V(4, q)$. Then $M$ is a bijection from $\mathcal{A}(4, q)$ to $\mathcal{S}(4, q)$. Moreover, the forms $a, b \in \mathcal{A}(4, q)$ are compatible iff $M(a), M(b) \in \mathcal{S}(4, q)$ are compatible.

Now the following theorem is an immediate consequence of [5, (2.3)].

THEOREM 2.2. Let $S$ be a finite connected flat geometry of type $C_3$ of uniform order $q$. Assume that the planes and the lines of $S$ form a projective space $D(S)$. (The points of $D(S)$ are the planes of $S$ and the blocks of $D(S)$ are the lines of $S$.) Then we have the following:

(i) $q$ is a prime power and $D(S)$ is the projective space $PG(3, q)$.

(ii) $\mathcal{A}(4, q)$ possesses $q^2 + q + 1$ forms which are compatible in pairs.

(iii) $\mathcal{S}(4, q)$ possesses $q^2 + q + 1$ matrices which are compatible in pairs.

Our next step is to present the concept of compatibility in another context.

DEFINITION 2.3. (i) For all $X, Y \in \mathcal{S}(4, q)$ with $X = (x_{ij})_{ij}$ and $Y = (y_{ij})_{ij}$ let

$$f(X, Y) = -x_{41}y_{32} + x_{31}y_{42} - x_{21}y_{43} - x_{32}y_{41} + x_{42}y_{31} - x_{43}y_{21}.$$ 

Obviously $f$ is a bilinear form on $\mathcal{S}(4, q)$. 

(ii) For all \( X \in \mathcal{H}(4, q) \) with \( X = (x_{ij})_{ij} \) let
\[
p(X) := (x_{21}, x_{31}, x_{41}).
\]
Then \( p \) is a homomorphism onto \( V(3, q) \).

**PROPOSITION 2.4.** Let \( \mathcal{H} \subseteq \mathcal{H}(4, q) \) be a set of matrices which are compatible in pairs and denote by \( \mathcal{P}(q) \) the point set of the projective plane \( \text{PG}(2, q) \). Then the map
\[
\pi: \mathcal{H} \to \mathcal{P}(q)
\]
\[
X \mapsto \langle p(X) \rangle
\]
is injective.

**PROOF.** By Definition 2.3, \( \pi \) is well defined. Assume that \( X, Y \in \mathcal{H} \) and \( \pi(X) = \pi(Y) \). Then there exists \( t \in \text{GF}(q) \) such that \( p(X) = tp(Y) \). Thus \( X - ty \) is singular. Since by assumption the set \( \mathcal{H} \) consists of matrices which are compatible in pairs, we have \( X = Y \). \( \square \)

3. **The Case \( q = 5 \)**

The purpose of this section is the proof of the following theorem which, together with Theorem 2.2, proves Theorem 1.3.

**THEOREM 3.1.** Let \( V(4, 5) \) denote the 4-dimensional vector space over \( \text{GF}(5) \). Then we have the following:

(i) Every set of 31 non-degenerate alternating bilinear forms on \( V(4, 5) \) contains two elements \( a \) and \( b \) such that the alternating form \( a - tb \) is degenerate for at least one \( t \in \text{GF}(5) \).

(ii) Every set of 31 non-singular skew symmetric \( 4 \times 4 \) matrices over \( \text{GF}(5) \) has two elements \( X \) and \( Y \) such that the matrix \( X - ty \) is singular for at least one \( t \in \text{GF}(5) \).

For the rest of this paper let \( \mathcal{H} \) be a set of 31 matrices of \( \mathcal{H}(4, 5) \) which are compatible in pairs.

For any \( n \)-tuple \((X_1, \ldots, X_n)\) of elements of \( \mathcal{H} \) the matrix
\[
(f(X_i, X_j))_{ij}
\]
will be called the Gram matrix of \((X_1, \ldots, X_n)\).

Finally, let \( \mathcal{H}_0 \) denote the subset of those matrices \( X \) of \( \mathcal{H} \) for which \( f(X, X) \) is non-square.

The proof of the following lemma is obvious.

**LEMMA 3.2.** If \( X, Y \in \mathcal{H} \) and \( f(X, X) = 2 = f(Y, Y) \), then \( f(X, Y) \in \{1, -1\} \).

**LEMMA 3.3.** Let \( \{X_1, X_2, X_3\} \subseteq \mathcal{H} \) and assume that \( \pi(\{X_1, X_2, X_3\}) \) is collinear. If \((X_1, X_2, X_3)\) has Gram matrix
\[
\begin{pmatrix}
2 & -1 & 1 \\
-1 & 2 & 1 \\
1 & 1 & 2
\end{pmatrix},
\]

then \( X_1 + X_2 - X_3 \in \ker p \).
PROOF. Let $X := X_1 + X_2 - X_3$. Then

$$f(X, X) = 0.$$ 

In particular, since $X_1, X_2, X_3$ are compatible in pairs, $\{X_1, X_2, X_3\}$ is linearly independent. Since the reduction of $f$ on $\langle X_1, X_2, X_3 \rangle$ has $\langle X_1, X_2 \rangle$ as an exterior line, the classification of the bilinear forms on $V(3, q)$ implies that $\langle X \rangle$ is the only totally isotropic subspace of dimension 1 contained in $\langle X_1, X_2, X_3 \rangle$.

On the other hand, $\pi(\{X_1, X_2, X_3\})$ is collinear. Hence $\langle X_1, X_2, X_3 \rangle \cap \ker p \neq 0$, and we must have $X \in \ker p$. 

**Lemma 3.4.** Let $\mathcal{C} \subseteq \mathcal{H}_0$ be a 4-set and assume that $\pi(\mathcal{C})$ is collinear. Then there exists a 4-set consisting of multiples of elements of $\mathcal{C}$ having Gram matrix

$$
\begin{pmatrix}
2 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 1 & 2 & 1 \\
1 & 1 & 1 & 2
\end{pmatrix}.
$$

**Proof.** Let

$$\mathcal{C} = \{X_1, X_2, X_3, X_4\}.$$ 

For each $i \in \{1, \ldots, 4\}$, $f(X_i, X_i) \in \{2, -2\}$. If $f(X_i, X_i) = -2$, replace $X_i$ with $2X_i$. Then

$$f(X_i, X_i) = 2$$

for each $i \in \{1, \ldots, 4\}$.

By Lemma 3.2 we may replace $X_1, X_2, X_3$ with their negatives if necessary to have Gram matrix

$$
\begin{pmatrix}
2 & \alpha & \beta & 1 \\
\alpha & 2 & \gamma & 1 \\
\beta & \gamma & 2 & 1 \\
1 & 1 & 1 & 2
\end{pmatrix}
$$

for $(X_1, X_2, X_3, X_4)$, where $\alpha, \beta, \gamma \in \{1, -1\}$.

Assume $\alpha = -1, \beta = \gamma = 1$. Then, by Lemma 3.3, we have

$$p(X_3) = p(X_1 + X_2) = p(X_4).$$

Now Proposition 2.4 yields $X_3 = X_4$, contrary to the choice of $\mathcal{C}$.

Without loss of generality we now may assume $\alpha = \gamma = -1, \beta = 1$. In this case Lemma 3.3 yields

$$p(X_3) = p(X_4 - X_2) = p(X_1).$$

Now Proposition 2.4 yields $X_3 = X_1$, contrary to the choice of $\mathcal{C}$.  

**Lemma 3.5.** $\mathcal{H}_0$ does not contain a 5-set $\mathcal{C}$ such that $\pi(\mathcal{C})$ is collinear.

**Proof.** Let

$$\mathcal{C} = \{X_1, X_2, X_3, X_4, X_5\}.$$
By Lemma 3.4 we may assume that \((X_1, X_2, X_3, X_4, X_5)\) has Gram matrix
\[
\begin{pmatrix}
2 & 1 & 1 & 1 & x \\
1 & 2 & 1 & 1 & \beta \\
1 & 1 & 2 & 1 & \gamma \\
1 & 1 & 1 & 2 & \delta \\
x & \beta & \gamma & \delta & 2
\end{pmatrix}.
\]

By Lemma 3.2 we have \(\alpha, \beta, \gamma, \delta \in \{1, -1\}\). By replacing \(X_5\) with its negative if necessary and reorganizing the first four elements of \(\mathcal{C}\) we may assume that \(\alpha = \beta = 1\).

If \(\delta = -1\), then Lemma 3.3 yields
\[
p(X_1) = p(X_4 + X_5) = p(X_2)
\]
which, by Proposition 2.4, implies the contradiction \(X_1 = X_2\).

For \(\gamma = -1\) we have the same contradiction.

Now \(\alpha = \beta = \gamma = \delta = 1\). This implies that \(f\) is non-degenerate on the subspace \(<\mathcal{C}\>\) of dimension 5 of \(\mathcal{P}(4, 5)\).

Assume that \(\pi(\mathcal{C})\) is collinear. Then \(p^{-1}(\pi(\mathcal{C}))\) has dimension 5 and contains \(<\mathcal{P}\>). But clearly \(f\) is degenerate on \(p^{-1}(\pi(\mathcal{C}))\). This contradiction proves the lemma. \(\square\)

**Lemma 3.6.** \(\mathcal{X} - \mathcal{X}_0\) does not contain a 5-set \(\mathcal{C}\) such that \(\pi(\mathcal{C})\) is collinear.

**Proof.** Assume by contradiction that \(\mathcal{X} - \mathcal{X}_0\) contains a 5-set \(\mathcal{C}\) such that \(\pi(\mathcal{C})\) is collinear, and define
\[
A := \begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Then \(\{AXA : X \in \mathcal{C}\}\) is a 5-set of \(\mathcal{X}_0\), the image of which under \(\pi\) is collinear. Thus we have a contradiction to Lemma 3.5. \(\square\)

**Lemma 3.7.** Let \(\mathcal{P}(5)\) denote the point set of the projective plane \(\text{PG}(2, 5)\) and let \(\varepsilon : \mathcal{P}(5) \to \{1, -1\}\) be a function. Then \(\mathcal{P}(5)\) contains a collinear 5-set on which \(\varepsilon\) is constant.

**Proof.** For each \(i \in \{0, 1, \ldots, 6\}\) let \(\mathcal{G}_i\) be the set of those lines of \(\text{PG}(2, 5)\) which are incident with precisely \(i\) elements of \(\varepsilon^{-1}(1)\). Further, let \(x_i := |\mathcal{G}_i|\).

Assume by contradiction that \(\mathcal{P}(5)\) does not contain a collinear 5-set, on which \(\varepsilon\) is constant.

Then \(x_0 = x_1 = x_4 = x_6 = 0\).

Count the lines of \(\text{PG}(2, 5)\), the lines of \(\text{PG}(2, 5)\) incident with some point of \(\varepsilon^{-1}(1)\), and the ordered pairs of points in \(\varepsilon^{-1}(1)\) to obtain
\[
\Sigma x_i = 31,
\]
\[
\Sigma ix_i = 6s,
\]
\[
\Sigma i(i - 1)x_i = s(s - 1),
\]
where $s := |e^{-1}(1)|$. Then the resulting system of linear equations has no solution for positive integers $s$, $x_3$, $x_4$, $x_5$.

Now the proof of Theorem 3.1 follows from the last three lemmata.

REFERENCES


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