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# Totally frustrated states in the chromatic theory of gain graphs

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**Abstract**

We generalize proper coloring of gain graphs to totally frustrated states, where each vertex takes a value in a set of ‘qualities’ or ‘spins’ that is permuted by the gain group. In standard coloring the group acts trivially or regularly on each orbit (an example is the Potts model), but in the generalization the action is unrestricted. We show that the number of totally frustrated states satisfies a deletion–contraction law. It is not matroidal except in standard coloring, but it does have a formula in terms of fundamental groups of edge subsets. One can generalize chromatic polynomials by constructing spin sets out of repeated orbits. The dichromatic and Whitney-number polynomials of standard coloring generalize to evaluations of an abstract partition function that lives in the edge ring of the gain graph.


## 1. States, colorations, and all that

When counting the proper colorations of a graph, whose number is given by the chromatic polynomial, all one needs to know is the number of colors in the color set; the nature of the individual colors is immaterial. When coloring a *gain graph*, where the edges are labelled by elements of a group (called the *gain group*), that is no longer so. In order for the concept of propriety of a coloration to be meaningful the group must have a permutation action on the set of colors, and then the exact way the group acts is a crucial factor in counting proper colorations. In this article we want to generalize properties of the chromatic polynomial to gain graphs.

As we shall see, one of the major properties, that the number of proper colorations in $\lambda$ colors is a polynomial function of $\lambda$ that depends on the graphic matroid, has an analog for gain graphs

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only when the group action on the color set is sharply restricted. Nevertheless, an even more basic property, the law of deletion and contraction, holds good for every color set with any group action, and when the group is finite there is also a generalized chromatic polynomial, though unlike the ordinary chromatic polynomial it seems not to be associated to a matroid.

We call a labelling of the vertices of a gain graph by elements of a set acted upon by the gain group a *state* and the elements of the color set *spins*. (We eschew standard coloring terminology because the properties of states are so much weaker than those of colorations.) The names come from the theory of spin glasses in physics [14]. A *signed graph* [8] has edges labelled from the two-element group, $\{+, -\}$, which acts upon the spin set $\{+1, -1\}$ in the obvious way. In one of the simplest spin glass models, which we call the *mixed Ising model* (see [1] or [4, Section 2.5]), a spin glass is a signed graph and a state is an assignment of a spin, $+1$ or $-1$, to each vertex. (In the standard Ising model all edges are positive or all are negative.) In the mixed Ising model an edge wants to have the same spin at both ends if it is positive and opposite spins if it is negative; if it does, it is 'satisfied', but if not, it is 'frustrated' [14] and has a higher energy. The most basic questions are these: Given the edge signs (which are fixed by the physical material), what state has the fewest frustrated edges and hence the lowest energy (see, e.g., [7])? And, how many states have the fewest frustrated edges?

It occurred to me that, turning the question around to *totally frustrated states*, in which no edge is satisfied, one has a generalization of graph coloring. If the gain group is trivial, then the spin set can be any set and a state is a proper coloration precisely when it is totally frustrated; this is ordinary graph coloring by another name. Turning to general gain graphs, there are many reasons to be interested in states when the gain group is not limited to one or even two members; for instance, spins from a finite set appear in the Potts model (as explained later) and from a real vector space in the work of Rybnikov and Zaslavsky on lifting of piecewise-linear cell complexes [10, Section 5], and then there is the entire theory of gain-graph coloring from [19, Section 5] and [21, Section 4], where the spin set consists mostly of copies of the gain group (and the number of proper colorations is matroidal). It therefore seemed desirable to study the general properties of totally frustrated states, and that is what we do here.

### *1.1. States of total frustration*

A *gain graph* is a graph with a function $\varphi$ that assigns to each oriented edge $e$ an element $\varphi(e)$ of a group, called the *gain group*, in such a way that reorienting the edge inverts the gain. A *state* $s$ of the gain graph (introduced in [10, Section 5]) is an assignment to each vertex of an element of some set $\mathbf{Q}$ upon which the gain group acts; $\mathbf{Q}$ is called the set of *qualities* (in [11]) or *spins* (in physics). A *state* $s$ of a gain graph (introduced in [10, Section 5]) is an assignment to each vertex of an element of the spin set $\mathbf{Q}$. We shall be investigating states of gain graphs.

With a gain graph and a state, we can classify the edges as *satisfied* or *frustrated*: the former if, taking the edge $e$ to be oriented from vertex $v$ to vertex $w$, the equation $s_w = s_v\varphi(e)$ is satisfied, and the latter if the equation is unsatisfied. What has been studied heretofore in connection with states has been principally the question of whether a state is satisfied (i.e., has no frustrated edges) or not and, if not, just how unsatisfied it is. However, if we turn to states in which *no* edge is satisfied, we discover a generalization of a classic problem of graph theory, the problem of proper coloring. Our objective is to examine coloring of gain graphs from the point of view of these *totally frustrated states*, and in particular, the behavior of the $\mathbf{Q}$-*chromatic function* $\chi_\Phi(\mathbf{Q})$, which is the number of such states, as a function of $\mathbf{Q}$.

### 1.2. Properly colored

In the standard theory of gain-graph coloring, from [21, Section 4] (the source for all properties cited herein), the color set consists of $k$ copies of the gain group $\mathfrak{G}$ and an extra fixed point; it is

$$\mathbf{C}_k := \mathbf{C}_k^* \cup \{0\}, \qquad \text{where } \mathbf{C}_k^* := \mathfrak{G} \times [k],$$

for $k$ a nonnegative integer, with $[k] := \{1, 2, \ldots, k\}$ (which is the void set if $k = 0$). The number of proper colorations is a polynomial function $\chi_\Phi(\lambda)$ of $\lambda = |\mathbf{C}_k| = k|\mathfrak{G}| + 1$, naturally called the *chromatic polynomial*, that satisfies the standard deletion–contraction relation

$$f(\Phi) = f(\Phi \setminus e) - f(\Phi/e) \tag{1.1}$$

for all edges $e$. The number of proper colorations with colors taken only from $\mathbf{C}_k^*$ is another polynomial function $\chi_\Phi^{\mathrm{b}}(\lambda)$, the *zero-free chromatic polynomial*, where now $\lambda = |\mathbf{C}_k^*| = k|\mathfrak{G}|$. The zero-free chromatic polynomial obeys the deletion–contraction rule for edges that are not loops, and its value is not changed by the deletion of nonidentity loops.

We want to relax the definition by admitting any finite spin set $\mathbf{Q}$, defining the state chromatic function which counts the number of totally frustrated states, and find out which properties are preserved and which are lost. Once we have done so, in order to understand better the algebraic properties of the state chromatic function we develop an abstract partition function, which lies in the polynomial ring generated by the edge set and which contains by evaluation and specialization not only the state chromatic function but state generalizations of the dichromatic and Whitney-number polynomials of a gain graph.

### 1.3. Coloring from equivalence classes

The example that inspired this thought is set coloring. A *proper set coloration* of a graph $\Delta$ is an assignment to each vertex of a set $S_v \subseteq [k]$ in such a way that adjacent vertices have sets of different sizes. This is a totally frustrated state of an associated gain graph. To form the gain graph, let the gain group be $\mathfrak{S}_k$, the group of permutations of $[k]$, and let the spin set $\mathbf{Q}$ be the class $\mathcal{P}([k])$ of subsets of $[k]$ with the natural action of $\mathfrak{S}_k$. Then a proper set coloration is a totally frustrated state of the gain graph $\mathfrak{S}_k\Delta$, called the $\mathfrak{S}_k$-*expansion* of $\Delta$, which has an edge of every possible gain between each pair of adjacent vertices. Let $\chi_\Delta^{\mathrm{set}}(k)$ be the number of proper set colorations of $\Delta$. This quantity is not a polynomial in any of $k$, $|\mathbf{Q}| = 2^k$, or $|\mathfrak{S}_k| = k!$, so we lose something from the standard theory of graph coloring. Not all is lost, however. There is still a deletion–contraction formula, though only in terms of the gain graph, and $\chi_\Delta^{\mathrm{set}}(k)$ is multiplicative on connected components, so $\chi_\Delta^{\mathrm{set}}(k)$ is what is called a Tutte invariant of gain graphs. Our first theorem will be that this is true for any group and any finite set of spins.

Set colorations exemplify a kind of coloring we call *coloring from equivalence classes*. Say a group $\mathfrak{G}$ acts on a finite spin set $\mathbf{Q}$, with orbits $\mathbf{Q}_1, \ldots, \mathbf{Q}_r$. The $\mathfrak{G}$-*expansion* of a graph $\Delta$ is the gain graph $\mathfrak{G}\Delta$ in which each edge of $\Delta$ is replaced by edges having every gain in $\mathfrak{G}$. Then a totally frustrated state of $\mathfrak{G}\Delta$ is the same as a coloration of the vertices of $\Delta$ such that adjacent vertices have inequivalent colors under the equivalence relation defined by the orbits.

### 1.4. Potts

Another example – in fact, it is an example of zero-free gain-graph coloring – is the *mixed Potts model*, which abstracts a partially disordered physical system such as a spin glass. There is a signed graph $(\Delta, \sigma)$ and there is a finite set of spins, with which we can form a state $s : V \to \mathbf{Q}$. A positive edge is satisfied when it has the same spin at both ends; a negative edge is satisfied when its endpoints have different spins. (When there are two spins this is the mixed Ising model. When all edges are positive it is the usual Potts model.) As in the Ising model a state has an 'energy' which is a decreasing function of the number of satisfied edges. One of the important questions is to find the lowest energy of a state, and especially whether there exists a completely satisfied state. (This account is very abbreviated. For a proper exposition with only positive edges see [18, Section 4.4]. The case of all negative edges is discussed in [12]. The generalization to two kinds of edges is found in the physics literature and also in [3] as interpreted in [22].)

To turn the mixed Potts model into a gain graph, assume $\mathbf{Q}$ is a group with identity element 1. The Potts gain graph $\Phi$ has an edge with gain 1 where $\Delta$ has a negative edge and it has edges with all nonidentity gains wherever $\Delta$ has a positive edge. A lowest-energy state of the mixed Potts model is a state with the most frustrated edges in $\Phi$; the Potts model is satisfied when $\Phi$ is totally frustrated; and the number of frustrated edges in the mixed Potts model is the number of satisfied edges of $\Phi$. And in particular, the number of ways to satisfy the mixed Potts model is the number of zero-free proper 1-colorations of $\Phi$, i.e., the value of $\chi^{\mathrm{b}}_{\Phi}(|\mathfrak{G}|)$.

## 2. General theory of total frustration

### 2.1. Technical basis

A graph $\Gamma = (V, E)$ may have loops and multiple edges. All our graphs have finite order $|V|$. A *link* is an edge that is not a loop. A vertex is *isolated* if it is not incident with any link (but it may support loops). The standard closure operator on the edge set of a graph is

$$\operatorname{cl} A := A \cup \{e \notin A : \text{ the endpoints of } e \text{ are connected in } A\}.$$

This is the closure operation of the graphic matroid (or 'cycle matroid') $G(\Gamma)$. The notation $e{:}vw$ means that $e$ is an edge whose endpoints are $v$ and $w$, which are equal for a loop. If $e$ needs to be oriented (e.g. when evaluating its gain), the notation implies an orientation from $v$ to $w$. The chromatic polynomial of $\Gamma$ is written $\chi_\Gamma(\lambda)$. When we speak of the *(connected) components* of an edge set $A$, we mean the connected components of the spanning subgraph $(V, A)$; $c(A)$ is the number of components.

A gain graph $\Phi = (\Gamma, \varphi)$ consists of an underlying graph $\Gamma = (V, E)$ and an orientable function $\varphi : E \to \mathfrak{G}$, where $\mathfrak{G}$ is the gain group. We call $\varphi$ the *gain function* and $\varphi(e)$ the *gain* of $e$. By calling $\varphi$ 'orientable' we mean that its value depends on the direction of $e$ and if the direction is reversed, the gain $\varphi(e)$ is inverted. Symbolically, letting $e^{-1}$ denote $e$ with the opposite orientation, $\varphi(e^{-1}) = \varphi(e)^{-1}$. A gain graph, or an edge set in it, is *balanced* if every simple closed walk has gain, obtained by multiplying the gains of its edges in cyclic order, equal to 1, the group identity. The number of connected components of $\Phi$ that are balanced is written $b(\Phi)$, and for an edge set $A$ in $\Phi$, $b(A)$ denotes the number of balanced components of $(V, A)$. The induced subgraph on a vertex set $W$ is written $\Phi{:}W$; the gains are as in $\Phi$. An *isomorphism* of gain graphs is a graph isomorphism that preserves the gains of edges.

The principal matroid in this work is the bias, or frame, matroid $G(\Phi)$ [20]. Its points are the edges of $\Phi$ and its rank function is $r(A) = |V| - b(A)$. The frame matroid generalizes the usual graphic matroid, since when $\Phi$ is balanced, $G(\Phi) = G(\Gamma)$. The class of flats determines $G(\Phi)$, of course. The class of balanced flats is a geometric semilattice [17] that determines what I call the *balanced semimatroid* of $\Phi$, which may be defined as the class of balanced edge sets with rank as in $G(\Phi)$. In general one may think of a semimatroid $S$ as the class of sets in a matroid $M$ on $E(S) \cup \{e_0\}$ whose closures do not contain $e_0$, together with the rank function on these sets. We call $M$ the *completion* of the semimatroid. According to [17, Theorem 3.2], the completion is unique. The completion of the balanced semimatroid of $\Phi$ is the complete lift matroid $L_0(\Phi)$ [20], whose contraction $L_0(\Phi)/e_o$ equals $G(\Gamma)$.

A *spin set* is a set $\mathbf{Q}$ upon which there is a right action of $\mathfrak{G}$. The action is *trivial* if every spin is fixed by every group element, *semiregular* if only the identity element has any fixed points, and *regular* if it is semiregular and transitive. A *state* is any function $s : V \to \mathbf{Q}$. It is *totally frustrated* if every edge is frustrated. What we are studying, a pair $(\Phi, \mathbf{Q})$ of a gain graph together with a spin set, is called a *permutation gain graph*. (This concept was introduced in [6], though our definition is slightly more general.)

A fundamental operation on gain graphs is *switching*. A *switching function* $\eta : V \to \mathfrak{G}$ gives a switched graph $\Phi^\eta$ whose underlying graph is the same as that of $\Phi$ and whose gain function is $\varphi^\eta$, defined by $\varphi^\eta(e) := \eta_v^{-1}\varphi(e)\eta_w$ for any edge $e{:}vw$. It is always possible to switch so that a given link has gain 1, and indeed so that the gains on a chosen forest or any balanced edge set are all 1. The *switching class* of a gain graph is the class of all gain graphs that are switchings of $\Phi$.

We must define deletion and contraction of an edge. Deletion is obvious. To contract a link $e$ we need switching. First we switch so $e$ has gain 1, then we delete it and identify its endpoints. The gains do not change except in the switching step. (Contraction of a loop will not be needed; for it one may consult [19].) The fact that a contraction $\Phi/e$ is uniquely defined only up to switching is a reason to consider the switching class of a gain graph to be more fundamental than the gain graph itself.

There is one more aspect of switching that is essential: switching acts on states as well as gains. We define $s^\eta$ by $s_v^\eta := s_v\eta_v$; in words, a switching function acts on a state in the obvious way. The point is that the set of satisfied edges, $I_\Phi(s)$, remains the same:

$$I_{\Phi^\eta}(s^\eta) = I_\Phi(s).$$

Obviously, therefore, the number of totally frustrated states is unaffected by switching and when contracting an edge we may assume that $\Phi$ is already switched so that $\varphi(e) = 1$.

### 2.2. The chromatic function

Let $\Phi$ be a gain graph with gain group $\mathfrak{G}$. Choosing a spin set $\mathbf{Q}$ (with a $\mathfrak{G}$-action), the *state chromatic function* of $\Phi$ is

$$\chi_\Phi(\mathbf{Q}) := \text{the number of totally frustrated states.}$$

This is a finite number if $\mathbf{Q}$ is finite.

**Proposition 2.1.** *If $\Phi$ is balanced, then $\chi_\Phi(\mathbf{Q}) = \chi_\Phi(|\mathbf{Q}|) = \chi_\Gamma(|\mathbf{Q}|)$.*

**Proof.** By switching we may assume all gains equal 1. Clearly, then $\chi_\Phi(\mathbf{Q}) = \chi_\Gamma(|\mathbf{Q}|) = \chi_\Phi(|\mathbf{Q}|)$. □

**Theorem 2.2.** *If $\mathbf{Q}$ is finite, the state chromatic function of gain graphs of finite order has the deletion–contraction property (1.1) with respect to all links e.*

**Proof.** Assume $\Phi$ switched so that $e$ has identity gain. We simply classify the totally frustrated states of $\Phi \setminus e$ according to whether $e$ is or is not frustrated. A state for which $e$ is frustrated is a totally frustrated state of $\Phi$. The criterion for $e$ to be satisfied is that its endpoints have the same spin. Hence a state in which $e$ is satisfied contracts to a totally frustrated state of $\Phi/e$, and conversely, any totally frustrated state of $\Phi/e$ defines a unique state of $\Phi$ in which $e$ and only $e$ is satisfied. This proves the theorem. □

### 2.3. States vs. colorations

The difference between a state, with an arbitrary spin set, and a coloration, whose spin set (or 'color set') is $\mathbf{C}_k$ or $\mathbf{C}_k^*$, is that in a coloration the spin set yields properties very similar to those of ordinary graph coloring. For instance, the set of frustrated edges in a coloration is closed in the frame matroid $G(\Phi)$ [21].

One could say that the crux of the difference is the behavior of loops—not surprisingly in view of Theorems 2.4 and 2.5, because knowing which spins on its supporting vertex satisfy a loop with gain $g$ is the same as knowing the fixed points of $g$ acting on $\mathbf{Q}$, and that is what decides whether $\chi_\Phi(\mathbf{Q})$ equals a chromatic polynomial. The proofs of those theorems show that the most basic question about loops is whether the number of totally frustrated states of a nonidentity loop is affected by the exact gain, or in other words, whether every $g \neq 1$ fixes the same number of spins.

### 2.4. Decomposition

A normalization of the state chromatic function is

$$p_\Phi(\mathbf{Q}) := |\mathbf{Q}|^{-b(\Phi)} \chi_\Phi(\mathbf{Q}).$$

The same normalization applied to the chromatic polynomial, i.e., $\lambda^{-b(\Phi)}\chi_\Phi(\lambda)$, gives the characteristic polynomial of $G(\Phi)$ [21, Section 5].

**Proposition 2.3.** *Assume finite $\mathbf{Q}$ and $\Phi$ and suppose $\Phi'$ and $\Phi''$ are subgraphs whose union is $\Phi$. If they are disjoint, or if their intersection is a single vertex and at least one of them is balanced, then*

$$p_\Phi(\mathbf{Q}) = p_{\Phi'}(\mathbf{Q}) p_{\Phi''}(\mathbf{Q}).$$

**Proof.** If $\Phi'$ and $\Phi''$ are vertex disjoint, then multiplicativity is obvious. From this one can see that it suffices to assume $\Phi'$ and $\Phi''$ are connected.

Suppose the intersection is a vertex $v$ and $\Phi'$ is balanced. A state of $\Phi$ is totally frustrated if and only if it is assembled from a totally frustrated state $s''$ of $\Phi''$ and a totally frustrated state $s'$ of $\Phi'$ that agrees with $s''$ on $v$. The question is how the number of such $s'$ depends on $s''_v$. The number is independent of $s''_v$, indeed it equals $\chi_{\Phi'}(\mathbf{Q})/|\mathbf{Q}|$ (by switching as at Proposition 2.1), hence it equals $p_{\Phi'}(\mathbf{Q})$. Multiplicativity follows. □

A particular case is the obvious but important fact that the chromatic function is multiplicative on connected components: if $\Phi$ has components $\Phi_1, \Phi_2, \ldots, \Phi_m$ then

$$\chi_\Phi(\mathbf{Q}) = \chi_{\Phi_1}(\mathbf{Q})\chi_{\Phi_2}(\mathbf{Q}) \cdots \chi_{\Phi_m}(\mathbf{Q}). \tag{2.1}$$

This, together with the deletion–contraction law and the facts that $\chi_\Phi(\mathbf{Q})$ is an isomorphism invariant and $\chi_\varnothing(\mathbf{Q}) = 1$, means that the state chromatic function satisfies the definition of a Tutte invariant of gain graphs, thus being another one in a long list of such invariants.

### 2.5. *Chromatic or not chromatic*

Given $\mathbf{Q}$ and $\mathfrak{G}$, to say that $\chi_\Phi(\mathbf{Q})$ is an *evaluation* of a function $F_\Phi(\lambda)$ means there is a fixed value $\lambda_0$ such that $\chi_\Phi(\mathbf{Q}) = F_\Phi(\lambda_0)$ for every permutation gain graph $(\Phi, \mathbf{Q})$ with gain group $\mathfrak{G}$.

**Theorem 2.4.** *Assume $\mathbf{Q}$ is a finite set of spins and $\mathfrak{G}$ is a finite group acting on $\mathbf{Q}$.*

(a) *If there is $q_0 \in \mathbf{Q}$ such that every nonidentity group element has fixed set $\{q_0\}$, then $\chi_\Phi(\mathbf{Q})$ is the evaluation of the chromatic polynomial $\chi_\Phi(\lambda)$ at $\lambda = |\mathbf{Q}|$. If there is no such $q_0$, then $\chi_\Phi(\mathbf{Q})$ is not an evaluation of $\chi_\Phi(\lambda)$.*
(b) *If $\mathfrak{G}$ acts semiregularly on $\mathbf{Q}$, then $\chi_\Phi(\mathbf{Q})$ is the evaluation of the zero-free chromatic polynomial $\chi^{\mathrm{b}}_\Phi(\lambda)$ at $\lambda = |\mathbf{Q}|$. If the action is not semiregular, then $\chi_\Phi(\mathbf{Q})$ is not an evaluation of $\chi^{\mathrm{b}}_\Phi(\lambda)$.*
(c) *If the action of $\mathfrak{G}$ is trivial, then $\chi_\Phi(\mathbf{Q}) = \chi_\Gamma(|\mathbf{Q}|)$, the evaluation at $|\mathbf{Q}|$ of the chromatic polynomial of the underlying graph. If the action is nontrivial, then $\chi_\Phi(\mathbf{Q})$ is not an evaluation of $\chi_\Gamma(\lambda)$.*

**Proof.** We prove the first implication in part (a) in stages. The underpinning is that the chromatic polynomial satisfies deletion–contraction for all links. Thus, if we prove the theorem for graphs without links, it follows by induction on the number of edges using Theorem 2.2. The chromatic polynomial and the state chromatic function both equal zero when $\Phi$ has an identity loop, so we may assume $\Phi$ has no edges other than nonidentity loops. Furthermore, both chromatic polynomial and state chromatic function are multiplicative on connected components, so we may assume $\Phi$ is connected. That is, $\Phi$ has a single vertex with some number of nonidentity loops.

If there are no loops, $\chi_\Phi(\mathbf{Q}) = |\mathbf{Q}|$ and $\chi_\Phi(\lambda) = \lambda$; therefore $\lambda = |\mathbf{Q}|$. If there is at least one loop, then $\chi_\Phi(\lambda) = \lambda - 1$. Now, let $G$ be the set of gains of the loops of $\Phi$. To be totally frustrated, a state $s$ must have $s_v \notin \mathrm{Fix}(g)$, the fixed set of $g$, for every $g \in G$. The only way this can give $\lambda - 1$ totally frustrated states is for $\mathrm{Fix}(g)$ to be the same set $F$ for every nonidentity element of the gain group and for $\lambda = |\mathbf{Q}| - |F| + 1$. It follows that $|F| = 1$.

So, we have necessary conditions for the state chromatic function to be an evaluation of the chromatic polynomial, but the proof also shows their sufficiency. That concludes the proof of part (a).

The proof of part (b) is similar. A nonidentity loop is never satisfied so it can be discarded without altering the number of totally frustrated states.

For part (c), note that if the action is trivial, then the gains do not matter. Conversely, if there is a $g \in \mathfrak{G}$ with nontrivial action, consider the gain graph with one vertex and one loop, whose gain is $g$. Then $\chi_\Gamma(\lambda) = 0$ but $\chi_\Phi(\mathbf{Q}) = |\mathbf{Q}| - |\mathrm{Fix}(g)| \neq 0$. □

Theorem 2.4(a, b) demonstrate that the state chromatic function equals the chromatic polynomial or zero-free chromatic polynomial only when $\mathbf{Q}$ is essentially a color set of the form $\mathbf{C}_k$ or $\mathbf{C}^*_k$, respectively.

### 2.6. *Matroid invariance*

We strengthen the second halves in Theorem 2.4(a, b, c) to a characterization of when the number of totally frustrated states is a matroid or semimatroid invariant. The matroid involved is the frame matroid $G(\Phi)$.

**Theorem 2.5.** *Let* $\mathbf{Q}$ *be a finite spin set and* $\mathfrak{G}$ *a group acting on* $\mathbf{Q}$.

(a) *The state chromatic function, as a function of the gain graph, is determined by the frame matroid* $G(\Phi)$ *and the numbers of components and balanced components of* $\Phi$, *if and only if* $\mathbf{Q}$ *contains a point* $q_0$ *as in Theorem* 2.4(a).

(b) *The state chromatic function, as a function of the gain graph, is determined by the balanced semimatroid of* $\Phi$ *and the numbers of components and balanced components of* $\Phi$ *if and only if* $\mathfrak{G}$ *acts semiregularly or trivially upon* $\mathbf{Q}$, *as in Theorem* 2.4(b)*or* (c).

**Proof of (a).** If $\mathbf{Q}$ does contain a $q_0$, then it is an evaluation of $\chi_\Phi(\lambda)$ (by Theorem 2.4(a)), which in turn is equal to $\lambda^{b(\Phi)}$ times the characteristic polynomial of $G(\Phi)$ [21, Section 5].

For the converse, suppose the state chromatic function is determined by the stated information. Let $g, h \in \mathfrak{G}$, both not the identity, but possibly equal.

The gain graph $\Phi_g$ that consists of one vertex and one loop with gain $g$ has matroid isomorphic to a coloop. The chromatic function is $|\mathbf{Q}| - |\operatorname{Fix}(g)|$. Since $\Phi_h$ has the same matroid and component numbers as $\Phi_g$, it must have the same chromatic function. It follows that every group element other than 1 must have the same number $f$ of fixed points.

The gain graph $\Phi_{g,h}$ has vertex $v_1$ with a loop of gain $g$ and $v_2$ with a loop of gain $h$ and a link $e{:}v_1v_2$ with gain 1. The description of a totally frustrated state is that at $v_1$ the spin is $q_1 \notin \operatorname{Fix}(g)$ and at $v_2$ the spin is $q_2 \notin \{q_1\} \cup \operatorname{Fix}(h)$. The chromatic function is

$$\begin{aligned}\chi(\mathbf{Q}) &= \sum_{q_1 \in \mathbf{Q} \setminus \operatorname{Fix}(g)} [|\mathbf{Q}| - |\{q_1\} \cup \operatorname{Fix}(h)|] \\ &= \sum_{q_1 \notin \operatorname{Fix}(g) \cup \operatorname{Fix}(h)} [|\mathbf{Q}| - f - 1] + \sum_{q_1 \in \operatorname{Fix}(h) \setminus \operatorname{Fix}(g)} [|\mathbf{Q}| - f] \\ &= (|\mathbf{Q}| - f)(|\mathbf{Q}| - f - 1) + |\operatorname{Fix}(h) \setminus \operatorname{Fix}(g)|.\end{aligned}$$

This value cannot depend on the choices of $g, h \neq 1$ because those do not change the matroid. Since taking $g = h$ gives value $(|\mathbf{Q}| - f)(|\mathbf{Q}| - f - 1)$, it follows that every nonidentity element has the same fixed set.

In effect, $\mathbf{Q}$ is the disjoint union

$$(\mathbf{Q}_1 \times [k_1]) \cup (\mathbf{Q}_2 \times [k_2]) \quad \text{where } \mathbf{Q}_1 = \mathfrak{G} \quad \text{and} \quad |\mathbf{Q}_2| = 1. \tag{2.2}$$

(That is because the nontrivial orbits of $\mathbf{Q}$ have no fixed points of any nonidentity element of $\mathfrak{G}$.) $\mathfrak{G}$ acts on $\mathbf{Q}$ by acting on the first component of each pair in $\mathbf{Q}$. We compare two gain graphs.

Consider first the gain graph $\Phi_2$ that has vertices $v_1$ and $v_2$ with two links joining them, one having gain 1 and the other with gain $g \neq 1$. To get a totally frustrated state we choose spin $q_1$ for $v_1$. There are $|\mathbf{Q}| - 1$ choices for $q_2$ at $v_2$ if $q_1 \in \mathbf{Q}_2 \times [k_2]$ but $|\mathbf{Q}| - 2$ choices if $q_1 \in \mathbf{Q}_1 \times [k_1]$, since selecting $q_1$ in the latter set implies $q_2 \neq q_1, q_1 g$. Thus, $\chi_{\Phi_2}(\mathbf{Q}) = (|\mathbf{Q}| - 1)^2 + k_2 - 1$.

Second, consider $\Phi_1$ that has the same vertices and a link with gain 1, but also a loop at $v_1$ with gain $g \neq 1$. The matroid and the numbers of components and balanced components are the same, but the spin at $v_1$ must belong to $\mathbf{Q}_1 \times [k_1]$, so $\chi_{\Phi_1}(\mathbf{Q}) = (|\mathbf{Q}| - 1)^2 - (k_2 - 1)(|\mathbf{Q}| - 1)$. We conclude that the only case in which the state chromatic function can be determined by the information provided in part (a) is that in which $k_2 = 1$. □

**Proof of (b).** If $\mathfrak{G}$ has semiregular action, then $\chi(\mathbf{Q})$ is an evaluation of $\chi^{\mathrm{b}}_\Phi(\lambda)$ (by Theorem 2.4(b, c)), which in turn is equal to $\lambda^{b(\Phi)}$ times the characteristic polynomial of the semilattice of balanced flats [21, Section 5].

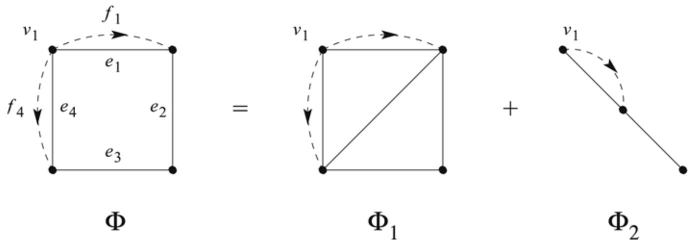


Fig. 1. A gain graph $\Phi$ with four identity edges $e_i$ and two adjacent edges that have gain $\varphi(f_i) = g \neq 1$, showing how its chromatic function decomposes by addition and contraction of an edge having gain 1. (In the contracted graph, multiple edges with identical gain are suppressed.)

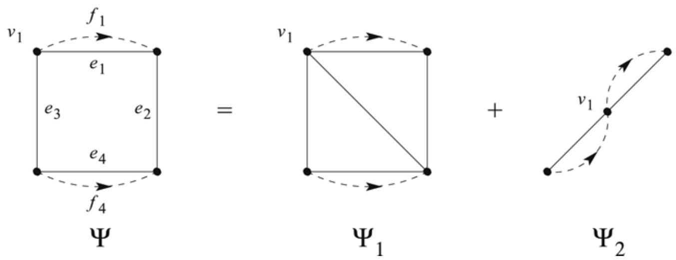


Fig. 2. A gain graph $\Psi$ with four identity edges $e_i$ and two nonadjacent edges that have gain $\varphi(f_i) = g \neq 1$.

If $\mathfrak{G}$ has trivial action, we know $\chi_\Phi(\mathbf{Q}) = \chi_\Gamma(k_2)$ by Proposition 2.1, but we have yet to prove that $\chi_\Gamma(\lambda)$ is determined by the balanced semimatroid. This follows from the fact that the completion of the balanced semimatroid is $L_0(\Phi)$, whose contraction by the extra point $e_0$ is the graphic matroid $G(\Gamma)$. Since the completion is unique, the balanced semimatroid determines $G(\Gamma)$; this in turn determines $\chi_\Gamma(\lambda)$ as $\lambda^{c(\Gamma)}$ times the characteristic polynomial of $G(\Gamma)$, where $c(\Gamma)$ is the number of components of $\Gamma$. Hence, the semimatroid does determine the number of totally frustrated states.

The proof of the converse proceeds in the same way as in part (a) to establish the form of $\mathbf{Q}$, i.e., Eq. (2.2), since the data of part (b) agree for all the graphs we compared in the steps leading to that conclusion.

The final step is more complicated than in part (a) because we need nonisomorphic gain graphs that have the same balanced sets and their ranks. That is impossible with only two vertices. Figs. 1 and 2 show two gain graphs with this property. The balanced sets are all those that do not contain a digon and, if they contain exactly one $f_i$, do not complete a circuit of four edges; these sets are the same in both graphs.

To calculate the chromatic function I used deletion–contraction in reverse, by means of which the chromatic function of each graph is expressed as the sum of two other chromatic functions that are easier to work with (see the figures). I calculated the number of totally frustrated states of each of the four graphs by the usual hand method of building the state from vertex to vertex, starting with spin $q_1$ at $v_1$ and treating $q_1 \in \mathbf{Q}_1 \times [k_1]$ and $q_1 \in \mathbf{Q}_2 \times [k_2]$ separately. I checked the result by comparing it, with $k_2 = 0$, to the zero-free chromatic polynomial computed from the semilattice of balanced flats as in [21, Section 5], which by Theorem 2.4(b) ought to be the same (and is). I omit the lengthy details. The conclusion is that

$$\chi_\Phi(\mathbf{Q}) = \lambda(\lambda - 2)[\lambda^2 - 4\lambda + 5] + k_2\left[2\lambda^2 - 7\lambda + 7\right],$$

$$\chi_{\Psi}(\mathbf{Q}) = \lambda(\lambda-2)[\lambda^2-4\lambda+5] + k_2\left[2\lambda^2-8\lambda+7+k_2\right],$$

where $\lambda = |\mathbf{Q}|$. The difference between these is $k_2(\lambda - k_2)$. Therefore, they are equal only when either $k_2 = 0$, in which case the action of $\mathfrak{G}$ is semiregular (and the number of totally frustrated states is $\chi^{\mathrm{b}}_{\Phi}(|\mathbf{Q}|)$), or $k_1 = 0$, where the action is trivial. □

### 2.7. Multiplicativity is not matroidal

In Proposition 2.3, the edge sets of the subgraphs are complementary separators of the frame matroid $G(\Phi)$. It is natural to ask whether $p(\mathbf{Q})$ is multiplicative on any subgraphs $\Phi'$ and $\Phi''$ whose edge sets are complementary matroid separators. The answer is no. We prove this with an example.

Suppose $\Phi$ is unbalanced and $e$ is a link such that $\Phi'' := \Phi \setminus e$ is balanced; then $\{e\}$ and its complement are separators of the matroid. Consider the specific example of $K_n$ with $\mathbf{Q}$ of order $\lambda$, where $\lambda \geq n \geq 3$, with a regular action (we assume $\mathbf{Q} = \mathfrak{G}$) and with gains all 1 except for one edge $e{:}v_1v_n$ having gain $g_0 \neq 1$. A totally frustrated state is the same as a proper $\mathfrak{G}$-coloration and the state chromatic function equals $\chi^{\mathrm{b}}_{\Phi}(\lambda)$.

It is easy to see that $p_{\Phi\setminus e}(\mathbf{Q}) = (\lambda-1)_{n-2}(\lambda-n+2)$, $(x)_k$ denoting the falling factorial, and $p_e(\mathbf{Q}) = (\lambda-1)$, $p_e(\mathbf{Q})$ being the normalized state chromatic function of the subgraph induced by $e$.

It is also true that $p_{\Phi}(\mathbf{Q}) = \chi^{\mathrm{b}}_{\Phi}(\lambda)$ (since $\Phi$ is unbalanced) $= \lambda(\lambda-2)_{n-3}(\lambda^2-n\lambda+2n-3)$. We prove it by counting proper $\mathfrak{G}$-colorations. First, color $v_1$ by a fixed color $q_1$. Then color $v_2, \ldots, v_{n-1}$; there are two ways to do so, either (a) using the color $q_1g_0$ or (b) not using it. Finally, color $v_n$ by $q_n$ under the restriction $q_n \neq q_1g_0, q_2, \ldots, q_{n-1}$. In (a) there are $n-2$ choices for where to put the color $q_1g_0$ and $(\lambda-2)_{n-3}$ ways to complete the coloring of all but $v_n$; then there are $\lambda-n+2$ colors $q_n$ available for $v_n$. In (b) there are $(\lambda-2)_{n-2}$ ways to color $v_2, \ldots, v_{n-1}$, as the color $q_1g_0$ cannot be used, and then $\lambda-n+1$ colors available for use at $v_n$. The total number of proper colorations is the sum of (a) and (b), multiplied by $\lambda$; it simplifies to $\lambda(\lambda-2)_{n-3}(\lambda^2-n\lambda+2n-3)$.

Equality of $p_e(\mathbf{Q})p_{\Phi\setminus e}(\mathbf{Q})$ with $p_{\Phi}(\mathbf{Q})$ implies, since $(\lambda-2)_{n-3} > 0$, that $\lambda^3-n\lambda^2+(2n-3)\lambda-(n-2) = \lambda^3-n\lambda^2+(2n-3)\lambda$. Because $n \geq 3$, equality is impossible.

## 3. A formula with the fundamental group

To get a more precise formula for the chromatic function we need a new concept.

### 3.1. The fundamental group

First we define the gain of a walk $W$: it has gain $\varphi(W)$ equal to the product of the gains of its edges, in the order and direction they are traversed by $W$.

Now, take an edge set $A \subseteq E$. Fix $v_0 \in V$ and let $A_0$ be the component of $A$ that contains $v_0$. The *fundamental group* of $A$ at the base vertex $v_0$ is

$$\mathfrak{F}(A, v_0) := \langle \varphi(W) : W \text{ is a walk in } A_0 \text{ from } v_0 \text{ to } v_0 \rangle,$$

where the angle brackets indicate the subgroup of $\mathfrak{G}$ generated by the gains $\varphi(W)$. (Should $v_0$ happen to be isolated in $A$, then its fundamental group is the subgroup of $\mathfrak{G}$ generated by the gains of loops at $v_0$.)

**Lemma 3.1.** *Fixing $A$ and $v_0$, switching $\Phi$ conjugates the fundamental group by $\eta_{v_0}$.*

**Proof.** Observe that $\varphi^\eta(W) = \eta_{v_0}^{-1}\varphi(W)\eta_{v_0}$. □

It is important to know that one can switch any balanced set $S$, such as a forest, to have all identity gains, by means of a switching function that takes a specified value, such as 1, on one specified vertex in each component of $S$. We make use of this fact repeatedly.

We give another definition of the fundamental group that depends on fewer generators. Let $T$ be a maximal forest in $A$. Let $T_0$ be the component of $T$ that contains $v_0$. For each $e \in A_0$, let $W_e$ be a minimal closed walk in $T_0 \cup \{e\}$ that starts at $v_0$ and contains $e$. The *fundamental group of $A$ with respect to $T$* at the base vertex $v_0$ is

$$\mathfrak{F}(A, v_0, T) := \langle \varphi(W_e) : e \in A_0 \rangle.$$

The generator $\varphi(W_e)$ is called the *fundamental generator of $e$*; it is 1 if and only if the edge set of $W_e$ is balanced, as for example when $e \in T$.

**Lemma 3.2.** *The fundamental group $\mathfrak{F}(A, v_0, T)$ with respect to any maximal forest $T$ equals $\mathfrak{F}(A, v_0)$.*

**Proof.** Obviously, $\mathfrak{F}(A, v_0, T) \subseteq \mathfrak{F}(A, v_0)$. On the other hand, if $W = e_1 \cdots e_l$ is any walk in $A$ from $v_0$ to $v_0$, then $\varphi(W) = \varphi(W_{e_1}) \cdots \varphi(W_{e_l}) \in \mathfrak{F}(A, v_0, T)$. □

A consequence of these lemmas is a criterion for the fundamental group to be trivial.

**Lemma 3.3.** *For a connected gain graph, its fundamental group is trivial if and only if it is balanced.*

**Proof.** By switching, assume the graph contains an identity spanning tree. There is a nonidentity edge if and only if the graph is unbalanced. Apply the definition of $\mathcal{F}(A, v_0, T)$ and Lemma 3.2. □

Should it happen that $T$ has identity gain on all edges, the definition of the fundamental group with respect to $T$ simplifies to

$$\mathfrak{F}(A, v_0, T) := \langle \varphi(e) : e \in A_0 \setminus T \rangle. \tag{3.1}$$

Even if not, there is a similar formula. Define $T_{vw}$ to be the unique path in $T$ from $v$ to $w$, if $v$ and $w$ are in the same component of $T$. Let $A$ have components $A_j$ and choose a fixed vertex $v_j$ in each component $A_j$. Switch $\varphi$ by $\eta$, the switching function defined by $\eta_v := \varphi(T_{vv_j})$ for $v \in V(A_j)$. Then

$$\mathfrak{F}(A, v_j, T) := \langle \varphi^\eta(e) : e \in A_j \setminus T \rangle, \tag{3.2}$$

because $\eta_{v_j} = 1$, so that switching by $\eta$ does not change the fundamental group, and by Eq. (3.1) applied to the switched gains. A more explicit definition of the switched gains is that

$$\varphi^\eta(e) = \varphi(W_e),$$

where $W_e$ is a minimal closed walk in $T \cup \{e\}$ from $v_j$ to $v_j$ that contains $e$ (with index $j$ such that $e \in A_j$).

One can define the fundamental group in terms of contraction. If we choose $T$ and contract it, employing a switching function as in Eq. (3.2) in the contraction to ensure $\eta_{v_0} = 1$, then

$\mathfrak{F}(A, v_0, T)$ is the subgroup of $\mathfrak{G}$ generated by the gains of the loops of the contracted graph that are incident with $v_0$ and belong to $A$. This is clear from Eq. (3.2).

**Lemma 3.4.** *Let $A \subseteq E$. The fundamental groups of $A$ with respect to any two vertices in the same component of $A$ are conjugate in $\mathfrak{G}$.*

**Proof.** If $v$ and $w$ are the vertices, choose $T$ and let $P = T_{vw}$, the path in $T$ from $v$ to $w$. Then the walk $W_e(w)$ based at $w$ is a reduced form of the walk $P^{-1}W_e(v)P$ (*reduced* by eliminating some or all consecutive edges of the form $ff^{-1}$), so $\varphi(W_e(w)) = \varphi(P)^{-1}\varphi(W_e(v))\varphi(P)$. It follows that $\mathfrak{F}(A, w, T) = \varphi(P)^{-1}\mathfrak{F}(A, v, T)\varphi(P)$. $\square$

A consequence of this lemma is that, given $A$, although the fixed set of the fundamental group may depend on the basepoint and switching, the size of the fixed set is independent of these choices as long as the basepoint stays in the same component. That is because conjugating a subgroup $\mathfrak{F}$ by $g \in \mathfrak{G}$ changes $\operatorname{Fix}\mathfrak{F}$ to $\operatorname{Fix}(\mathfrak{F}^g) = (\operatorname{Fix}\mathfrak{F})g$. Thus, we are justified in defining

$$f_{\mathbf{Q}}(A) := |\operatorname{Fix}(\mathfrak{F}(A, v))|$$

for a *connected* edge set $A \neq \varnothing$. We assume $v$ is chosen in the vertex set $V(A)$. Then $f_{\mathbf{Q}}(A)$ is invariant under switching of $\Phi$. If $A$ is empty we treat $V(A)$ as a single vertex; then the fundamental group is trivial, hence $f_{\mathbf{Q}}(\varnothing) = |\mathbf{Q}|$.

(Another consequence is that the fundamental groups at different basepoints form a fundamental groupoid in the obvious way, provided that $\Phi$ is connected. Indeed, our definition of the fundamental group(oid) is covertly topological via sewing on 2-cells; this will be treated elsewhere.)

Next we define fundamental closure. Again, $T$ is some maximal forest in $A$ and for each edge $e$ we choose a basepoint $v_0$ in $T_0$, the component of $T$ that contains $e$. The *fundamental closure* of $A$ is

$$\operatorname{fcl} A := \{e \in \operatorname{cl} A : \varphi(W_e) \in \mathfrak{F}(A, v_0, T)\}.$$

Note that $\operatorname{fcl} A \supseteq A$ and that $\operatorname{fcl} A$ is the union of the fundamental closures of the components of $A$. This union includes loops with identity gain at isolated vertices of $A$. We know from Lemmas 3.2 and 3.4 that this closure is independent of the choices. We mention $v_0$ and $T$ only because $W_e$ depends on which ones we pick. One can think of the fundamental closure of a component $A_0$ as the set of edges induced by $V(A_0)$ that lie in the inverse image $(\varphi^\eta)^{-1}(\mathfrak{F}(A_0, v_0))$, where $v_0 \in V(A_0)$ and $\eta$ is the switching function of Eq. (3.2). The definition simplifies if $T$ happens to have all identity gains; then

$$\operatorname{fcl} A = \{e \in \operatorname{cl} A : \varphi(e) \in \mathfrak{F}(A, v_0, T)\}.$$

A set that is its own fundamental closure is, of course, called *fundamentally closed*. We write $\mathcal{F}_\Phi$ for the class of fundamentally closed edge sets.

### *3.2. Satisfied edge sets*

An arbitrary state $s$ has a set $I(s)$ of satisfied edges; we ask what kind of set this can be. We want a characterization in terms of the gains and gain group, independent of the particular actions. The detailed formula we want for the chromatic function comes from Möbius inversion over the sets $I(s)$; knowing what they may be tells us the poset over which to invert.

Recall that the satisfied edges are invariant under switching.

**Lemma 3.5.** *The satisfied edges of a state constitute a fundamentally closed set.*

**Proof.** Take a state $s$ and an edge $e$ in the fundamental closure of $I(s)$. Choose a spanning tree $T$ of the component of $I(s)$ that contains the endpoints of $e$ and a base vertex $v_0 \in V(T)$. Assume by switching that $T$ has identity gains; that does not change $I(s)$ or its fundamental closure. Then $s$ is constant on $V(T)$; say $s_v = q \in \mathbf{Q}$ for every $v \in V(T)$. Also, then the fundamental group $\mathfrak{F}(I(s), v_0)$ is generated by $\varphi(f)$ for $f \in I(s)$.

Each fundamental generator $\varphi(f)$ lies in $\mathfrak{G}_q$, the stabilizer of $q$, because $f \in I(s)$ and $s = q$ on $V(T)$. Therefore, $\mathfrak{F}(I(s), v_0) \leq \mathfrak{G}_q$.

The fact that $e$ is in the fundamental closure of $I(s)$ means that $\varphi(e) \in \mathfrak{F}(I(s), v_0)$. This implies that $\varphi(e) \in \mathfrak{G}_q$, so that $e \in I(s)$. □

### 3.3. A very detailed formula

We are now ready to employ the standard method of Möbius inversion [9,13] to get an exact formula for the state chromatic function. In ordinary coloring theory the formula is quite simple because the number of all colorations, not necessarily proper, is simply a power of the number of colors (see [9, Section 9]), but in state coloring that is not the case; rather, the result has to be expressed in terms of fixed sets of fundamental groups. We state two versions of the formula. The first has fewer terms but involves the Möbius function of the lattice $\mathcal{F}_\Phi$ of fundamentally closed sets, about which nothing is known. The second, which is just an inclusion–exclusion formula, is simpler but has more terms.

**Theorem 3.6.** *For a finite gain graph $\Phi$ with a finite spin set $\mathbf{Q}$,*

$$\chi_\Phi(\mathbf{Q}) = \sum_{A \in \mathcal{F}_\Phi} \mu_{\mathcal{F}_\Phi}(\varnothing, A) \prod_{j=1}^{m} f_{\mathbf{Q}}(A_j)$$
$$= \sum_{A \subseteq E} (-1)^{|A|} \prod_{j=1}^{m} f_{\mathbf{Q}}(A_j),$$

*where $A_1, \ldots, A_m$ are the connected components of $A$.*

Remember that a component of $A$ which is an isolated vertex $v$ without loops has $f_{\mathbf{Q}}(v) = |\mathbf{Q}|$.

**Proof.** We shall prove the first formula, summing over the class $\mathcal{F}_\Phi$, but that of the second is identical except for replacing $\mathcal{F}_\Phi$ by $\mathcal{P}(E)$ with its Möbius function $\mu(\varnothing, A) = (-1)^{|A|}$.

Let $f(A)$ be the number of states for which $I(s) \supseteq A$ and let $g(A)$ be the number such that $I(s) = A$. Since every possible set of satisfied edges belongs to $\mathcal{F}_\Phi$, we see that

$$f(B) = \sum_{A \in \mathcal{F}_\Phi : A \supseteq B} g(A)$$

for every $B \in \mathcal{F}_\Phi$. By Möbius inversion,

$$g(B) = \sum_{A \in \mathcal{F}_\Phi} \mu_{\mathcal{F}_\Phi}(B, A) f(A).$$

Setting $B = \varnothing$ we get

$$\chi_\Phi(\mathbf{Q}) = g(\varnothing) = \sum_{A \in \mathcal{F}_\Phi} \mu_{\mathcal{F}_\Phi}(\varnothing, A) f(A).$$

To finish the proof we have to interpret $f(A)$. Let $T$ be a maximal forest in $A$ and switch so $T$ has identity gains. Then any state counted by $f(A)$ is constant on each component $A_j$, having let us say spin $q_j$. For each edge $e \in A_j \setminus T$ we must have $q_j\varphi(e) = q_j$; thus, $q_j$ can be any spin that is fixed by every gain $\varphi(e)$ for $e \in A_j$. These gains are the generators of $\mathfrak{F}(A_j, v_0, T)$, so the possible spins $q_j$ are precisely those that lie in Fix $\mathfrak{F}(A_j, v_0, T)$. The number of these is $f_{\mathbf{Q}}(A_j)$. The value of $f(A)$ is the number of ways to choose one spin for each component, i.e., the product of all $f_{\mathbf{Q}}(A_j)$. That proves the formula. □

Theorem 5.1(iv) is a generalization with a different proof.

## 4. A grand polynomial

Despite the difficulties about matroids, there is a way to make the state chromatic function into a polynomial that generalizes the chromatic polynomial.

### *4.1. A multichromatic polynomial*

Let us have spin sets $\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_p$, that is, each is a set with a $\mathfrak{G}$-action, and to avoid notational difficulty suppose that all the sets $\mathbf{Q}_i$ and $\mathbf{Q}_i \times \mathbb{N}$ are pairwise disjoint. ($\mathbb{N}$ is the set of nonnegative integers.) Write $\mathrm{Fix}_i\, \mathfrak{F}$ for the fixed set of the action of $\mathfrak{F}$ on $\mathbf{Q}_i$, when $\mathfrak{F}$ is a subgroup of $\mathfrak{G}$. Set

$$\mathbf{Q} := \mathbf{Q}_{k_1,k_2,\ldots,k_p} := (\mathbf{Q}_1 \times [k_1]) \cup (\mathbf{Q}_2 \times [k_2]) \cup \cdots \cup \big(\mathbf{Q}_p \times [k_p]\big),$$

where $k_1, k_2, \ldots, k_p \in \mathbb{N}$.

**Theorem 4.1.** *Given $\mathbf{Q}_1, \ldots, \mathbf{Q}_p$, the number $\chi_\Phi(\mathbf{Q}_{k_1,k_2,\ldots,k_p})$ of totally frustrated states with spins from $\mathbf{Q}_{k_1,k_2,\ldots,k_p}$ is given by the multivariate polynomial*

$$\chi_{\Phi;\mathbf{Q}_1,\ldots,\mathbf{Q}_p}(k_1, \ldots, k_p) = \sum_{A \in \mathfrak{F}_\Phi} \mu_{\mathfrak{F}_\Phi}(\varnothing, A) \prod_{j=1}^{m} \left[ \sum_{i=1}^{p} k_i \,|\mathrm{Fix}_i\, \mathfrak{F}(A_j, v_j)| \right], \tag{4.1}$$

*where $A_1, \ldots, A_m$ denote the connected components of $A$ and $v_j$ denotes any one vertex of $A_j$.*

*If not identically* 0*, the polynomial has total degree n and the terms of highest degree are the terms of the expression*

$$\prod_{v \in V} \sum_{i=1}^{p} k_i \left[ |\mathbf{Q}_i| - \left| \bigcup_{l_v} \mathrm{Fix}_i(\varphi(l_v)) \right| \right], \tag{4.2}$$

*where $l_v$ ranges over all loops incident with $v$.*

An isolated vertex of $A$ with no loops, being a connected component, gives the bracketed factor $\sum_i k_i |\mathbf{Q}_i|$ in (4.1).

**Proof.** We first give a simple proof of polynomiality, degree, and highest terms, without the explicit formula (4.1). We use induction on the number of links, employing deletion and contraction. Let $\mathrm{Fix}_i(g)$ denote the fixed set of the action of $g$ on $\mathbf{Q}_i$.

First assume $\Phi$ has no links. Consider the case of a single vertex $v$. For each loop $l_v$, the spins in its fixed set are not allowed to color $v$. The number of totally frustrated states is therefore

$$\begin{aligned} t(v) :&= \sum_{i=1}^{p} \text{number of spins in } \mathbf{Q}_i \times [k_i] \text{ not fixed by any loop gain at } v \\ &= \sum_{i=1}^{p} k_i \left| \mathbf{Q}_i \setminus \bigcup_{l_v} \mathrm{Fix}_i(\varphi(l_v)) \right|. \end{aligned}$$

If $\Phi$ has several vertices, the number of totally frustrated states is the product $\prod_{v \in V} t(v)$, by Proposition 2.3. Thus, the polynomial is homogeneous with total degree $n$, unless there are no totally frustrated states at all.

Now suppose $\Phi$ has a link $e$ and apply Eq. (1.1). We may assume $\Phi$ does have a totally frustrated state. We find that $\chi_\Phi(\mathbf{Q})$ is the difference of one polynomial of total degree $n$ and another with total degree $n-1$. (The former cannot be identically zero, since that would mean $\Phi \setminus e$ has no totally frustrated states, hence the same would be true of $\Phi$, contrary to assumption.) The highest-degree terms of $\chi_\Phi(\mathbf{Q})$, having degree $n$, are those of $\chi_{\Phi \setminus e}(\mathbf{Q})$, which by induction are the ones specified in the statement.

The precise formula comes from Theorem 3.6. It depends on evaluating $f_{\mathbf{Q}}(A)$ for the special spin set $\mathbf{Q}$. Since $\mathrm{Fix}\, \mathfrak{F} = \bigcup_i (\mathrm{Fix}_i\, \mathfrak{F}) \times [k_i]$, for a connected subgraph with edge set $A_j$ we have

$$f_{\mathbf{Q}}(A_j) = \sum_{i=1}^{p} k_i |\mathrm{Fix}_i\, \mathfrak{F}(A_j, v_j)|,$$

where $v_j$ is any vertex of $A_j$. Thus we immediately obtain (4.1). □

The term of a set $A$ in (4.1) is homogeneous of degree $c(A)$, the number of components of $A$; thus, the terms of highest degree in (4.1) are those corresponding to sets $A$ that contain only loops. One can use this fact to prove (4.2) from the second formula of Theorem 3.6. Another proof of (4.2) is implied by Theorem 5.5.

### 4.2. *The grand chromatic polynomial*

There is a single most general polynomial of the form (4.1) when $\mathfrak{G}$ is finite. Defining a $\mathfrak{G}$-*set* as a set with a $\mathfrak{G}$-action, there are only finitely many nonisomorphic transitive $\mathfrak{G}$-sets. Let them be $\hat{\mathbf{Q}}_1, \ldots, \hat{\mathbf{Q}}_r$ (in an arbitrary but fixed order). With $p = r$ and $\mathbf{Q}_i = \hat{\mathbf{Q}}_i$, (4.1) defines an $r$-variable polynomial

$$\chi_{\Phi;\mathfrak{G}}(x_1, \ldots, x_r) := \chi_{\Phi;\hat{\mathbf{Q}}_1,\ldots,\hat{\mathbf{Q}}_r}(x_1, \ldots, x_r)$$

that we call the *grand chromatic polynomial* of $\Phi$. (We write $\chi_{\Phi,\mathfrak{G}}$ because the $\hat{\mathbf{Q}}_i$ are completely determined by $\mathfrak{G}$.) Every polynomial $\chi_{\Phi;\mathbf{Q}_1,\ldots,\mathbf{Q}_p}(k_1, \ldots, k_p)$, hence every state chromatic function, is an evaluation of the grand chromatic polynomial. To see why, first suppose that, say, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ have isomorphic actions of $\mathfrak{G}$. Then

$$\chi_{\Phi;\mathbf{Q}_1,\mathbf{Q}_2,\mathbf{Q}_3,\ldots,\mathbf{Q}_p}(k_1, \ldots, k_p) = \chi_{\Phi;\mathbf{Q}_1,\mathbf{Q}_3,\ldots,\mathbf{Q}_p}(k_1 + k_2, k_3, \ldots, k_p).$$

On the other hand, suppose that $\mathbf{Q}_1$, say, has several orbits $\mathbf{Q}_{11}, \ldots, \mathbf{Q}_{1m}$. Then

$$\chi_{\Phi;\mathbf{Q}_1,\ldots,\mathbf{Q}_p}(k_1, \ldots, k_p) = \chi_{\Phi;\mathbf{Q}_{11},\ldots,\mathbf{Q}_{1m},\mathbf{Q}_2,\ldots,\mathbf{Q}_p}(k_1, \ldots, k_1, k_2, \ldots, k_p).$$

Splitting all the sets $\mathbf{Q}_i$ into their orbits, we may assume that each is a copy of a $\hat{\mathbf{Q}}_j$, then combine them to obtain an evaluation of the grand chromatic polynomial in which the variable $x_j$ corresponding to any $\hat{\mathbf{Q}}_j$ that is not an orbit of some $\mathbf{Q}_i$ is equal to 0.

**Theorem 4.2.** *Let $\Phi$ be a gain graph with finite gain group $\mathfrak{G}$. The number of totally frustrated states of any permutation gain graph whose gain graph is $\Phi$ is an evaluation of the grand chromatic polynomial of $\Phi$.*

### 4.3. Many zeroes

Naturally, the chromatic polynomials are special cases of the grand chromatic polynomial. The zero-free chromatic polynomial corresponds to $p = 1$ and $\mathbf{Q}_1 = \mathfrak{G}$ with variable $\lambda := k_1|\mathbf{Q}_1|$, and the chromatic polynomial corresponds to $p = 2$, $\mathbf{Q}_1 = \mathfrak{G}$, $|\mathbf{Q}_2| = 1$, and $k_2 = 1$ with variable $\lambda := k_1|\mathbf{Q}_1| + |\mathbf{Q}_2|$.

The two-variable generalization with $\mathbf{Q}_1 = \mathfrak{G}$ and $|\mathbf{Q}_2| = 1$ is near enough to standard gain-graph coloring to be interesting. By analogy with standard coloring, one might think of $\mathbf{Q}_2 \times [k_2]$ as the set $\{0\} \times [k_2]$ consisting of $k_2$ different zeroes.

The fixed sets of the fundamental groups have sizes $f_{\mathbf{Q}_2}(A_j) = |\mathbf{Q}_2| = 1$ and

$$f_{\mathbf{Q}_1}(A_j) = |\mathbf{Q}_1| \quad \text{or } 0$$

depending on whether $A_j$ is balanced or unbalanced. A connected edge set that is balanced is fundamentally closed if and only if it is a maximal balanced set on its vertices. A connected edge set that is unbalanced is fundamentally closed if and only if it is closed in the underlying graph, i.e., it is a connected induced subgraph. Consequently, a set is fundamentally closed if and only if each connected component is either an induced subgraph or a maximal balanced set on its vertex set. These sets include all flats of the frame matroid $G(\Phi)$ and also sets obtained by taking one or more unbalanced components of a flat, partitioning each component's vertex set, and taking the induced subgraphs on the blocks of the partition. From Eq. (4.1), the formula is

$$\begin{aligned}\chi_{\Phi;\mathbf{Q}_1,\mathbf{Q}_2}(k_1,k_2) &= \sum_{A\in\mathcal{F}_\Phi} \mu_{\mathcal{F}_\Phi}(\varnothing, A) \prod_{j=1}^{m} \left[k_1 f_{\mathbf{Q}_1}(A_j) + k_2\right] \\ &= \sum_{A\in\mathcal{F}_\Phi} \mu_{\mathcal{F}_\Phi}(\varnothing, A)[k_1|\mathfrak{G}| + k_2]^{b(A)} k_2^{c(A)-b(A)}, \qquad (4.3)\end{aligned}$$

where $c(A)$ is the number of components of $A$.

## 5. An abstract partition function

We wish to expose the underlying structure of state chromatic functions. Thus we develop a formal algebra inspired by Tutte's ring for graph polynomials [15] and its modern descendants, as well as his dichromatic polynomial and the later Whitney-number polynomial of a gain graph.

### 5.1. Dichromatic polynomial

As Tutte discovered long ago [16], adding a second variable to the chromatic polynomial produces a new graph polynomial, satisfying a variant of the deletion–contraction law (1.1), known as the *dichromatic polynomial*. Generalizing to gain graphs, we call it $Q_\Phi(u, v)$

[21, Section 3, p. 25]. We define it here by way of the normalized dichromatic polynomial, $\bar{Q}_\Phi(\lambda, v) = v^n Q_\Phi(\lambda/v, v)$, which can be defined in terms of colorations:

$$\bar{Q}_\Phi(1 + k|\mathfrak{G}|, v) := \sum_{s:V\to\mathbf{C}_k} (v+1)^{|I(s)|}$$

(from [21, Corollary 4.4]). The obvious states analog is the *normalized state dichromatic function* of $\Phi$, namely,

$$\bar{Q}_\Phi(\mathbf{Q}, v) := \sum_{s:V\to\mathbf{Q}} (v+1)^{|I(s)|}.$$

Setting $v = -1$ gives the state chromatic function. We shall see as a special case of Theorem 5.1 that the normalized state dichromatic function satisfies the modified deletion–contraction formula

$$\bar{Q}_\Phi(\mathbf{Q}, v) = \bar{Q}_{\Phi\setminus e}(\mathbf{Q}, v) + v\bar{Q}_{\Phi/e}(\mathbf{Q}, v)$$

for any link $e$.

### 5.2. *Whitney-number polynomial*

The *Whitney-number polynomial* of a gain graph [21, p. 26] can be defined by either of the sums

$$w_\Phi(x, \lambda) = \sum_{A\subseteq E} x^{n-b(A)}\chi_{\Phi/A}(\lambda) = \sum_{s:V\to\mathbf{C}_k} x^{n-b(I(s))}; \tag{5.1}$$

the first by applying [21, Equation (3.1d)] with $w = 1$ and $v = -1$; the second by noting that $\chi_{\Phi/A}(\lambda)$ counts the colorations of which the satisfied edge set equals $A$, cf. [21, Corollary 4.5]. One may try to generalize either of these two sums.

The expression $\chi_{\Phi/A}(\lambda)$ in the first sum arises from the fact that if a coloration $s$ has satisfied edge set $I(s)$, it corresponds to a proper coloration of $\Phi/I(s)$. It seems impossible to generalize that fact to states, even by modifying the definition of contraction.

An obvious states generalization of the second part of Eq. (5.1) is

$$w_\Phi(x, \mathbf{Q}) := \sum_{s:V\to\mathbf{Q}} x^{n-b(I(s))}.$$

This has an incongruous aspect. The sets $I(s)$ are fundamentally closed. There exist fundamentally closed sets $A \subset B$ having the same rank in the frame matroid, and this formula treats them identically since the exponent of $x$ is the rank; but one would expect a good generalization to treat them differently. For instance, perhaps there should be a second variable whose exponent reflects the difference between $I(s)$ and its matroid closure. However, rather than trying to decide how to modify the polynomial, we go directly to basics by proving that both $\bar{Q}_\Phi(\mathbf{Q}, v)$ and $w_\Phi(x, \mathbf{Q})$ are obtained from a more abstract polynomial.

### 5.3. *Formal algebra and the partition function*

We abstract both state generalizations within the *edge ring* $\mathbb{Z}[E]$. In the edge ring a subset $A \subseteq E$ is identified with the monomial $\prod_{e\in A} e$; the multiplicative identity is therefore $1 = \varnothing$.

Within $\mathbb{Z}[E]$ is the $\mathbb{Z}$-submodule $\mathbb{Z}\mathcal{P}(E)$, the span of the power set of $E$. The *abstract partition function* of $(\Phi, \mathbf{Q})$ is

$$Z(\Phi, \mathbf{Q}) := \sum_{s:V\to\mathbf{Q}} I(s) \in \mathbb{Z}\mathcal{P}(E) \subseteq \mathbb{Z}[E].$$

The normalized state dichromatic function $\bar{Q}_\Phi(\mathbf{Q}, v)$ is the image of $Z(\Phi, \mathbf{Q})$ under the ring homomorphism $\mathbb{Z}[E] \to \mathbb{Z}[v]$ defined by $e \mapsto v+1$ for every edge. The state chromatic function is obtained by $e \mapsto 0$, since the coefficient of $\varnothing$ is $\chi_\Phi(\mathbf{Q})$. The state Whitney-number function $w_\Phi(x, \mathbf{Q})$ is the image of $Z(\Phi, \mathbf{Q})$ under the $\mathbb{Z}$-module mapping $\mathbb{Z}\mathcal{P}(E) \to \mathbb{Z}[x]$ defined by $A \mapsto x^{n-b(A)}$.

The essential facts about $Z(\Phi, \mathbf{Q})$ are its reduction properties, Parts (i–iii) of the next theorem, by which it can be calculated recursively in terms of gain graphs with only one vertex, whose partition functions are the initial conditions of the recurrence. Part (iv) generalizes Theorem 3.6; it is analogous to the subset expansion of the Tutte polynomial of a matroid [2].

**Theorem 5.1.** *The abstract partition function of a permutation gain graph $\Phi$ satisfies the following properties:*

(i) $Z(\Phi, \mathbf{Q}) = Z(\Phi \setminus e, \mathbf{Q}) + (e-1)Z(\Phi/e, \mathbf{Q})$ *for a link $e$ in $\Phi$.*
(ii) $Z(\Phi, \mathbf{Q}) = Z(\Phi_1, \mathbf{Q})Z(\Phi_2, \mathbf{Q})$ *if $\Phi$ is the disjoint union of $\Phi_1$ and $\Phi_2$.*
(iii) $Z(\varnothing, \mathbf{Q}) = \varnothing$.
(iv) $Z(\Phi, \mathbf{Q})$ *has the closed form*

$$Z(\Phi, \mathbf{Q}) = \sum_{A\subseteq E} \prod_{f\in A} (f-1) \cdot \prod_{j=1}^{m} f_{\mathbf{Q}}(A_j),$$

*where $A_1, \ldots, A_m$ are the connected components of $(V, A)$.*
(v) *If $\theta : \Phi_1 \to \Phi_2$ is an isomorphism, then $\theta(Z(\Phi_1, \mathbf{Q})) = Z(\Phi_2, \mathbf{Q})$.*
(vi) *If $\alpha$ is an automorphism of $\Phi$, then $\alpha(Z(\Phi, \mathbf{Q})) = Z(\Phi, \mathbf{Q})$.*

For instance, the abstract partition function of a gain graph $\Phi{:}\{v\}$ with one vertex is straightforward. Let $E_v$ denote the set of loops incident with $v$. Then

$$Z(\Phi{:}\{v\}, \mathbf{Q}) = \sum_{q\in\mathbf{Q}} \{l_v \in E_v : q \in \operatorname{Fix}\varphi(l_v)\}.$$

Another example is the gain graph $\Phi$ that consists of two parallel links $e$ and $f$ with gains 1 and $g$, respectively. Then $\Phi/e$ is the gain graph with one vertex and the one edge $l_f$, which is $f$ as a loop with gain $g$. The abstract partition function is

$$\begin{aligned} Z(\Phi, \mathbf{Q}) &= Z(f, \mathbf{Q}) + (e-1)Z(\Phi/e, \mathbf{Q}) \\ &= Z(K_1, \mathbf{Q})^2 + (f-1)Z(K_1, \mathbf{Q}) + (e-1)Z(\Phi/e, \mathbf{Q}) \\ &= |\mathbf{Q}|^2 + (f-1)|\mathbf{Q}| + (e-1)\left[|\mathbf{Q}| + (f-1)|\operatorname{Fix} g|\right] \\ &= ef|\mathbf{Q}| - (e-1)(f-1)\left[|\mathbf{Q}| - |\operatorname{Fix} g|\right]. \end{aligned}$$

**Proof of Theorem 5.1.** *Part* (i). We may assume by switching that the link $e{:}vw$ has gain 1. We compare the satisfied edge sets of states in the three graphs. Let $v_e$ be the vertex of $\Phi/e$ that results from contracting $e$. Write $V'' := V(\Phi/e)$; then $V'' = V \setminus \{v, w\} \cup \{v_e\}$. The task is to prove that

$$\sum_{s:V\to\mathbf{Q}} I_\Phi(s) = \sum_{s:V\to\mathbf{Q}} I_{\Phi\setminus e}(s) + \sum_{s'':V''\to\mathbf{Q}} (e-1)I_{\Phi/e}(s'').$$

The states $s''$ of $\Phi/e$ correspond to the states $s$ of $\Phi$ in which $s_v = s_w$, by the correspondence $s_v = s_w = s''_{v_e}$. Then $I_{\Phi\setminus e}(s) = I_{\Phi/e}(s'')$. For a state of this type, the terms of the three summations are $I_\Phi(s) = eI_{\Phi\setminus e}(s)$ on the left and $I_{\Phi\setminus e}(s) + (e-1)I_{\Phi/e}(s'') = eI_{\Phi\setminus e}(s)$ on the right.

For a state in which $s_v \neq s_w$, the terms are the equal quantities $I_\Phi(s)$ on the left and $I_{\Phi\setminus e}(s)$ on the right, since there is no corresponding state of $\Phi/e$.

*Part* (ii) is obvious from the definition.

*Part* (iii). There are no vertices or edges so there is one state with null domain, empty range, and no satisfied edges.

*Part* (iv). We employ the expression

$$\zeta(\Phi, \mathbf{F}) := \sum_{A\subseteq E} \prod_{f\in A} (f-1) \cdot \prod_{j=1}^{m} |\mathbf{F} \cap \operatorname{Fix} \mathfrak{F}(A_j, v_j)|$$

for $\mathbf{F} \subseteq \mathbf{Q}$, where $A_1, \ldots, A_j$ are the components of $(V, A)$ and $v_j$ is any vertex of $A_j$.

First we settle the case where $\Phi$ has a single vertex $v_0$. We make use of a restricted partition function,

$$Z(\Phi, \mathbf{F}) := \sum_{s:V\to\mathbf{F}} I(s) \quad \text{for } \mathbf{F} \subseteq \mathbf{Q}.$$

**Lemma 5.2.** *The restricted partition function of a gain graph $\Phi$ with one vertex satisfies*

$$Z(\Phi, \mathbf{F}) = Z(\Phi \setminus e, \mathbf{F}) + (e-1)Z(\Phi \setminus e, \mathbf{F} \cap \operatorname{Fix} \varphi(e))$$

*for any edge $e$.*

**Proof.** A state $s$ is equivalent to its value $q = s(v_0)$. Consequently,

$$\begin{aligned} Z(\Phi, \mathbf{F}) &= e \sum_{q\in\mathbf{F}\cap\operatorname{Fix}\varphi(e)} \{f \in E \setminus e : q \in \operatorname{Fix} \varphi(f)\} \\ &\quad + \sum_{q\in\mathbf{F}\setminus\operatorname{Fix}\varphi(e)} \{f \in E \setminus e : q \in \operatorname{Fix} \varphi(f)\} \\ &= (e-1) \sum_{q\in\mathbf{F}\cap\operatorname{Fix}\varphi(e)} \{f \in E \setminus e : q \in \operatorname{Fix} \varphi(f)\} \\ &\quad + \sum_{q\in\mathbf{F}} \{f \in E \setminus e : q \in \operatorname{Fix} \varphi(f)\} \\ &= (e-1)Z(\Phi, \mathbf{F} \cap \operatorname{Fix} \varphi(e)) + Z(\Phi \setminus e, \mathbf{F}). \quad \square \end{aligned}$$

**Lemma 5.3.** *For a gain graph $\Phi$ with one vertex there is the reduction formula*

$$\zeta(\Phi, \mathbf{F}) = \zeta(\Phi \setminus e, \mathbf{F}) + (e-1)\zeta(\Phi \setminus e, \mathbf{F} \cap \operatorname{Fix} \varphi(e)).$$

**Proof.** Let us calculate.

$$\zeta(\Phi, \mathbf{F}) - \zeta(\Phi \setminus e, \mathbf{F}) = (e-1) \sum_{S \subseteq E \setminus e} \prod_{f \in S} (f-1) \cdot |\mathbf{F} \cap \operatorname{Fix} \mathfrak{F}(S \cup \{e\}, v_0)|,$$

where $S \cup \{e\}$ stands for any set $A$ that contains $e$ in the definition of $\zeta(\Phi, \mathbf{F})$,

$$= (e-1) \sum_{S \subseteq E \setminus e} \prod_{f \in S} (f-1) \cdot |\mathbf{F} \cap \operatorname{Fix} \varphi(e) \cap \operatorname{Fix} \mathfrak{F}(S, v_0)|$$

because $\mathfrak{F}(S \cup \{e\}, v_0)$ is generated by $\mathfrak{F}(S, v_0)$ and $\varphi(e)$. □

Since $Z(\Phi, \mathbf{F})$ and $\zeta(\Phi, \mathbf{F})$ obey the same edge reduction identity, and since $Z(K_1, \mathbf{F}) = \varnothing = 1 = \zeta(K_1, \mathbf{F})$, we conclude that $Z(\Phi, \mathbf{F})$ and $\zeta(\Phi, \mathbf{F})$ are equal. Setting $\mathbf{F} = \mathbf{Q}$, we have part (iv) for a one-vertex graph $\Phi$.

Now we treat larger graphs $\Phi$.

**Lemma 5.4.** *For a link e in $\Phi$ we have the deletion–contraction identity*

$$\zeta(\Phi, \mathbf{Q}) = \zeta(\Phi \setminus e, \mathbf{Q}) + (e-1)\zeta(\Phi/e, \mathbf{Q}).$$

**Proof.** Assume $e$ switched to have gain 1.

First we compute

$$\zeta(\Phi, \mathbf{Q}) - \zeta(\Phi \setminus e, \mathbf{Q}) = (e-1) \sum_{e \in A \subseteq E} \prod_{f \in A \setminus e} (f-1) \prod_{j=1}^{m} |\operatorname{Fix} \mathfrak{F}(A_j, v_j)|.$$

The components of $A$ in $\Phi$ are the same as the components of $A \setminus e$ in $\Phi/e$ with the exception of that component which contains $e$, call it $A_1$.

We compare this quantity with

$$\zeta(\Phi/e, \mathbf{Q}) = \sum_{B \subseteq E \setminus e} \prod_{f \in B} (f-1) \cdot \prod_{j=1}^{m} |\operatorname{Fix} \mathfrak{F}(B_j, v_j)|,$$

with $B = A \setminus e$. The only difference is that the former expression has $\mathfrak{F}(A_1, v_1)$ in $\Phi$ where the latter has $\mathfrak{F}(B_1, v_1)$ in $\Phi/e$. These groups are equal by the definition of fundamental groups and the fact that $\varphi(e) = 1$. □

We compare $Z(\Phi, \mathbf{Q})$ to $\zeta(\Phi, \mathbf{Q})$. They agree for gain graphs with one vertex and they satisfy the same deletion–contraction recurrence. Hence, they are equal.

*Part* (v) follows from the fact that $\theta$ acts on states and preserves satisfaction or frustration of edges.

*Part* (vi). First we observe that for any state $s$, $\alpha(I(s)) = I(s^\alpha)$, where by $s^\alpha$ we mean the state defined by $(s^\alpha)_v := s_{\alpha^{-1}v}$. The reason for this is that, for an edge $e{:}vw$, $e \in I(s) \iff s_w = s_v\varphi(e) \iff s^\alpha_{\alpha w} = s^\alpha_{\alpha v}\varphi(\alpha e)$ (by the definition of $s^\alpha$ and the fact that $\alpha$ preserves gains) $\iff \alpha e \in I(s^\alpha)$.

Now, applying $\alpha$ to $Z$, represented as a sum over all states $s$ or equivalently all states $s^\alpha$, we see that $Z = \sum_{s^\alpha} I(s^\alpha) = \sum_s \alpha(I(s)) = \alpha(Z)$. □

### 5.4. The grand partition function

The idea behind the grand chromatic polynomial works equally well for the abstract partition function. As in Section 4.2, we assume $\mathfrak{G}$ is finite. We define $\hat{\mathbf{Q}}$ to be the disjoint union of the $r$ sets $\hat{\mathbf{Q}}_i \times [k_i]$, where $\hat{\mathbf{Q}}_1, \ldots, \hat{\mathbf{Q}}_r$ are the nonisomorphic transitive $\mathfrak{G}$-sets. The abstract partition function of $\hat{\mathbf{Q}}$ is the *grand partition function* of $\Phi$. We write it as $Z(\Phi, \mathfrak{G})(k_1, \ldots, k_r) := Z(\Phi, \hat{\mathbf{Q}})$, since $\hat{\mathbf{Q}}$ is completely determined by $\mathfrak{G}$ and $k_1, \ldots, k_r$. Just as with the grand chromatic polynomial, the abstract partition function of $\Phi$ for any finite spin set $\mathbf{Q}$ is obtained from $Z(\Phi, \mathfrak{G})(k_1, \ldots, k_r)$ by appropriate evaluation of $k_1, \ldots, k_r$.

We denote by $\Pi_V$ the lattice of all partitions of the set $V$. Recall that $\Phi{:}W$ is the subgraph induced by $W$, and that $(x)_r$ is the falling factorial.

**Theorem 5.5.** *The grand partition function has the form*

$$Z(\Phi, \mathfrak{G})(k_1, \ldots, k_r) = \sum_{\pi \in \Pi_V} \sum_{\mathbf{i}} \prod_{W \in \pi} \sum_{\tau \in \Pi_W} \prod_{B \in \tau} Z(\Phi{:}B, \hat{\mathbf{Q}}_{\mathbf{i}(W)}) \cdot (k_{\mathbf{i}(W)})_{|\tau|},$$

*where* $\mathbf{i}$ *ranges over all injective mappings* $\pi \to [r]$. *This is a polynomial in the nonnegative integers* $k_1, \ldots, k_r$. *The terms of highest total degree, which is n, are those of the expression*

$$\prod_{v \in V} \sum_{i=1}^{r} k_i Z(\Phi{:}\{v\}, \hat{\mathbf{Q}}_i).$$

**Proof.** Consider a state $s : V \to \hat{\mathbf{Q}}$. For each $i \in [r]$, there is a vertex set $W_i := s^{-1}(\hat{\mathbf{Q}}_i \times [k_i])$. Some of these sets may be empty, but the nonvoid ones form a partition $\pi$ of $V$. Conversely, given a partition $\pi$, there are $(r)_{|\pi|}$ injective functions $\mathbf{i} : \pi \to [r]$, which correspond to the ways to label the blocks of $\pi$ with distinct transitive $\mathfrak{G}$-sets $\hat{\mathbf{Q}}_i$, by taking $i = \mathbf{i}(W)$. Thus,

$$Z(\Phi, \mathfrak{G})(k_1, \ldots, k_r) = \sum_{s:V \to \hat{\mathbf{Q}}} I(s) = \sum_{\pi \in \Pi_V} \sum_{\mathbf{i}} \sum_{s_W} \prod_{W} I_{\Phi:W}(s_W),$$

where $s_W$ means $s|_W$ reinterpreted as an independently varying state of $\Phi{:}W$ with values in $\hat{\mathbf{Q}}_{\mathbf{i}(W)} \times [k_{\mathbf{i}(W)}]$, the summation $\sum_{s_W}$ is over all such states, and we keep in mind that multiplication in the edge algebra is union. The latter two operations of summation and product can be reversed, yielding

$$Z(\Phi, \mathfrak{G})(k_1, \ldots, k_r) = \sum_{\pi \in \Pi_V} \sum_{\mathbf{i}} \prod_{W} Z\left(\Phi{:}W, \hat{\mathbf{Q}}_{\mathbf{i}(W)} \times [k_{\mathbf{i}(W)}]\right).$$

Now we focus on $\Phi{:}W$, writing $i := \mathbf{i}(W)$ for brevity. The analysis is similar, except that $\hat{\mathbf{Q}}_i \times [k_i]$ and the isomorphic copies $\hat{\mathbf{Q}}_i \times \{j\}$ of $\hat{\mathbf{Q}}_i$, for $j \in [k_i]$, replace $\hat{\mathbf{Q}}$ and the $\hat{\mathbf{Q}}_i \times [k_{\mathbf{i}(W)}]$. Thus,

$$Z\left(\Phi{:}W, \hat{\mathbf{Q}}_i \times [k_i]\right) = \sum_{\tau \in \Pi_W} \sum_{\mathbf{j}} \prod_{B \in \tau} Z(\Phi{:}B, \hat{\mathbf{Q}}_i),$$

where $\mathbf{j}$ denotes an injection $\mathbf{j} : \tau \to [k_i]$. Now a difference appears. As the spin sets are the same for every block of $\tau$, it no longer matters exactly what $\mathbf{j}$ is; the important fact is that there are $(k_i)_{|\tau|}$ different ones, so that

$$Z\left(\Phi{:}W, \hat{\mathbf{Q}}_i \times [k_i]\right) = \sum_{\tau \in \Pi_W} (k_i)_{|\tau|} \prod_{B \in \tau} Z(\Phi{:}B, \hat{\mathbf{Q}}_i).$$

This is the first part of the theorem.

The highest-degree terms appear when $\tau = 0_W$, the total partition of $W$, and we keep only the highest term, $k_{\mathbf{i}(W)}^{|\tau|} = k_{\mathbf{i}(W)}^{|W|}$, of $(k_i)_{|\tau|}$. Then we have

$$\sum_{\pi \in \Pi_V} \sum_{\mathbf{i}} \prod_{W \in \pi} \prod_{w \in W} Z(\Phi{:}\{w\}, \hat{\mathbf{Q}}_{\mathbf{i}(W)}) \cdot k_{\mathbf{i}(W)}^{|W|}$$
$$= \sum_{\pi \in \Pi_V} \sum_{\mathbf{i}} \prod_{W \in \pi} \prod_{w \in W} k_{\mathbf{i}(W)} Z(\Phi{:}\{w\}, \hat{\mathbf{Q}}_{\mathbf{i}(W)})$$
$$= \sum_{\pi \in \Pi_V} \sum_{\mathbf{i}} \prod_{w \in V} k_{\mathbf{i}(W)} Z(\Phi{:}\{w\}, \hat{\mathbf{Q}}_{\mathbf{i}(W)}),$$

where $W$ denotes the block of $\pi$ that contains $w$,

$$= \sum_{\mathbf{j}:V \to [r]} \prod_{w \in V} k_{\mathbf{j}(w)} Z(\Phi{:}\{w\}, \hat{\mathbf{Q}}_{\mathbf{j}(w)})$$

summed over all functions $\mathbf{j} : V \to [r]$, because an injection $\mathbf{i} : \pi \to [r]$ is equivalent to an arbitrary function $\mathbf{j} : V \to [r]$ that happens to have $\mathbf{j}(w) = \mathbf{i}(W)$ when $w \in W$,

$$= \prod_{w \in V} \sum_{i=1}^{r} k_i Z(\Phi{:}\{w\}, \hat{\mathbf{Q}}_i). \quad \square$$

The second part of Theorem 4.1 follows from that of Theorem 5.5 by substituting $e \mapsto 0$.

The main part of Theorem 5.5 specializes to a formula for the grand chromatic polynomial.

**Corollary 5.6.** *The grand chromatic polynomial has the form*

$$\chi_{\Phi;\hat{\mathbf{Q}}_1,\ldots,\hat{\mathbf{Q}}_r}(k_1, \ldots, k_r) = \sum_{\pi \in \Pi_V} \sum_{\mathbf{i}} \prod_{W \in \pi} \sum_{\tau \in \Pi_W} \prod_{B \in \tau} \chi_{\Phi:B}(\hat{\mathbf{Q}}_{\mathbf{i}(W)}) \cdot (k_{\mathbf{i}(W)})_{|\tau|},$$

*where* $\mathbf{i}$ *ranges over injective mappings* $\pi \to [r]$.

## 6. Edge-spin algebras

Theorem 5.1 finds its natural expression in an even more abstract setting. The trick is to define the partition function by means of Theorem 5.1(iv) and then replace $|\operatorname{Fix} \mathfrak{F}|$ by $\operatorname{Fix} \mathfrak{F}$. One even gets a slight simplification in part of the proof. We sketch the ideas.

We indicate an orbit of the right action of $\mathfrak{G}$ on subsets of $\mathbf{Q}$ by square brackets, such as $[\mathbf{C}]$ for the orbit of $\mathbf{C} \in \mathcal{P}(\mathbf{Q})$. The quotient set, denoted by $\bar{\mathcal{P}}(\mathbf{Q})$, generates a free $\mathbb{Z}[E]$-module $\bar{\mathbf{A}} := \mathbb{Z}[E]\bar{\mathcal{P}}(\mathbf{Q})$. The *symmetric edge-spin algebra* $\mathbf{S}\bar{\mathbf{A}}$ is the symmetric algebra of $\bar{\mathbf{A}}$. The symbol for multiplication in this algebra is $\otimes$. Define the *hyperabstract partition function* to be

$$\bar{Z}(\Phi, \mathbf{Q}) = \sum_{A \subseteq E} \prod_{f \in E} (e - 1) \cdot \bigotimes_{j=1}^{m} [\operatorname{Fix} \mathfrak{F}(A_j, v_j)] \in \mathbf{S}\bar{\mathbf{A}}.$$

(We are forced to work with orbits instead of individual fixed sets because different choices of basepoint $v_j$ conjugate the fundamental group and correspondingly translate the fixed set.) We are generalizing the formula of Theorem 5.1(iv), as $Z(\Phi, \mathbf{Q})$ is the image of $\bar{Z}(\Phi, \mathbf{Q})$ under the cardinality mapping $\mathbf{C} \mapsto |\mathbf{C}|$ for $\mathbf{C} \in \mathcal{P}(\mathbf{Q})$. One can prove properties of $\bar{Z}(\Phi, \mathbf{Q})$ just as we established those of $Z(\Phi, \mathbf{Q})$ and with no greater difficulty.

**Theorem 6.1.** *The hyperabstract partition function satisfies*

(i) $\bar{Z}(\Phi, \mathbf{Q}) = \bar{Z}(\Phi \setminus e, \mathbf{Q}) + (e-1)\bar{Z}(\Phi/e, \mathbf{Q})$ *for a link e in* $\Phi$,
(ii) $\bar{Z}(\Phi, \mathbf{Q}) = \bar{Z}(\Phi_1, \mathbf{Q})\bar{Z}(\Phi_2, \mathbf{Q})$ *if* $\Phi$ *is the disjoint union of* $\Phi_1$ *and* $\Phi_2$, *and*
(iii) $\bar{Z}(\varnothing, \mathbf{Q}) = \varnothing$.
(iv) *If* $\theta : \Phi_1 \to \Phi_2$ *is an isomorphism, then* $\theta(\bar{Z}(\Phi_1, \mathbf{Q})) = \bar{Z}(\Phi_2, \mathbf{Q})$.
(v) *If* $\alpha$ *is an automorphism of* $\Phi$, *then* $\alpha(\bar{Z}(\Phi, \mathbf{Q})) = \bar{Z}(\Phi, \mathbf{Q})$.

Unfortunately, the partition function loses its natural interpretation as a state sum. The obvious question of whether one can find such a sum is open.

Now, make $\mathcal{P}(\mathbf{Q})$ into a semigroup by defining multiplication to be set intersection. The *singleton edge-spin algebra* $\mathbf{A}$ is the semigroup algebra $\mathbb{Z}[E]\mathcal{P}(\mathbf{Q})$, with multiplication symbolized by a raised dot. (It happens to be the $\mathbb{Z}[E]$-Möbius algebra of $\mathbf{Q}$ as described in [5] but we do not make use of that fact.) For a singleton gain graph $\Phi_1$, with vertex $v_0$, we can ignore switching and define a hyperabstract partition function that lies in $\mathbf{A}$, call it

$$\hat{Z}_1(\Phi_1, \mathbf{Q}) := \sum_{A \subseteq E} \operatorname{Fix} \mathfrak{F}(A, v_0) \prod_{f \in A} (f-1).$$

Thus, $Z(\Phi_1, \mathbf{F})$ is obtained from $\mathbf{F} \cdot \hat{Z}_1(\Phi_1, \mathbf{Q})$ by expressing the latter as a $\mathbb{Z}[E]$-linear combination of sets $\mathbf{C} \in \mathcal{P}(\mathbf{Q})$ and evaluating $\mathbf{C} \mapsto |\mathbf{C}|$. A simple calculation shows that

$$\hat{Z}_1(\Phi_1, \mathbf{Q}) = \prod_{f \in E} [1 + (f-1) \operatorname{Fix} \varphi(f)] . \tag{6.1}$$

Now the $\hat{Z}_1$-analog of Lemma 5.2 is immediate from (6.1).

The edge-spin algebras would be more satisfactory if they were combined into one, but that would entail combining the semigroup product in the power set with the orbit partition, which seems impossible.

## 7. Two questions

### 7.1. Negative numbers?

The striking parallelism with the ordinary theory of graph and gain-graph coloring omits one remarkable feature of the latter theories, the interpretation of the chromatic polynomials at negative arguments $\lambda$. Can this be repeated for the grand chromatic polynomial? It is not clear even how to make sense of such a question because there is no variable that corresponds directly to $\lambda$.

### 7.2. Matroids?

The arguments for Theorem 2.5, showing which choices of gain group $\mathfrak{G}$ and spin set $\mathbf{Q}$ make the state chromatic function essentially a function of a matroid of $\Phi$, only apply to the frame matroid and the balanced semimatroid, the same ones associated with the two chromatic polynomials. It is an open question whether some other choices of group and spin set could make $\chi_\Phi(\mathbf{Q})$ a function of another matroid on the edges of $\mathfrak{G}$-gain graphs. (Only one such general gain-graphic matroid is presently known: the lift matroid $L(\Phi)$, or – what is nearly the same – its extension the complete lift matroid $L_0(\Phi)$ [20].)

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