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General Properties of Certain Non-linear Integro-Differential Equations

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1. INTRODUCTION

In this paper some properties of the following non-linear integro-differential equation shall be studied:

$$\tau\dot{\phi}(t) = -\phi(t) - \int_0^t m(t-t')\dot{\phi}(t')\,dt', \tag{1.1a}$$

$$m(t) = \mathcal{F}(\phi(t)). \tag{1.1b}$$

Here ϕ is a real function on $\mathbf{R}^+$ which is continuous and has a continuous derivative $\dot{\phi}$. The initial condition $\phi(0) = 1$ is imposed. Function $\mathcal{F}$ is required to be absolutely monotone [13] on the interval $[0, 1 + \delta^*)$, where $\delta^* > 0$. Function $\mathcal{F}$ may also depend smoothly on some control parameter vector $\mathbf{V} \in \mathbf{K} \subset \mathbf{R}_N$, and if the dependence on $\mathbf{V}$ is emphasized $\mathcal{F}(\mathbf{V}, f)$ shall be written instead of $\mathcal{F}(f)$. The scale τ is required to be positive. It shall be transformed to unity by the substitution $\phi(t) = \hat{\phi}(\hat{t})$, $\hat{t} = t/\tau$, whenever this simplifies notation in the following. The formulated equations appear to be of some interest since they exhibit bifurcation scenarios which differ drastically from those known, e.g., for ordinary differential equations. A special example for $\mathcal{F}$, exhibiting some of the new bifurcation features, is given by

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$$\mathcal{F} = v_1 f + v_2 f^2. \tag{1.2}$$

In this case $\mathbf{V} = (v_1, v_2)$ can be considered as a control parameter vector chosen from the quadrant $\mathbf{K} = \{(v_1, v_2); v_1 \geq 0, v_2 \geq 0\}$.

Equations (1.1) together with the initial condition are equivalent to the integral equation

$$\phi(t) = 1 + \int_0^t [\mathcal{F}(\phi(t')) - \phi(t')]\, dt' - \int_0^t \phi(t - t')\mathcal{F}(\phi(t'))\, dt'. \tag{1.3}$$

The difference of (1.3) compared to previously studied non-linear Volterra integral equations [8] is that ϕ occurs not only as non-linear function $\mathcal{F}(\phi(t))$ but also as kernel $\phi(t - t')$ in the convolution integral. Equations similar to (1.1) appear also in probability theory in connection with non-linear renewal equations and branching processes [3, 4]. In fact the solution $\phi(t)$ to our equations is a monotonic or even a completely monotone function, and can therefore be interpreted as a distribution function or the characteristic function of such a distribution. The essential feature of (1.1) is that the kernel $m(t)$ is not given, but depends on $\phi(t)$.

There is the obvious generalization of the problem to one dealing with M-dimensional vectors $\phi(t) = \{\phi_q(t); q = 1, ..., M\}$. The integro-differential equation then reads

$$\tau_q \dot{\phi}_q(t) = -\phi_q(t) - \int_0^t m_q(t - t')\dot{\phi}_q(t')\, dt'. \tag{1.4}$$

The initial conditions are chosen as before: $\phi_q(0) = 1$. All scales τ_q are positive. The equations are coupled via kernels m_q, which are given as $m_q(t) = \mathcal{F}_q(\phi(t))$. Here the $\mathcal{F}_q(f)$ are absolutely monotone in each of the M variables f_q on the interval $0 \leq f_q < 1 + \delta^*$, $q = 1, ..., M$.

Some heuristic remarks on the motivation of (1.1), (1.4) might be adequate. Let us consider a dynamics specified by M real functions of non-negative time t: ϕ_q, $q = 1, 2, ..., M$. For the equations of motion we have the differential equations with retardation:

$$\ddot{\phi}_q(t) + \nu_q \dot{\phi}_q(t) + \Omega_q^2 \phi_q(t) + \Omega_q^2 \int_0^t m_q(t - t')\dot{\phi}_q(t')\, dt' = 0. \tag{1.5}$$

The first three terms constitute the equation of motion of some oscillator, characterized by its frequency $\Omega_q > 0$ and friction constant $\nu_q > 0$. The integral term is now recognized as a generalization of a Newtonian friction so that the force at time t is proportional to the velocity $\dot{\phi}(t')$ for all times t', preceding t. The derivation of equations like (1.5) is standard in the statistical mechanics theory of many particle systems. The retardation effects come about because certain degrees of freedom, which are not described explicitly by the M functions $\phi_q(t)$, are eliminated. We are interested in long time or low frequency phenomena. Then it is plausible that one can ignore the inertia term $\ddot{\phi}$ in (1.5), and this yields (1.4) with $\tau_q = \nu_q/\Omega_q^2$.

Equations (1.4), (1.5) have been discussed extensively in the physical literature in recent years for polynomial kernels:

$$\mathcal{F}_q(f) = \sum_l \frac{1}{l!} \sum_{p_1 \dots p_l} V^{(l)}(q, p_1 \dots p_l) f_{p_1} \dots f_{p_l}. \tag{1.6}$$

The coefficients $V^{(l)}$ appear as transition probabilities and hence they are non-negative.

Equations (1.4)–(1.6) were studied because their bifurcation scenarios correlate with a number of experimental findings on glass forming systems. A review of these findings [10] summarizes the discussion in the physics literature and in Ref. [9] the reader can find a summary of the mathematical discussions carried out so far. The physical theories do not deal with a coordinate description of the dynamics, but rather with a statistical one in terms of so-called correlation functions. These are positive definite functions $\phi_q(t)$, and hence they can be written as a Fourier transform of some measure $\sigma_q(\omega)$ [5]:

$$\phi_q(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\, d\sigma_q(\omega). \tag{1.7}$$

Proving representation (1.7) for the solution of (1.4) is one of the aims of this paper.

Section 2 describes the general properties of Eqs. (1.1), (1.4). Sections 3–7 present the proofs of the general properties. In Section 8 it is explained by means of example (1.2) what the novel features of the bifurcations consist of. Generalization of the proofs presented for (1.1) to (1.4) is trivial except for two points. The latter are considered in Sections 5 and 7 while in the other cases the discussions are restricted to (1.1).

2. The General Properties

THEOREM 1. *Equation* (1.1) *has a unique solution* ϕ; *it is defined for all* $t \geq 0$ *and is completely monotone.*

Equation (1.3) is quite similar to the Picard equation studied for ordinary differential equations; and it is left to the reader to adopt the standard uniqueness proof to our problem. The global existence proof and complete monotonicity is obtained in Section 3 from a study of the iteration sequence $\phi^{(n)}(t)$, $n = 0, 1, \ldots$:

$$\dot{\phi}^{(n+1)}(t) + \phi^{(n+1)}(t) + \int_0^t m^{(n)}(t-t')\dot{\phi}^{(n+1)}(t')\,dt' = 0 \tag{2.1}$$

$$m^{(n)}(t) = \mathcal{F}(\phi^{(n)}(t)). \tag{2.2}$$

The procedure is started with $\phi^{(0)}(t) = \exp(-t\Gamma)$ with any $\Gamma \geq 0$. For every iteration step one has to solve the analogue of (1.1), but now for a given kernel $m^{(n)}(t)$. This is done most efficiently by using Laplace transforms. For the latter we adopt the convention

$$\hat{F}(s) = \int_0^\infty e^{-st}F(t)\,dt. \tag{2.3}$$

We shall apply (2.3) to functions F which are continuous and bounded. Then $\hat{F}(s)$ is holomorphic for all s with Re $s > 0$. If one assumes ϕ, $\dot{\phi}$ to have these properties, Eqs. (1.1) are equivalent to [13]

$$\hat{\phi}(s)[1 + s(1 + \hat{m}(s))] = 1 + \hat{m}(s). \tag{2.4}$$

As a by-product of the proof one obtains a procedure for solving the equations of motion numerically, which we have applied repeatedly [11]:

COROLLARY 1.1. *The sequence $\phi^{(n)}(t)$ converges uniformly towards the solution $\phi(t)$ for every fixed finite time interval* $0 \leq t \leq T < \infty$.

The proof ensures also uniform convergence for $\phi^{(n)}(\mathbf{V}, t)$ with respect to control parameters $\mathbf{V}$, and it can be extended trivially to prove the same for all derivatives of $\phi^{(n)}(\mathbf{V}, t)$ with respect to V_i. Hence

COROLLARY 1.2. *The solution depends smoothly on* $\mathbf{V}$ *for fixed finite time intervals* $0 \leq t \leq T < \infty$.

Bernstein's theorem [2, 5, 13] ensures that the solution can be written as a Laplace transform of some measure α, which is concentrated on the half axis $\gamma \geq 0$:

$$\phi(t) = \int_0^\infty e^{-\gamma t}\,d\alpha(\gamma). \tag{2.5}$$

This implies that the Laplace transform $\hat{\phi}(s)$ can be analytically continued on the whole s-plane with the possible exception of the negative real axis

[8]. Furthermore, there is a representation (1.7) and there is a spectral density $\phi''(\omega) = 2\pi\, d\sigma(\omega)/d\omega$, which is a smooth function of frequency ω for all $|\omega| \geq \omega_0 > 0$. The spectral density is an even function

$$\phi''(\omega) = \int_0^\infty \frac{\gamma}{\omega^2 + \gamma^2}\, d\alpha(\gamma), \qquad \omega \neq 0. \tag{2.6}$$

Completely monotone functions have a holomorphic extension on the half plane Re $t > 0$ [13]. The following theorem shows that this domain can be extended to include a neighbourhood of the origin $|t| < t_0$. It provides also a convenient means of evaluating the solution for an initial interval $0 \leq t \leq t_0$ numerically. Such an initial solution is needed as a start for efficient numerical procedures to solve (1.1) [11, 6].

THEOREM 2. *There is some radius of convergence $t_0 > 0$ so that the solution for $0 \leq t < t_0$ is given by a power series*

$$\phi(t) = \sum_{n=0}^{\infty} c^{(n)} t^n. \tag{2.7}$$

The coefficients $c^{(n)}$ for $n \geq 1$ can be determined recursively starting with $c^{(0)} = 1$.

It is well known that bifurcation scenarios for non-linear differential equations depend in a very subtle manner on the dimensionality M. This is not so for the generic cases of Eq. (1.4). To indicate the origin for this peculiarity, Theorem 3 and its corollary shall be formulated in their general forms.

The analysis of the long time behaviour will lead to the following set of M implicit equations for M-component vectors f^*:

$$f_q^* = \mathcal{T}_q(f^*); \quad \text{with } \mathcal{T}_q(f^*) = \mathcal{F}_q(f^*)/[1 + \mathcal{F}_q(f^*)]. \tag{2.8}$$

We will be interested only in solutions from the unit cubes $\mathbf{R}_M^C: 0 \leq f_q^* < 1$. Obviously, $0 < 1 - f_q^* \leq 1$. Solutions of (2.8) shall be referred to as fixed points. The linearization of the fixed point equation is specified by an $M \times M$ matrix A, formed with the derivatives as

$$A_{qp}(f^*) = (1 - f_q^*) \frac{\partial \mathcal{F}_q(f^*)}{\partial f_p} (1 - f_p^*); \qquad q, p = 1, \ldots, M. \tag{2.9}$$

The numbers A_{qp} are non-negative. Thus the Frobenius–Perron theory [7, 12] ensures that the matrix A has a real eigenvalue $E(f^*) \geq 0$, which equals its spectral radius.

THEOREM 3. *There is a uniquely determined fixed point* $f(\mathbf{V})$ *so that* $f_q^* \leq f_q(\mathbf{V}), q = 1, \ldots, M$, *for all fixed points* f^*. *For* $E(\mathbf{V}) = E(f(\mathbf{V}))$ *one gets*

$$E(\mathbf{V}) \leq 1. \tag{2.10}$$

The distinguished fixed point $f(\mathbf{V})$ shall be called maximum fixed point and $E(\mathbf{V})$ maximum eigenvalue. The proof is done in Section 5 by studying the iterated mapping

$$f_q^{(n+1)} = \mathcal{T}_q(f^{(n)}), n = 0, 1 \ldots; \qquad f_q^{(0)} = 1, q = 1, \ldots, M. \tag{2.11}$$

As a by-product one obtains a procedure for calculating the maximum fixed point:

COROLLARY 3.1. *The sequence* $f^{(n)}$ *converges monotonically towards* $f(\mathbf{V})$: $f_q^{(n+1)} \leq f_q^{(n)}, \lim_{n\to\infty} f_q^{(n)} = f_q(\mathbf{V})$.

The Jacobian of the implicit equations (2.8) is the same as the determinant of matrix $1 - A$. This motivates the introduction of two concepts: $\mathbf{V} \in \mathbf{K}$ is called a regular point if $E(\mathbf{V}) < 1$ and $\mathbf{V}_c \in \mathbf{K}$ is called a critical point if $E(\mathbf{V}_c) = 1$. Straightforward application of the implicit function theorem shows that the set $\mathcal{D}_c$ of all critical points is closed. The set of regular points is open in the sense that, for $\mathbf{V} \notin \mathcal{D}_c$, there exists some $\varepsilon > 0$ so that all $\mathbf{V}'$ with $|\mathbf{V}' - \mathbf{V}| < \varepsilon$ and $\mathbf{V}' \in \mathbf{K}$ are regular. Similarly, $f(\mathbf{V}')$ is a smooth function of $\mathbf{V}'$ in the specified neighbourhood of $\mathbf{V}$. Furthermore, there is some $\varepsilon' > 0$ so that there is no fixed point f^* for $\mathbf{V}'$ in the specified neighbourhood, obeying $|f_q^* - f_q(\mathbf{V})| < \varepsilon'$ for all q.

The relevance of the preceding considerations for the equation under study is now established by the following two results:

THEOREM 4. *The long time limit of the solutions is given by the maximum fixed point*: $\phi(t \to \infty) = f(\mathbf{V})$.

THEOREM 5. *Regular points are characterized by some frequency* $s_0 > 0$ *so that*

$$\phi(t) - f(\mathbf{V}) = O(e^{-s_0 t}).$$

The proof of this result in Section 7 proceeds in two steps. First, one shows that the moments $\omega^{(n)}$, $n = 0, 1, \ldots$, of the functions $\tilde{\phi}(t) = \phi(t) - f(\mathbf{V})$ are finite:

$$\omega^{(n)} = \int_0^\infty t^n \tilde{\phi}(t)\, dt. \tag{2.12}$$

Then one proves that the power series expansion of the Laplace transform $\hat{\phi}(s)$ has a radius of convergence $s_0 > 0$. Thus the analytical continuation of the Laplace transform reads, for $|s| < s_0$,

$$\hat{\phi}(s) - [f(\mathbf{V})/s] = \sum_n (-s)^n \omega^{(n)}/n!. \tag{2.13}$$

This ensures that the measure α in (2.5) has an atom of mass $f(\mathbf{V})$ at $\gamma = 0$, that α is constant within the interval $0 < \gamma < s_0$, and that s_0 is a point of increase for α. One can write

$$\phi(t) = f(\mathbf{V}) + \int_{s_0}^{\infty} e^{-\gamma t}\, d\alpha(\gamma), \tag{2.14}$$

and this implies $\phi(t) - f(\mathbf{V}) = O(e^{-s_0 t})$, a result which can be sharpened for $M = 1$. Similarly, the measure $\sigma(\omega)$ in (1.6) consists of an atom at $\omega = 0$ and a part with a density $\tilde{\phi}''(\omega)$, which is smooth for all frequencies ω. The susceptibility spectrum $\chi''(\omega) = \omega\phi''(\omega)$, which is of direct relevance for applications in physics [10], is a smoothly varying function of frequency ω at regular points:

$$\chi''(\omega) = \int_{s_0}^{\infty} \frac{\omega\gamma}{\omega^2 + \gamma^2}\, d\alpha(\gamma). \tag{2.15}$$

One concludes that the bifurcations of Eqs. (1.1), (1.4) are caused by those of the special solutions $f(\mathbf{V})$ of the implicit equations (2.8). The Jacobian matrix of the latter equations exhibits generically only non-degenerate vanishing eigenvalues, because of the Frobenius–Perron theorems [7, 12]. The bifurcations occur at the critical point $\mathbf{V}_c$ and generically they are of type A_l, $l = 2, 3, \ldots$ [1]. It is easy to show that all A_l can occur for the restriction to $M = 1$, i.e., for (1.1), where $\mathcal{F}$ can be specialized to a polynomial of a degree not larger than l.

3. Existence of Completely Monotone Solutions

If $\phi^{(n)}(t)$ is completely monotone the same is true for $m^{(n)}(t)$ defined in (2.2) [13, IV, Theorem 2b]. Therefore the Laplace transform $\hat{m}^{(n)}(s)$ exhibits the following four properties: (i) it can be extended to a function holomorphic on $\mathbf{C}\backslash\mathbf{R}^-$, (ii) $\hat{m}(s)^* = \hat{m}(s^*)$, (iii) $\hat{m}^{(n)}(s) \to 0$ for $\operatorname{Re} s \to \infty$, and (iv) $\operatorname{Im} m^{(n)}(s) \leq 0$ for $\operatorname{Im} s > 0$ [8, Chap. 5, Theorem 2.6]. It is now elementary to show that the functions $\hat{\phi}^{(n+1)}(s)$, defined by

$$\hat{\phi}^{(n+1)}(s) = \frac{1 + \hat{m}^{(n)}(s)}{1 + s[1 + \hat{m}^{(n)}(s)]}, \tag{3.1}$$

exhibit the same properties (i)–(iv). Therefore the quoted theorem [8] ensures that $\hat{\phi}^{(n+1)}(s)$ is the analytic continuation of the Laplace transform of a function $\phi^{(n+1)}(t)$, which is completely monotone. Because of (2.4) the so constructed $\phi^{(n+1)}(t)$ is the unique solution of Eq. (2.1).

The preceding paragraph can be summarised: Equations (2.1), (2.2) define uniquely a sequence of functions $\phi^{(n)}(t)$, which are completely monotone and normalised to $\phi^{(n)}(0) = 1$. This implies in particular

$$0 < \phi^{(n)}(t) \leq 1. \tag{3.2}$$

The proof of Theorem 1 and its corollaries is now completed by showing the uniform convergence of the sequence $\phi^{(n)}(t)$. This can be done by modifying the standard proof of the theory of differential equations slightly so that a Picard majorant is identified. Let us first rewrite (2.1), (2.2) in a form corresponding to (1.3):

$$\phi^{(n+1)}(t) = 1 + \int_0^t K(\phi^{(n)}(t'), \phi^{(n)}(t - t'), \phi^{(n+1)}(t'))\, dt'. \tag{3.3}$$

Here the kernel K is a smooth function of its three variables: $K(x, y, z) = \{\mathcal{F}(x) - [1 + \mathcal{F}(y)]z\}$. Because of (3.2), K is needed only on the closed domain: $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$. Thus there exists a Lipschitz constant L, so that

$$|K(x_1, y_1, z_1) - K(x_2, y_2, z_2)| \leq L(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|). \tag{3.4}$$

Below we shall restrict the time also to a finite interval $0 \leq t \leq T$. Let us also restrict the control parameters to a finite closed domain, e.g., $0 \leq V_i \leq V_0$. Then L can be chosen so large that (3.4) holds for all variables within the specified closed domains and the following considerations establish uniformity of the convergence with respect to all variables t, $\mathbf{V}$. One gets for $X_n(t) = |\phi^{(n+1)}(t) - \phi^{(n)}(t)|$, $n = 0, 1, \ldots$, the inequalities

$$X_n(t) \leq 1, \tag{3.5a}$$

$$X_n(t) \leq L \int_0^t (2X_{n-1}(t') + X_n(t'))\, dt'. \tag{3.5b}$$

This sequence shall be compared with a sequence $a_n(t)$, defined recursively by

$$a_n(t) = 3L \int_0^t a_{n-1}(t')\,dt', \qquad n \geq 1; \qquad a_0 = 1. \tag{3.6}$$

Obviously, $X_0(t) \leq a_0(t)$ and $X_1(t) \leq a_1(t)$. Similarly, first, one gets $X_2(t) \leq L \int_0^t (2X_1(t') + 1)\,dt' \leq a_1(t)$ and then, $X_2(t) \leq 3L \int_0^t a_1(t')\,dt' = a_2(t)$. This procedure can be continued by induction to get $X_n(t) \leq a_n(t)$ for all n. The majorant a_n ensures the uniform convergence of $X_n(t)$ and this guarantees the desired properties of the sequence $\phi^{(n)}(t)$.

4. The Power Series Solution

Let us start with an auxiliary problem, obtained from (1.1) by changes of signs: we look for a continuous function $\Phi(t)$ with continuous derivative and $\Phi(t = 0) = 1$ so that

$$\dot{\Phi}(t) = \Phi(t) + \int_0^t M(t - t')\dot{\Phi}(t')\,dt', \tag{4.1}$$

where $M(t) = \mathcal{F}(\Phi(t))$. Since $\mathcal{F}(f)$ is holomorphic for $|f| < 1 + \delta^*$ [13, IV, Theorem 3a] one can write for $|f - 1| < \delta^*$

$$\mathcal{F}(f) = M_0 + \sum_{l=1}^{\infty} F_l(f - 1)^l \tag{4.2a}$$

with $F_l > 0$. Suppose now that there is a solution given for sufficiently small t as power series $\Phi(t) = 1 + \sum_{l=1}^{\infty} C_l t^l$. Then one gets also the expansion $M(t) = M_0 + \sum_{l=1}^{\infty} M_n t^n$, where the non-negative coefficients M_n are determined by

$$M_n = \sum_{l=1}^{n} F_l \sum_{n_1+n_2+\cdots+n_l=n} C_{n_1}C_{n_2}\cdots C_{n_l}. \tag{4.2b}$$

Substitution of the series for Φ and M into (4.1) brings out the fact that this equation is solved iff

$$C_{n+1} = \left[C_n + \sum_{k+l=n} M_k C_l k!l!/(k + l)!\right] \Big/ (n + 1). \tag{4.3}$$

Equations (4.2), (4.3) define recursively a sequence of positive coefficients $C_n > 0$. One repeats a similar discussion for the original equation (1.1).

By induction one finds that a solution of the equations of motion is given by

$$\phi(t) = \sum_{n=0}^{\infty} C_n(-t)^n. \tag{4.4}$$

The formula shows explicitly that the constructed transient solution is completely monotone.

The proof of Theorem 2 shall now be completed by establishing the existence of a non-zero radius of convergence t_0 for the specified power series. This will be achieved by adopting a proof done by Lindelöf for the corresponding problem for ordinary differential equations. It is convenient to work with function $X(t) = \Phi(t) - 1$. Equation (4.1) can be cast into the equivalent form via the analogy of (1.3), (3.3),

$$X(t) = \int_0^t K(X(t'), X(t - t'))\, dt', \tag{4.5}$$

where we use the abbreviation $K(x, y) = [1 + x + x\mathcal{F}(1 + y)]$. The nth approximant shall be denoted by

$$X_n(t) = \sum_{l=1}^{n} C_l t^l, \qquad n = 1, 2, ... \tag{4.6}$$

Evidently,

$$X_{n+1}(t) \le \int_0^t K(X_n(t'), X_n(t - t'))\, dt', \tag{4.7}$$

since the rhs contains only positive monomials $a_l t^l$ with $l > (n + 1)$ in addition to those monomials of order $l \le (n + 1)$, which are necessary to establish (4.2), (4.3). Kernel K is monotonically increasing with x and y; and the approximants are increasing with t. Thus in (4.7) one can write $K(X_n(t'), X_n(t - t')) \le K(X_n(t), X_n(t))$. Hence,

$$X_{n+1} \le A(t, X_n(t)), \tag{4.8}$$

where $A(t, y) = t[1 + y + y\mathcal{F}(1 + y)]$. Consider the smooth mapping of the positive half axis into itself, defined for every parameter t by $y \to A(t, y)$. For $t = 0$ there is a fixed point $y = 0$. Since $\partial A(t, y)/\partial y = 0$ for $t = 0$ there are some $\delta > 0$ and some $\varepsilon > 0$ so that for all $|t| < \delta$ and $|y| < \varepsilon$, $\partial A(t, y)/\partial y < 1$. Therefore the iterated mapping $y_{n+1} = A(t, y_n)$, started with some y_0, converges towards some $y(t)$ provided $|y_0| < \varepsilon$. One chooses t_0 as the minimum of ε/C_1 and δ and derives from (4.8) by induction: $0 \le$

$X_n(t) \leq y_n(t)$ for all n provided $t \leq t_0$. Thus y_n provide a convergent majorant for the X_n.

5. Maximum Fixed Point and Maximum Eigenvalue

Let us introduce a semi-ordering in $\mathbf{R}_M$: $f \leq g$ iff $f_q \leq g_q$ for $q = 1, ..., M$; $f < g$ iff $f \leq g$ and $f \neq g$. With the particular vector e, defined by $e_q = 1$ for all q, the unit cube is the set of all f obeying $0 \leq f \leq e$. Let us note the following properties of the mappings $\mathcal{F}$ and $\mathcal{T}$ defined in (2.8):

$$0 \leq f \Rightarrow 0 \leq \mathcal{F}(f), \qquad 0 \leq \mathcal{T}(f) < e, \tag{5.1a}$$

$$0 \leq f \leq g \Rightarrow \mathcal{F}(f) \leq \mathcal{F}(g); \qquad \mathcal{T}(f) \leq \mathcal{T}(g). \tag{5.1b}$$

For a given fixed point f^* we define the invertible linear transformation in $\mathbf{R}_M$, $f \rightarrow \tilde{f}$, by

$$f_q = f_q^* + (1 - f_q^*)\tilde{f}_q, \qquad q = 1, ..., M. \tag{5.2}$$

This transformation maps f^* to 0, leaves e invariant, and preserves the ordering: $f \leq g$ iff $\tilde{f} \leq \tilde{g}$. One checks easily that the transformation leaves the fixed point equation (2.8) covariant in the following sense: $f_q = \mathcal{T}_q(f)$ holds iff $\tilde{f}_q = \tilde{\mathcal{T}}_q(\tilde{f})$ with $\tilde{\mathcal{T}}_q(\tilde{f}) = \tilde{\mathcal{F}}_q(\tilde{f})/[1 + \tilde{\mathcal{F}}_q(\tilde{f})]$ and

$$\tilde{\mathcal{F}}_q(\tilde{f}) = [\mathcal{F}_q(f) - \mathcal{F}_q(f^*)](1 - f_q^*). \tag{5.3}$$

With (5.2) one checks that the $\tilde{\mathcal{F}}_q$ are absolutely monotone on the intervals $0 \leq \tilde{f}_q < 1 + \delta^*$. Furthermore the derivative matrix of $\tilde{\mathcal{F}}_q$ at the origin is given by the matrix (2.9),

$$\partial\tilde{\mathcal{F}}_q/\partial\tilde{f}_p|_{\tilde{f}=0} = A_{qp}(f^*). \tag{5.4}$$

Consider the sequence $f^{(n)}$ defined in (2.11). Obviously $f^{(1)} \leq f^{(0)} = e$. By complete induction one gets from (5.1): $0 \leq f^{(n+1)} \leq f^{(n)}$ for all n. The sequence is monotone and bounded and so there exists some $g \geq 0$ with $f^{(n)} \rightarrow g$. Since $\mathcal{T}$ is continuous, g is a fixed point. One gets $\tilde{f}^{(n)} \rightarrow \tilde{g}$ and concludes from the found covariance that $\tilde{g} \geq 0$. From (5.2) one gets $g \geq f^*$ so that g is some maximum fixed point. Since $g \geq g'$ and $g' \geq g$ implies $g = g'$, there is no other maximum fixed point, and this proves Corollary 3.1 and the first part of Theorem 3.

Let f^* be a fixed point with $E(f^*) = 1 + \delta$, $\delta > 0$. Equation (2.10) shall be proved by showing that there exists some fixed point g with $g > f^*$. We consider the transformation (5.2). Then $\tilde{f}^* = 0$ is a fixed point of $\tilde{\mathfrak{T}}$. Matrix A_{qp} characterizes the linearization of $\tilde{\mathfrak{F}}$ and $\tilde{\mathfrak{T}}$ as discussed above in connection with (2.9), (5.4). Its spectral radius is $E(f^*)$. The Frobenius–Perron theory guarantees that there is some vector $r > 0$ with

$$\sum_p A_{qp} r_p = (1 + \delta) r_q, \qquad q = 1, ..., M. \tag{5.5}$$

The proof shall be completed by showing that there is some fixed point $\tilde{g} > 0$ for $\tilde{\mathfrak{T}}$. There is some $\varepsilon > 0$ so that $0 < \xi r < e$ for all ξ within the interval $0 < \xi \leq \varepsilon$. $\tilde{\mathfrak{F}}_q(\xi r)$ vanishes for $\xi = 0$, it is non-negative for $\xi > 0$, and it is smooth in ξ; thus one can find some $C > 0$ so that $0 \leq \tilde{\mathfrak{F}}_q(\xi r) \leq C\xi$ for ξ within the stated interval. Now we choose $\xi_0 = \mathrm{Min}(\varepsilon, \delta/2C)$ and introduce $\tilde{f}^{(0)} = \xi_0 r$. One gets

$$\tilde{\mathfrak{F}}_q(\tilde{f}^{(0)}) \geq \sum_p A_{qp} \tilde{f}_p^{(0)} = (1 + \delta) \tilde{f}_q^{(0)},$$

$$1 + \mathfrak{F}_q(\tilde{f}^{(0)}) \leq 1 + C\xi_0 \leq (1 + \delta/2).$$

If one introduces $\tilde{f}^{(1)} = \tilde{\mathfrak{T}}(\tilde{f}^{(0)}) = \tilde{\mathfrak{F}}_g(\tilde{f}^{(0)})(1 + \tilde{\mathfrak{F}}_q(\tilde{f}^{(0)}))$ one gets $\tilde{f}^{(1)} > \tilde{f}^{(0)}$. Now one defines recursively a sequence $\tilde{f}^{(n)}$ by $\tilde{f}^{(n+1)} = \tilde{\mathfrak{T}}(\tilde{f}^{(n)})$ for $n = 0, 1, ...$ From (5.1) one finds by complete induction $e > \tilde{f}^{(n+1)} \geq \tilde{f}^{(n)}$ for all n. Thus the sequence is monotonic and bounded and must therefore converge; $\tilde{f}^{(n)} \to \tilde{g} > 0$. Because of the continuity of $\tilde{\mathfrak{T}}$, $\tilde{g}$ is a fixed point. Now choosing $f^* = f(\mathbf{V})$ we find that $g = f(\mathbf{V}) + (1 - f(\mathbf{V}))\tilde{g} > f(\mathbf{V})$. Since there cannot be a fixed point larger than $f(\mathbf{V})$ the assumption that $E(\mathbf{V}) = 1 + \delta > 1$ leads to a contradiction, and we must have $E(\mathbf{V}) \leq 1$.

6. The Long Time Limit

Equations (1.1) or Eq. (1.4) exhibit a covariance property similar to the one discussed above for the fixed point equation. We will need this result for the case in which f^* in (5.2) is specified to the maximum fixed point $f(\mathbf{V})$. So, let us map the solutions $\phi(t)$ on another completely monotone function $\tilde{\phi}(t)$ according to

$$\phi(t) = f(\mathbf{V}) + (1 - f(\mathbf{V}))\tilde{\phi}(t). \tag{6.1}$$

The new function obeys also the normalization: $\tilde{\phi}(t = 0) = 1$. One checks easily that (1.1) holds for $\tilde{\phi}$ if the time scale τ is replaced by $\tilde{\tau} = (1 -$

$f(\mathbf{V}))\tau$ and if the kernel $m(t)$ is replaced by $\tilde{m}(t) = \tilde{\mathcal{F}}(\tilde{\phi}(t))$. Here the absolutely monotone function $\tilde{\mathcal{F}}$ is given by (5.3), with $f^* = f(\mathbf{V})$.

For the bounded monotone function $\phi(t)$ there exists the long time limit $\phi(t \to \infty) = g$, obeying $0 \le g \le 1$. Because of the covariance property the analogous results hold for the transformed quantity $\tilde{\phi}(t \to \infty) = \tilde{g}$ with $0 \le \tilde{g} \le 1$. This with (6.1) implies, in particular, that $g \ge f(\mathbf{V})$, where the equality holds iff $\tilde{g} = 0$. All derivatives of completely monotone functions vanish for large times, e.g., $\dot{\phi}(t \to \infty) = 0$. Hence one can carry out the long time limit in (1.1) with the result $0 = -g - \mathcal{F}(g)(g - 1)$. Thus g is a fixed point. Since $f(\mathbf{V})$ is the maximum fixed point one gets Theorem 4: $g = f(\mathbf{V})$. Because of (6.1) this is equivalent to

$$\lim_{t\to\infty} \tilde{\phi}(t) = 0. \tag{6.2}$$

7. The Long Time Behavior at Regular Points

Let us start with a preliminary remark. The absolutely monotone function $\tilde{\mathcal{F}}(\tilde{f})$ is holomorphic for $|\tilde{f}| < 1 + \delta^*$ [13, IV, Theorem 3a] so that

$$\tilde{m}(t) = \left[A^{(1)} + \sum_{l=2}^{\infty} A^{(l)} \tilde{\phi}(t)^{l-1} \right] \tilde{\phi}(t). \tag{7.1}$$

The coefficients $A^{(l)}$ are non-negative and $A^{(1)} = E(\mathbf{V})$ is the maximum eigenvalue. Since the $\tilde{\phi}$ decrease monotonically from 1 to 0 if t increases from 0 to ∞ one gets

$$\tilde{m}(t) \le E(t_0)\tilde{\phi}(t), \qquad t \ge t_0. \tag{7.2}$$

Here $E(t)$ abbreviates the non-negative and monotonically decreasing function

$$E(t) = E(\mathbf{V}) + \sum_{l=2}^{\infty} A^{(l)} \tilde{\phi}(t). \tag{7.3}$$

We are interested in regular points $\mathbf{V}$, so that $E(\mathbf{V}) < 1$. Hence there is some $\varepsilon > 0$ and $t_0 \ge 0$ so that

$$0 \le E(t) \le 1 - \varepsilon, \quad t \ge t_0. \tag{7.4}$$

Now we consider the truncated moments, defined for $n = 0, 1, \ldots$ and $x > x_0 \geq 0$:

$$\omega^{(n)}(x, x_0) = \int_{x_0}^{x} t^n \tilde{\phi}(t)\, dt; \qquad \mu^{(n)}(x, x_0) = \int_{x_0}^{x} t^n \tilde{m}(t)\, dt. \tag{7.5}$$

Because of (7.2) one gets

$$\mu^{(n)}(x, x_0) \leq E(x_0)\omega^{(n)}(x, x_0). \tag{7.6}$$

In the following all quantities will be used after the transformation explained in connection with (6.1). The tilde will be dropped to simplify the notations.

If one multiplies (1.1) by t^n and integrates over t between x_0 and x one gets

$$\begin{aligned} \omega^{(n)}(x, x_0) = {} & \mu^{(n)}(x, x_0) \\ & + C^{(n)}(x_0) + n\left[\tau\omega^{(n-1)}(x, x_0) + J(x, x_0)\right] \\ & - \left\{x^n \int_0^x m(x - t')\phi(t')\, dt' + \tau x^n \phi(x)\right\}. \end{aligned} \tag{7.7}$$

Here one uses the abbreviations

$$C^{(n)}(x_0) = \tau x_0^n \phi(x_0) + x_0^n \int_0^{x_0} m(x_0 - t)\phi(t)\, dt, \tag{7.8}$$

$$J(x, x_0) = \int_{x_0}^{x} \int_0^t t^{n-1} m(t - t')\phi(t')\, dt'\, dt. \tag{7.9a}$$

Obviously, the integration can be transformed to

$$\begin{aligned} J(x, x_0) = {} & \sum_{i=0}^{n-1} \binom{n-1}{i} \omega^{(n-1-i)}(x, 0)\mu^{(i)}(x, 0) \\ & - \left\{\int_0^{x_0} t^{n-1} \int_0^t m(t - t')\phi(t')\, dt'\, dt \right. \\ & \left. + \int_0^x \phi(t') \int_{x-t'}^{x} (t + t')^{n-1} m(t)\, dt\, dt' \right\}. \end{aligned} \tag{7.9b}$$

The expressions within the curly brackets in (7.7), (7.9b) are non-negative. By dropping them, (7.7) is changed to an inequality. In addition $\mu^{(n)}(x, x_0)$

shall be eliminated in (7.7) and (7.9b) by using (7.6). The result shall be used with $x = t$, $x_0 = t_0$,

$$[1 - E(t_0)]\omega^{(n)}(t, t_0) \leq C^{(n)}(t_0) + nD^{(n)}(t, t_0), \tag{7.10a}$$

where

$$D^{(n)}(t, t_0) = \tau\omega^{(n-1)}(t, t_0) + E(0) \sum_{i=1}^{n-1} \binom{n-1}{i} \omega^{(n-1-i)}(t, 0)\omega^{(i)}(t, 0). \tag{7.10b}$$

With (7.4) one finds finally

$$\omega^{(n)}(t, t_0) \leq [C^{(n)}(t_0) + nD^{(n)}(t, t_0)]/\varepsilon. \tag{7.11}$$

The number $\omega^{(0)}(t, 0) = \omega^{(0)}(t_0, 0) + \omega^{(0)}(t, t_0)$ is bounded by the finite number $\omega^{(0)}(t_0, 0) + C^{(0)}(t_0)/\varepsilon$, and since it increases monotonically with t one concludes that the zeroth moment $\omega^{(0)} = \omega^{(0)}(\infty, 0)$ exists. Complete induction leads with (7.10b) to the same result for all other moments. So, the numbers $\omega^{(n)}$ in (2.12) are shown to be finite and the same holds for $\mu^{(n)} = \mu^{(n)}(\infty, 0)$.

Let us use (7.7)–(7.9) with $x_0 = 0$ for $t \to \infty$. Since $t^l\phi(t)$ and $t^l m(t)$ vanish for large times one gets

$$\omega^{(n)} = \mu^{(n)} + \tau[\delta_{n0} + n\omega^{(n-1)}] + n \sum_{i=1}^{n-1} \binom{n-1}{i} \omega^{(n-1-i)}\mu^{(i)} \tag{7.12}$$

as a starting equation for the following estimations.

One gets $\mu^{(n)} = \mu^{(n)}(\infty, t_0) + \mu^{(n)}(t_0, 0)$. The first term is estimated with (7.6) as $\mu^{(n)}(\infty, t_0) \leq E(t_0)\omega^{(n)}$. Similarly one estimates the moments in the last term of (7.12); $\mu^{(i)} \leq E(0))\omega^{(i)}$. Finally, one uses (7.6) again to get $\mu^{(n)}(t_0, 0) \leq E(0)\omega^{(n)}(t_0, 0)$ and estimates $\omega^{(n)}(t_0, 0) \leq t_0^{n+1}/(n+1)$, which gives

$$[1 - E(t_0)]\omega^{(n)} \leq \tau\delta_{n0} + E(0)t_0^{n+1}/(n+1) + n\tau\omega^{(n-1)} + nE(0) \sum_{i=1}^{n-1} \binom{n-1}{i} \omega^{(n-1-i)}\omega^{(i)}. \tag{7.13}$$

Let us simplify somewhat by introducing instead of the moments the numbers

$$a_n = \omega^{(n)}/n!, \tag{7.14}$$

so that

$$a_n \leq \delta_{n0} T + X t_0^n/(n+1)! + Y a_{n-1} + Z \sum_{i=1}^{n-1} a_{n-1-i} a_i. \tag{7.15}$$

Here the numbers T, X, Y, and Z are finite and non-negative; they depend on the specified t_0. The result is noted so that it holds for the general case; for the special case $M = 1$ we obviously have $Y = T$ and $X = t_0 Z$.

To construct a first majorant we define a sequence c_n, $n = 0, 1, \ldots$, recursively by: $c_0 = a_0$ and for $n \geq 1$

$$c_n = X t_0^n/(n+1)! + Y c_{n-1} + Z \sum_{k=1}^{n-1} c_{n-1-k} c_k. \tag{7.16}$$

By induction one shows with (7.15) that $0 \leq a_n \leq c_n$. Therefore one obtains for the absolutely monotone polynomial $g^{(n)}(s)$ defined for $s \geq 0$ by $g^{(n)}(s) = \sum_{k=0}^{n} c_k s^k$ the inequality

$$\sum_{k=0}^{n} a_k s^k \leq g^{(n)}(s). \tag{7.17}$$

To construct a Lindelöf majorant for $g^{(n)}(s)$, first we notice that

$$G^{(0)}(s) = T + X \sum_{k=0}^{\infty} (s t_0)^k/(k+1)! \tag{7.18a}$$

is defined for $s \geq 0$ as an absolutely monotone function. Second, we show by induction that a sequence of absolutely monotone functions $G^{(n)}(s)$, $n = 1, 2, \ldots$, is defined for $s \geq 0$ by

$$G^{(n)}(s) = G^{(0)}(s) + s[Y + Z G^{(n-1)}(s)] G^{(n-1)}(s). \tag{7.18b}$$

Third if we write $G^{(n)}(s) = \sum_k C_k^{(n)} s^k$ we show by induction that $c_k = C_k^{(n)}$ for $n \geq k$ and $C_k^{(n-1)} \leq C_k^{(n)}$ for $n \leq k$. This result establishes the desired inequality

$$g^{(n)}(s) \le G^{(n)}(s). \tag{7.19}$$

Consider now the mapping $G \to A(s, G)$, of the positive half axis $G \ge 0$ into itself, defined for all $s \ge 0$ by $A(s, G) = G^{(0)}(s) + s[Y + ZG]G$. For $s = s^* = 0$ this mapping has a fixed point $G^* = G^{(0)}(0)$. Since $\partial A(s^*, G^*)/\partial G = 0$ one concludes that there is some $\sigma > 0$ so that the iterated mapping $G^{(n)} = A(s, G^{(n-1)})$, started with $G^{(0)} = G^{(0)}(s)$, converges to some fixed limit $G(s)$ for all $0 \le s < \sigma$. From (7.19), (7.17), (7.14) one infers that the series (2.13) has some radius of convergence s_0, where $s_0 \ge \sigma$.

The series (2.13) is constructed with the Taylor coefficients of the Laplace transform $\hat{\phi}(s)$ of $\phi(t)$. The latter is positive and can therefore be considered as a density of a measure, concentrated on $0 \le t$ and normalized by $\omega^{(0)}$. Therefore the Taylor series (2.13) converges to the Laplace transform for $s > 0$ [5]. Thus (2.13) describes the analytic continuation of $\hat{\phi}(s)$ till the singularity $-s_0$. This is the singularity nearest to the origin. Inversion of the Laplace transform ensures $\phi(t) = O(\exp(-s_0 t))$.

From (7.1) one gets $m(t) = E(\mathbf{V})\phi(t) + R(t)$, where $R(t) = O(\exp(-2s_0 t))$. Thus $\hat{m}(s) = E(\mathbf{V})\hat{\phi}(s) + \hat{R}(s)$, where the analytic continuation $\hat{R}(s)$ of the Laplace transform of the completely monotone function $R(t)$ is holomorphic except, possibly, for negative real s obeying $s \le -2s_0$. From (2.4) one gets with $Q(s) = 1 + \hat{R}(s)$

$$\hat{\phi}(s) = 2Q(s)/\{1 - E(\mathbf{V}) + sQ(s) + [(1 - E(\mathbf{V}) + sQ(s))^2 + 4sE(\mathbf{V})Q(s)]^{1/2}\}. \tag{7.20}$$

Let us ignore the trivial case $\hat{m} = 0$. Then $\hat{\phi}$ is holomorphic for Re $s > -2s_0$ except for a square root branch point at $s = -s_0$. The Tauberian theorem [5] yields Theorem 5 in a sharpened form:

$$\phi(t) - f(\mathbf{V}) = ae^{-s_0 t}t^{-3/2}(1 + O(1/t)), \qquad 0 < a < \infty. \tag{7.21}$$

The generalization of the proof of Theorem 5 to $M > 1$ is based on the obvious modification of (7.2)–(7.3) to

$$\tilde{m}_q(t) \le \sum_p A_{qp}(t_0)\tilde{\phi}_p(t), \qquad t \ge t_0, \tag{7.22a}$$

$$A_{qp}(t) = A_{qp} + \sum_k A_{qpk}\tilde{\phi}_k(t) \tag{7.22b}$$

Here A_{qp} specifies the $M \times M$ matrix discussed in (5.4) and the non-negative numbers A_{qpk} are sums over the Taylor coefficients of $\tilde{\mathcal{F}}$. The maximum eigenvalue $E(\mathbf{V})$ is the spectral radius ρ of matrix A. Since we consider regular points, there is some ε, $0 < \varepsilon \le \frac{1}{2}$, so that $\rho = 1 - 2\varepsilon$.

The $A_{qp}(t)$ decrease monotonically with increasing t to A_{qp}. Hence one can choose t_0 so large that the spectral radius $\rho(t_0)$ of $A(t_0)$ obeys $0 \leq \rho(t) \leq 1 - \varepsilon$ for $t \geq t_0$. Thus the resolvent $R = (1 - A(t_0))^{-1}$ can be evaluated as a Neumann series $R = 1 + \sum_l A(t_0)^l$. So one gets

$$\sum_p R_{qp}[\delta_{pk} - A_{pk}(t_0)] = \delta_{qk}, \tag{7.23}$$

where $R_{qp} \geq 0$.

The generalization of (7.6)–(7.9) is obvious and leads to the one for (7.10a),

$$\sum_p [\delta_{qp} - A_{qp}(t_0)]\omega_p^{(n)}(t, t_0) \leq C_q^{(n)}(t_0) + nD_q^{(n)}(t, t_0). \tag{7.24}$$

With (7.23) one proceeds to the analogy of (7.11),

$$\omega_q^{(n)}(t, t_0) \leq \sum_p R_{qp}[C_p^{(n)}(t_0) + nD_p^{(n)}(t, t_0)]. \tag{7.25}$$

The rest of the proof follows the same pattern.

8. Conclusions

Let us consider the example specified in (1.2). One checks easily that all points are regular except the ones on two curves. The first curve is the straight line segment $v_1 = 1$, $0 \leq v_2 \leq 1$. The second one is the piece of a parabola given in parameter representation by $v_1^c = (2 - 1/\lambda)/\lambda$, $v_2^c = 1/\lambda^2$, $\frac{1}{2} \leq \lambda \leq 1$. Within the domain $0 \leq v_1 < 1$, $0 \leq v_2 < v_2^c$ the long time limit of $\phi(t)$ is zero. Outside the specified domain the long time limit according to Theorem 4 is given by

$$f(\mathbf{V}) = \frac{v_2 - v_1 + [(v_1 + v_2)^2 - 4v_2]^{1/2}}{2v_2}.$$

At the mentioned parabola piece, $f(\mathbf{V})$ experiences a Whitney fold bifurcation. With increasing coupling parameters it jumps from zero to $f^c = 1 - \lambda$ and then it grows continuously. Let us consider a sequence of parameter points placed on a straight line: $\mathbf{V}_{\pm}^{(n)} = \mathbf{V}^c(1 \pm \varepsilon_n)$, $\varepsilon_n = 1/4^n$, $n = 3, 4, \ldots.$

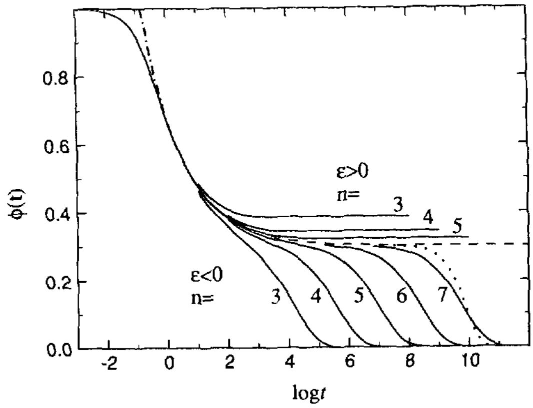

FIG. 1. $\phi(t)$ versus $\log t$ from (1.1), (1.2). The solid curves correspond to parameter points $\mathbf{V} = \mathbf{V}^c(1 \pm \varepsilon_n)$ with $\varepsilon_n = 1/4^n$, $n = 3, 4, \ldots$. The dashed curve is the solution for $\mathbf{V} = \mathbf{V}^c$, and the dotted curve shows $\phi_D(t) = 0.3 \exp(-t/\tau_2)$ with $\tau_2 = 8.9 \times 10^9$. The chain curve shows $\phi(t) - f^c = (t_0/t)^a$, $a = 0.327$, $t_0 = 0.05$.

Thus ε measures the separation from the bifurcation point. Figure 1 exhibits the evolution of the solutions $\phi(t)$ versus $\log(t/\tau)$ evaluated numerically for $\lambda = 0.7$. Figure 2 shows the corresponding susceptibility spectra in double logarithmic representation. By $\log t$ and $\log w$ the Briggsien logarithm is meant.

The decay curves in Fig. 1 for $t/\tau \simeq 1$ are not sensitive to ε-changes,

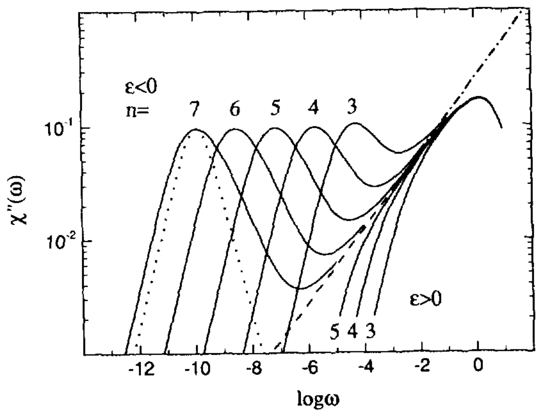

FIG. 2. The susceptibility $\chi''(\omega) = \omega\phi''(\omega)$ versus $\log \omega$ for the same data shown in Fig. 1. Here the dotted curve has been normalized to the same height as that of the full curve representing $n = 7$.

exemplifying Corollary 1.2. This transient dynamics leads to the ε-independent susceptibility peak for $\omega\tau \sim 1$ in Fig. 2. For longer times and small ε, $\phi(t)$ decays towards the plateau f^c so that the $\phi(t)$ versus log t graph is convex. For even larger t, the solutions depend sensitively on ε, in particular on its sign. Either they decay exponentially to $f > f^c$ for $\varepsilon > 0$, exemplifying Theorem 5, or, for $\varepsilon < 0$, they decay from f^c to zero so that the $\phi(t)$ versus log t graph exhibits two inflection points. This long time decay produces the ε-sensitive low frequency susceptibility peaks in Fig. 2. The general features described would also appear for a fold bifurcation of an ordinary differential equation. As a simple example for the latter one might consider the evolution equation $\tau\dot{\phi}(t) = -K(\phi(t))$,

$$K(f) = f[(f - f^c)^2 - \varepsilon].$$

As a first remarkable feature we consider the long time decay at the critical point $\varepsilon = 0$. Ordinary differential equations yield for fold bifurcations in that case $\phi(t) \sim (t_0/t)$, so that the low frequency spectrum diverges logarithmically $\phi''(\omega) \sim \ln \omega$. One can show that also for the solutions of (1.1), (1.2) the spectrum $\phi''(\omega)$ diverges for $\varepsilon = 0$ and $\omega \to 0$. Assuming regular variation one can show that $\phi(t) \sim (t_0/t)^a$ with exponent a the solution of $\Gamma(1 - a)^2/\Gamma(1 - 2a) = \lambda$, $0 < a \leq \frac{1}{2}$. This holds also for the general case, but then the relation of λ to the polynomial $\mathcal{F}$ in (1.6) is more involved. The dashed lines in the figures exhibit the solution for $\varepsilon = 0$ and they follow closely the laws $f^c + (t_0/t)^a$ and $\chi''(\omega) = (\omega/\omega_0)^a$, respectively, with $a = 0.327$ ..., and shown as the chain curves. So the divergency of the spectrum for low frequencies, $\phi''(\omega) \sim 1/\omega^{1-a}$, is much stronger than that in conventional theories. Moreover, different fold singularities can differ in λ, thereby exhibiting different dynamics.

Let us consider $\varepsilon < 0$ and characterize the long time decay by some time scales. Be τ_1 the time for reaching the plateau is $\phi(\tau_1) = f^c$. Another time τ_1' shall specify the position $\omega_{\min}$ of the susceptibility minimum, $\tau_1' = 2\pi/\omega_{\min}$. Both times increase strongly with decreasing $|\varepsilon|$ but so that τ_1/τ_1' is constant. Further scales can be defined by $\phi(\tau_2) = f^c/2$ and $\tau_2' = 2\pi/\omega_{\max}$, where $\omega_{\max}$ is the susceptibility maximum. Again one finds that they increase so that τ_2'/τ_2 is ε-independent. For ordinary differential equations one finds also that the ratio τ_1/τ_2 remains finite for $\varepsilon \to 0$. In this sense one can say that the slow dynamics is specified by a single scale, say τ_1, which is found to diverge like $\tau_1 \sim 1/|\varepsilon|$. As a second remarkable feature of the present results one finds $\tau_1 \sim 1/|\varepsilon|^{\gamma_1}$, $\tau_2 \sim 1/|\varepsilon|^{\gamma_2}$; $\gamma_1 \approx 1.5$, $\gamma_2 \approx 2.3$. If one can extrapolate these results to $\varepsilon \to 0$, one can conclude that the slow dynamics is characterized by two different scales, τ_1 and τ_2. Both diverge upon approaching the singularity, but so that the ratio τ_2/τ_1 diverges as well.

Furthermore one finds that the exponents γ_1, γ_2 depend on λ, i.e., different fold bifurcations exhibit quite different dynamics.

Theorem 5 does not imply that the decay of $\phi(t)$ from f^c to zero is similar to an exponential, say $\phi_D(t) = f^c \exp(-t/\tau_2)$. The latter curve is added as a dotted line in Fig. 1. The $\phi(t)$ versus $\log t$ graph is stretched to a time interval larger than expected for ϕ_D. Similarly, the low frequency susceptibility peak is quite different from the one described by the Fourier transform of ϕ_D: $\chi''_D(\omega) = f^c\omega\tau_2/[1 + (\omega\tau_2)^2]$. Such a function, renormalized to match the lowest susceptibility peak, is also added as a dotted line in Fig. 2. The peak of the $\log \chi''$ versus $\log \omega$ graph is stretched to a frequency interval much larger than expected for the exponential decay. As a third remarkable feature we note that the long time decay and the equivalent low frequency peak shown are very close to the results expected for a Lévy stable distribution specified by its characteristic function $f^c \exp(-t/\tau_2)^\beta$, $\beta = 0.58$.

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