Journal of Algebra 359 (2012) 35–48

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Lie bialgebra structures on the twisted Heisenberg–Virasoro algebra

Dong Liu [a,*], Yufeng Pei [b], Linsheng Zhu [c]

[a] Department of Mathematics, Huzhou Teachers College, Zhejiang Huzhou 313000, China
[b] Department of Mathematics, Shanghai Normal University, Shanghai 200234, China
[c] Department of Mathematics, Changshu Institute of Technology, Jiangsu Changshu 215500, China

ARTICLE INFO

Article history:
Received 7 November 2010
Available online 28 March 2012
Communicated by Vera Serganova

MSC:
17B05
17B37
17B62
17B68

Keywords:
Lie bialgebra
Yang–Baxter equation
Twisted Heisenberg–Virasoro algebra

ABSTRACT

In this paper we investigate Lie bialgebra structures on the twisted Heisenberg–Virasoro algebra. With the determination of certain Lie bialgebra structures on the Virasoro algebra, we determine certain structures on the twisted Heisenberg–Virasoro algebra. Moreover, some general and useful results are obtained. With our methods and results we also can easily determine certain structures on some Lie algebras related to the twisted Heisenberg–Virasoro algebra.

© 2012 Elsevier Inc. All rights reserved.

1. Introduction

Witt type Lie bialgebras were studied in [15,16,20,17], whose generalized cases were considered in [19,21]. Lie bialgebra structures on some Lie (super)algebras including the Schrödinger–Virasoro Lie algebra, the Lie algebra of Weyl type, the $N = 2$ superconformal algebra, were investigated case by case in [8,6,24,12,23], etc. However, the calculations in these papers are very complicated. These algebras are all related to the twisted Heisenberg–Virasoro algebra, which has been first studied by Arbarello et al. in [1], where a connection is established between the second cohomology of certain moduli spaces of curves and the second cohomology of the Lie algebra of differential operators of order at most one. Moreover, the twisted Heisenberg–Virasoro algebra has some relations with

* Corresponding author.
E-mail addresses: liudong@hutc.zj.cn (D. Liu), peiyufeng@gmail.com (Y. Pei), lszhu@cslg.edu.cn (L. Zhu).

0021-8693/$ – see front matter © 2012 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jalgebra.2012.03.009

the full-toroidal Lie algebras and the $N=2$ super conformal algebra, which is one of the most important algebraic objects in superstring theory. The structure and representations for the twisted Heisenberg–Virasoro algebra was studied in [1,2,18,14,10,11], etc. However, Lie bialgebra structures on the twisted Heisenberg–Virasoro Lie algebra have not yet been considered. Drinfel'd [4] posed the problem whether or not there exists a general way to quantize all Lie bialgebras. Although Etingof and Kazhdan [5] gave a positive answer to the question, they did not provide a uniform method to realize quantizations of all Lie bialgebras. As a matter of fact, investigating Lie bialgebras and quantizations is a complicated problem.

In this paper, we shall obtain some Lie bialgebra structures on the twisted Heisenberg–Virasoro algebra. Moreover, we obtain some general results for this kinds of Lie algebras and provide a uniform method (no need some complicated calculations) to obtain Lie (super)bialgebra structures on a series of Lie (super)algebras related to the twisted Heisenberg–Virasoro algebra, including the Lie algebra of differential operators, the Schrödinger–Virasoro algebra, the $N=2$ superconformal algebra, etc.

We note that the center of the twisted Heisenberg–Virasoro algebra is 4-dimensional, which is different from the above algebras whose centers are no more than one-dimensional. So our construction is new and it has the potential to construct more Lie bialgebra structures.

Throughout the paper, we denote by $\mathbb{Z}_+$ the set of all nonnegative integers and $\mathbb{Z}^*$ (resp. $\mathbb{C}^*$) the set of all nonzero elements of $\mathbb{Z}$ (resp. $\mathbb{C}$).

2. Basics

2.1. The twisted Heisenberg–Virasoro algebra

By definition, as a vector space over $\mathbb{C}$, the twisted Heisenberg–Virasoro algebra $\mathcal{L}$ has a basis $\{L_m, I_m, C_L, C_I, C_{LI} \mid m \in \mathbb{Z}\}$, subject to the following relations:

$$
\begin{aligned}
&[L_m, L_n] = (n-m)L_{m+n} + \delta_{m+n,0}\frac{1}{12}\big(m^3-m\big)C_L;\\
&[I_m, I_n] = n\delta_{m+n,0}C_I;\\
&[L_m, I_n] = nI_{m+n} + \delta_{m+n,0}\big(m^2-m\big)C_{LI};\\
&[\mathcal{L}, C_L] = [\mathcal{L}, C_I] = [\mathcal{L}, C_{LI}] = 0.
\end{aligned} \tag{2.1}
$$

Clearly the Heisenberg algebra $H=\mathbb{C}\{I_m, C_I \mid m\in\mathbb{Z}\}$ and the Virasoro algebra $\mathfrak{v}=\mathbb{C}\{L_m, C_L \mid m\in\mathbb{Z}\}$ are subalgebras of $\mathcal{L}$. Moreover, $\mathcal{L}=\bigoplus_{m\in\mathbb{Z}}\mathcal{L}_m$, where $\mathcal{L}_m=\mathbb{C}\{L_m, I_m\}\oplus\delta_{m,0}\mathbb{C}\{C_I, C_L, C_{LI}\}$, is a $\mathbb{Z}$-graded Lie algebra. Clearly $\mathcal{Z}=\{I_0, C_{LI}, C_L, C_I\}$ is a basis of the center $\mathcal{C}$ of $\mathcal{L}$.

Let $\mathbb{C}[t,t^{-1}]$ be the algebra of Laurent polynomials over $\mathbb{C}$. Denote $\mathcal{G}=\operatorname{Diff}\mathbb{C}[t,t^{-1}]$ by the Lie algebra of differential operators over $\mathbb{C}[t,t^{-1}]$. Denote by $D=t\partial$ then as a vector space over $\mathbb{C}$, $\mathcal{G}=\operatorname{Span}_{\mathbb{C}}\{t^mD^n \mid m\in\mathbb{Z},\ n\in\mathbb{N}\}$ with Lie bracket

$$
\big[t^mD^n, t^{m_1}D^{n_1}\big] = t^{m+m_1}\left(\sum_{i=1}^{n}\binom{n}{i}m_1^iD^{n+n_1-i} - \sum_{j=1}^{n_1}\binom{n_1}{j}m^jD^{n+n_1-j}\right). \tag{2.2}
$$

Let $\mathcal{G}_1$ be subalgebra of the Lie algebra $\mathcal{G}$ of differential operators generated by $\{t^m, t^mD \mid m\in\mathbb{Z}\}$. Then $\mathcal{G}_1$ is isomorphic to the centerless twisted Heisenberg–Virasoro algebra by t^mD to L_m and t^m to I_m.

Let M be a $\mathbb{Z}$-graded $\mathcal{L}$-module. Denote by $\operatorname{Der}(\mathcal{L}, M)$ the set of *derivations* $\phi:\mathcal{L}\to M$, namely, ϕ is a linear map satisfying

$$
\phi\big([x,y]\big) = x\cdot\phi(y) - y\cdot\phi(x), \tag{2.3}
$$

and the set $\mathrm{Inn}(\mathcal{L}, M)$ consisting of the derivations v_{inn}, $v \in M$, where v_{inn} is the *inner derivation* defined by $v_{\mathrm{inn}} : x \mapsto x \cdot v$. Then it is well known that $H^1(\mathcal{L}, M) \cong \mathrm{Der}(\mathcal{L}, M)/\mathrm{Inn}(\mathcal{L}, M)$, where $H^1(\mathcal{L}, M)$ is the *first cohomology group* of the Lie algebra $\mathcal{L}$ with coefficients in the $\mathcal{L}$-module M.

A derivation $\phi \in \mathrm{Der}(\mathcal{L}, M)$ is *homogeneous of degree* $m \in \mathbb{Z}$ if $\phi(\mathcal{L}_p) \subset M_{m+p}$ for all $p \in \mathbb{Z}$. Denote $\mathrm{Der}(\mathcal{L}, M)_m = \{\phi \in \mathrm{Der}(\mathcal{L}, M) \mid \deg\phi = m\}$ for $m \in \mathbb{Z}$. Let ϕ be an element of $\mathrm{Der}(\mathcal{L}, M)$. For any $m \in \mathbb{Z}$, define the linear map $\phi_m : \mathcal{L} \to M$ as follows: For any $u \in \mathcal{L}_q$ with $q \in \mathbb{Z}$, write $\phi(u) = \sum_{p\in\mathbb{Z}} u_p$ with $u_p \in M_p$, then we set $\phi_m(u) = u_{q+m}$. Obviously, $\phi_m \in \mathrm{Der}(\mathcal{L}, M)_m$ and we have

$$\phi = \sum_{m\in\mathbb{Z}} \phi_m, \tag{2.4}$$

which holds in the sense that for every $u \in \mathcal{L}$, only finitely many $\phi_m(u) \neq 0$, and $\phi(u) = \sum_{m\in\mathbb{Z}} \phi_m(u)$ (we call such a sum in (2.4) *summable*).

It is well known that $\mathcal{L}$ is the universal central extension of $\mathcal{G}_1$ (see [1]).

Lemma 2.1. *(See [18].) $H^1(\mathcal{L}, \mathcal{L}) = \mathfrak{D}$, where $\mathfrak{D}$ is consist of the following derivations χ:*

$$\begin{aligned}
&\chi(L_n) = (\alpha n + \gamma) I_n + \delta_{n,0}(\gamma + \alpha) C_{LI},\\
&\chi(I_n) = \beta I_n + \delta_{n,0}(\alpha + \gamma) C_I,\\
&\chi(C_L) = -24\alpha C_{LI}, \qquad \chi(C_{LI}) = \beta C_{LI} - \alpha C_I, \qquad \chi(C_I) = 2\beta C_I,
\end{aligned}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, $n \in \mathbb{Z}$.

The following lemma is very useful to investigate Lie bialgebra structures for some Lie algebras related to the Virasoro algebra.

Lemma 2.2. *Suppose that $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ is a $\mathbb{Z}$-graded Lie algebra with a finite-dimensional center $\mathcal{C}_\mathfrak{g}$, and $\mathfrak{g}_0$ is generated by $\{\mathfrak{g}_n,\ n \neq 0\}$, then*

$$H^1(\mathfrak{g}, \mathcal{C}_\mathfrak{g} \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathcal{C}_\mathfrak{g})_0 = \mathcal{C}_\mathfrak{g} \otimes H^1(\mathfrak{g}, \mathfrak{g})_0 + H^1(\mathfrak{g}, \mathfrak{g})_0 \otimes \mathcal{C}_\mathfrak{g},$$

where for any $z_1 \otimes \sigma' + \sigma'' \otimes z_2 \in \mathcal{C}_\mathfrak{g} \otimes \mathrm{Der}(\mathfrak{g}, \mathfrak{g}) + \mathrm{Der}(\mathfrak{g}, \mathfrak{g}) \otimes \mathcal{C}_\mathfrak{g}$, $z_1 \otimes \sigma' + \sigma'' \otimes z_2$ is an element of $\mathrm{Der}(\mathfrak{g}, \mathcal{C}_\mathfrak{g} \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathcal{C}_\mathfrak{g})$ by $(z_1 \otimes \sigma' + \sigma'' \otimes z_2)(x) = z_1 \otimes \sigma'(x) + \sigma''(x) \otimes z_2$.

Proof. Suppose that $r = \dim \mathcal{C}_\mathfrak{g}$, and $\mathcal{C}_\mathfrak{g} = \mathbb{C}\{z_1, z_2, \ldots, z_r\}$, then for any $n \neq 0$, $x \in \mathfrak{g}_n$, $\sigma(x) = \sum_{i=1}^r \sigma_i'(x) \otimes z_i + \sum_{i=1}^r z_i \otimes \sigma_i''(x)$ for some $\sigma_i', \sigma_i'' \in \mathrm{Hom}_\mathbb{C}(\mathfrak{g}, \mathfrak{g})$. Then applying σ to $[x, y]$ for any $x \in \mathfrak{g}_m$, $y \in \mathfrak{g}_n$ with $m + n \neq 0$, we have

$$\begin{aligned}
\sigma_i'\big([x, y]\big) \otimes z_i + z_i \otimes \sigma_i''\big([x, y]\big) = {} & \big[\sigma_i'(x), y\big] \otimes z_i + \big[x, \sigma_i'(y)\big] \otimes z_i\\
& + z_i \otimes \big[\sigma_i''(x), y\big] + z_i \otimes \big[x, \sigma_i''(y)\big].
\end{aligned}$$

Then

$$\begin{aligned}
\sigma_i'\big([x, y]\big) &= \big[\sigma_i'(x), y\big] + \big[x, \sigma_i'(y)\big],\\
\sigma_i''\big([x, y]\big) &= \big[\sigma_i''(x), y\big] + \big[x, \sigma_i''(y)\big].
\end{aligned}$$

Since $\mathfrak{g}_0$ is generated by $\{\mathfrak{g}_n,\ n \neq 0\}$, then σ induces derivations $\sigma_i', \sigma_i'' \in \mathrm{Der}(\mathfrak{g}, \mathfrak{g})$ for $i = 1, 2, \ldots, r$. Moreover σ is an inner derivation if and only if all σ_i', σ_i'', $i = 1, 2, \ldots, r$ are inner derivations. $\square$

2.2. Lie bialgebras

Let us recall the definitions related to Lie bialgebras. Let L be any vector space. Denote ξ the *cyclic map* of $L\otimes L\otimes L$, namely, $\xi(x_1\otimes x_2\otimes x_3)=x_2\otimes x_3\otimes x_1$ for $x_1,x_2,x_3\in L$, and τ the *twist map* of $L\otimes L$, i.e., $\tau(x\otimes y)=y\otimes x$ for $x,y\in L$. The definitions of a Lie algebra and Lie coalgebra can be reformulated as follows. A *Lie algebra* is a pair (L,δ) of a vector space L and a bilinear map $\delta: L\otimes L\to L$ with the conditions:

$$\operatorname{Ker}(1-\tau)\subset\operatorname{Ker}\delta,\qquad \delta\cdot(1\otimes\delta)\cdot\big(1+\xi+\xi^2\big)=0: L\otimes L\otimes L\to L.$$

Dually, a *Lie coalgebra* is a pair (L,Δ) of a vector space L and a linear map $\Delta: L\to L\otimes L$ satisfying:

$$\operatorname{Im}\Delta\subset\operatorname{Im}(1-\tau),\qquad \big(1+\xi+\xi^2\big)\cdot(1\otimes\Delta)\cdot\Delta=0: L\to L\otimes L\otimes L. \tag{2.5}$$

We shall use the symbol "·" to stand for the *diagonal adjoint action*:

$$x\cdot\Big(\sum_i a_i\otimes b_i\Big)=\sum_i\big([x,a_i]\otimes b_i+a_i\otimes[x,b_i]\big).$$

A *Lie bialgebra* is a triple (L,δ,Δ) such that (L,δ) is a Lie algebra, (L,Δ) is a Lie coalgebra, and the following compatible condition holds:

$$\Delta\delta(x\otimes y)=x\cdot\Delta y-y\cdot\Delta x,\quad \forall x,y\in L. \tag{2.6}$$

Denote U the universal enveloping algebra of L, and 1 the identity element of U. For any $r=\sum_i a_i\otimes b_i\in L\otimes L$, define $\boldsymbol{c}(r)$ to be elements of $U\otimes U\otimes U$ by

$$\boldsymbol{c}(r)=\big[r^{12},r^{13}\big]+\big[r^{12},r^{23}\big]+\big[r^{13},r^{23}\big],$$

where $r^{12}=\sum_i a_i\otimes b_i\otimes 1$, $r^{13}=\sum_i a_i\otimes 1\otimes b_i$, $r^{23}=\sum_i 1\otimes a_i\otimes b_i$. Obviously,

$$\boldsymbol{c}(r)=\sum_{i,j}[a_i,a_j]\otimes b_i\otimes b_j+\sum_{i,j}a_i\otimes[b_i,a_j]\otimes b_j+\sum_{i,j}a_i\otimes a_j\otimes[b_i,b_j].$$

Definition 2.3. (1) A *coboundary Lie bialgebra* is a 4-tuple (L,δ,Δ,r), where (L,δ,Δ) is a Lie bialgebra and $r\in\operatorname{Im}(1-\tau)\subset L\otimes L$ such that $\Delta=\Delta_r$ is a *coboundary of* r, where Δ_r is defined by

$$\Delta_r(x)=x\cdot r\quad\text{for } x\in L. \tag{2.7}$$

(2) A coboundary Lie bialgebra (L,δ,Δ,r) is called *triangular* if it satisfies the following *classical Yang–Baxter equation* (CYBE):

$$\boldsymbol{c}(r)=0. \tag{2.8}$$

(3) An element $r\in\operatorname{Im}(1-\tau)\subset L\otimes L$ is said to satisfy the *modified Yang–Baxter equation* (MCYBE) if

$$x\cdot\boldsymbol{c}(r)=0,\quad\forall x\in L. \tag{2.9}$$

Lemma 2.4. *Regard $\mathcal{L}^{\otimes n}$ (the tensor product of n copies of $\mathcal{L}$) as an $\mathcal{L}$-module under the adjoint diagonal action of $\mathcal{L}$. Suppose $r \in \mathcal{L}^{\otimes n}$ satisfying $x \cdot r = 0$, $\forall\, x \in L$. Then $r \in \mathcal{C}_{\mathcal{L}}^{\otimes n}$, where $\mathcal{C}_{\mathcal{L}}$ is the center of $\mathcal{L}$.*

Proof. It can be proved directly by using the similar arguments as those presented in the proof of Lemma 2.2 of [22] (also see the proof Lemma 2.5 of [9]). □

Lemma 2.5. *Let L be a Lie algebra and $r \in \mathrm{Im}(1-\tau) \subset L \otimes L$, then*

$$\left(1+\xi+\xi^2\right) \cdot (1 \otimes \Delta_r) \cdot \Delta_r(x) = x \cdot \boldsymbol{c}(r), \quad \forall x \in L, \tag{2.10}$$

and the triple $(L, [\cdot,\cdot], \Delta_r)$ is a Lie bialgebra if and only if r satisfies MCYBE (2.9).

Proof. The result can be found in [3,4,17]. □

The Lie bialgebra structures over the Witt algebra W (the centerless Virasoro algebra) and the Virasoro algebra $\mathfrak{v}$ were determined in [17].

Proposition 2.6. *(See [17,19].) For the Witt algebra W and the Virasoro algebra $\mathfrak{v}$, $H^1(W, W \otimes W) = H^1(\mathfrak{v}, \mathfrak{v} \otimes \mathfrak{v}) = 0$, and every Lie bialgebra structure on W or $\mathfrak{v}$ is triangular coboundary.*

3. Lie bialgebra structures on the twisted Heisenberg–Virasoro algebra

Regard $\mathcal{V} = \mathcal{L} \otimes \mathcal{L}$ as a $\mathcal{L}$-module under the adjoint diagonal action,then $\mathcal{L}$ and $\mathcal{V}$ are both $\mathbb{Z}$-graded. Now we shall calculate $H^1(\mathcal{L}, \mathcal{V})$ for the twisted Heisenberg–Virasoro algebra $\mathcal{L}$ with Proposition 2.6, and then determine Lie bialgebra structures on the algebra.

For any 6 elements $\alpha, \alpha^\dagger, \beta, \beta^\dagger, \gamma, \gamma^\dagger \in \mathbb{C}$ and $z_1, z_1^\dagger, w_1, w_1^\dagger \in \mathcal{Z}$, one can easily verify that the linear map $\varrho : \mathcal{L} \to \mathcal{V}$ defined below is a derivation:

$$\begin{aligned}
\varrho(L_n) &= (n\alpha+\gamma)z_1 \otimes I_n + \left(n\alpha^\dagger+\gamma^\dagger\right)I_n \otimes z_1^\dagger + \delta_{n,0}\left((\gamma+\alpha)z_1 \otimes C_{LI} + \left(\gamma^\dagger+\alpha^\dagger\right)C_{LI} \otimes z_1^\dagger\right), \\
\varrho(I_n) &= \beta w_1 \otimes I_n + \beta^\dagger I_n \otimes w_1^\dagger + \delta_{n,0}\left((\gamma+\alpha)z_1 \otimes C_I + \left(\gamma^\dagger+\alpha^\dagger\right)C_I \otimes z_1^\dagger\right), \\
\varrho(C_L) &= -24\left(\alpha z_1 \otimes C_{LI} + \alpha^\dagger C_{LI} \otimes z_1^\dagger\right), \\
\varrho(C_{LI}) &= \left(\beta w_1 \otimes C_{LI} + \beta^\dagger C_{LI} \otimes w_1^\dagger\right) - \left(\alpha z_1 \otimes C_I + \alpha^\dagger C_I \otimes z_1^\dagger\right), \\
\varrho(C_I) &= 2\left(\beta w_1 \otimes C_I + \beta^\dagger C_I \otimes w_1^\dagger\right), \quad n \in \mathbb{Z},\ z_i \in \mathcal{Z},\ i = 1,2,3,4.
\end{aligned} \tag{3.1}$$

Clearly ϱ is an outer derivation of $\mathrm{Der}(\mathcal{L}, \mathcal{L} \otimes \mathcal{L})$ if $\alpha, \alpha^\dagger, \beta, \beta^\dagger, \gamma, \gamma^\dagger$ are not zeros. Denote $\mathcal{D}$ the vector space spanned by the such elements ϱ over $\mathbb{C}$. Let $\mathcal{D}^0$ be the subspace of $\mathcal{D}$ consisting of elements ϱ such that $\rho(\mathcal{L}) \subseteq \mathrm{Im}(1-\tau)$. Namely, $\mathcal{D}^0$ is a subspace of $\mathcal{D}$ consisting of elements ϱ with $\alpha = -\alpha^\dagger$, $\beta = -\beta^\dagger$, $\gamma = -\gamma^\dagger$, $z_1 = z_1^\dagger$, $w_1 = w_1^\dagger$.

The main results of this paper can be formulated as follows.

Theorem 3.1.

(i) $\mathrm{Der}(\mathcal{L}, \mathcal{V}) = \mathrm{Inn}(\mathcal{L}, \mathcal{V}) \oplus \mathcal{D}$ *and* $H^1(\mathcal{L}, \mathcal{V}) \cong \mathcal{D}$.

(ii) *Let $(\mathcal{L}, [\cdot,\cdot], \Delta)$ be a Lie bialgebra such that Δ has the decomposition $\Delta_r + \sigma$ with respect to* $\mathrm{Der}(\mathcal{L}, \mathcal{V}) = \mathrm{Inn}(\mathcal{L}, \mathcal{V}) \oplus \mathcal{D}$, *where $r \in \mathcal{V}$ (mod $\mathcal{C} \otimes \mathcal{C}$) and $\sigma \in \mathcal{D}$. Then, $r \in \mathrm{Im}(1-\tau)$ and $\sigma \in \mathcal{D}^0$. Furthermore, $(\mathcal{L}, [\cdot,\cdot], \sigma)$ is a Lie bialgebra provided $\sigma \in \mathcal{D}^0$.*

Proof. For any $\varphi \in \mathrm{Der}(\mathcal{L}, \mathcal{V})$, we first claim that if $m \in \mathbb{Z}^*$ then $\varphi_m \in \mathrm{Inn}(\mathcal{L}, \mathcal{V})$. To see this, denote $\gamma = m^{-1}\varphi_m(L_0) \in \mathcal{V}_m$. Then for any $x_n \in \mathcal{L}_n$, applying φ_m to $[L_0, x_n] = nx_n$ and using $\varphi_m(x_n) \in \mathcal{V}_{n+m}$, we obtain $(m+n)\varphi_m(x_n) - x_n \cdot \varphi_m(L_0) = L_0 \cdot \varphi_m(x_n) - x_n \cdot \varphi_m(L_0) = n\varphi_m(x_n)$, i.e., $\varphi_m(x_n) = \gamma_{\mathrm{inn}}(x_n)$. Thus $\varphi_m = \gamma_{\mathrm{inn}}$ is inner.

We can claim that $\varphi(L_0) \equiv 0 \pmod{\mathcal{C} \otimes \mathcal{C}}$. Indeed, for any $p \in \mathbb{Z}$ and $x_p \in \mathcal{L}_p$, applying φ to $[L_0, x_p] = px_p$, one has $x_p \cdot \varphi(L_0) = 0 \pmod{\mathcal{C} \otimes \mathcal{C}}$. Thus by Lemma 2.4, $\varphi(L_0) \equiv 0 \pmod{\mathcal{C} \otimes \mathcal{C}}$.

Now we claim that for any $\varphi \in \mathrm{Der}(\mathcal{L}, \mathcal{V})$, (2.4) is a finite sum. To see this, one can suppose $\varphi_n = (w_n^\dagger)_{\mathrm{inn}}$ for some $w_n^\dagger \in \mathcal{V}_n$ and $n \in \mathbb{Z}^*$. If $\mathbb{Z}' = \{n \in \mathbb{Z}^* \mid w_n^\dagger \neq 0\}$ is an infinite set, then $\varphi(L_0) = \varphi(L_0) + \sum_{n \in \mathbb{Z}'} L_0 \cdot w_n^\dagger = \varphi(L_0) + \sum_{n \in \mathbb{Z}'} n w_n^\dagger$ is an infinite sum, which is not an element in $\mathcal{V}$, contradicting the fact that φ is a derivation from $\mathcal{L}$ to $\mathcal{V}$. This together with Propositions 3.3, 3.4 below proves Theorem 3.1(i).

By definition, the algebra $\mathfrak{h} = \mathbb{C}\{I_n, C_I, C_L, C_{LI} \mid n \in \mathbb{Z}\}$ is an ideal of $\mathcal{L}$. Set $\mathcal{H} = \mathcal{L} \otimes \mathfrak{h} + \mathfrak{h} \otimes \mathcal{L}$, then $\mathcal{H}$ is a $\mathcal{L}$-submodule of $\mathcal{V}$. The exact sequence $0 \to \mathcal{H} \to \mathcal{V} \to \mathcal{V}/\mathcal{H} \to 0$ induces a long exact sequence

$$\to H^0(\mathcal{L}, \mathcal{K}) \to H^1(\mathcal{L}, \mathcal{H}) \to H^1(\mathcal{L}, \mathcal{V}) \to H^1(\mathcal{L}, \mathcal{K}) \to$$

of $\mathbb{Z}$-graded vector spaces, where all coefficients of the tensor products are in $\mathbb{C}$, and $\mathcal{K} = \mathcal{V}/\mathcal{H}$ is the quotient $\mathcal{L}$-module, on which $\mathfrak{h}$ acts trivially. Clearly $H^0(\mathcal{L}, \mathcal{K}) = \mathcal{K}^{\mathcal{L}} = \{x \in \mathcal{K} \mid \mathcal{L} \cdot x = 0\} = 0$. Then $H^1(\mathcal{L}, \mathcal{H}) \cong H^1(\mathcal{L}, \mathcal{V})$ if we prove that $H^1(\mathcal{L}, \mathcal{K}) = 0$.

Proposition 3.2. $H^1(\mathcal{L}, \mathcal{K}) = 0$.

Proof of Proposition 3.2. The exact sequence $0 \to \mathfrak{h} \to \mathcal{L} \to \mathcal{L}/\mathfrak{h} \to 0$ induces an exact sequence

$$0 \longrightarrow H^1(\mathcal{L}/\mathfrak{h}, \mathcal{K}) \longrightarrow H^1(\mathcal{L}, \mathcal{K}) \longrightarrow H^1(\mathfrak{h}, \mathcal{K})^{\mathcal{L}} \tag{3.2}$$

of the 5-term sequence associated to the Hochschild–Serre spectral sequence $H^p(\mathcal{L}/\mathfrak{h}, H^q(\mathfrak{h}, \mathcal{K}) \Rightarrow H^{p+q}(\mathcal{L}, \mathcal{K})$. Clearly, as $\mathcal{L}$-modules, the quotient modules $\mathcal{K} \cong W \otimes W$, on which $\mathfrak{h}$ acts trivially. Then $H^1(\mathcal{L}/\mathfrak{h}, \mathcal{K}) = 0$ by Proposition 2.6, and $H^1(\mathfrak{h}, \mathcal{K})^{\mathcal{L}}$ embeds into $\mathrm{Hom}_{U(W)}(\mathfrak{h}, W \otimes W)$. So we only need to prove that $\mathrm{Hom}_{U(W)}(\mathfrak{h}, W \otimes W) = 0$.

Let $f \in \mathrm{Hom}_{U(W)}(\mathfrak{h}, W \otimes W)$, for any $n \in \mathbb{Z}$,

$$0 = f\big([L_n, C_{LI}]\big) = \big[L_n, f(C_{LI})\big].$$

So $f(C_{LI}) = 0$. Similarly, $f(I_0) = f(C_L) = 0$. Moreover, $f(I_m) \in \mathcal{V}_m$ since $f([L_0, I_m]) = [L_0, f(I_m)]$.

By $f([L_{-m}, I_m]) = [L_{-m}, f(I_m)]$, we can suppose that $f(I_m) = a_m(L_{2m} \otimes L_{-m} - 3L_m \otimes L_0 + 3L_0 \otimes L_m - L_{-m} \otimes L_{2m})$. Moreover by $mf(I_{n+m}) = [L_n, f(I_m)]$ for any $n \in \mathbb{Z}$ we obtain that $a_m = 0$ for all $0 \neq m \in \mathbb{Z}$. Therefore, $f = 0$. □

Now we shall determine $H^1(\mathcal{L}, \mathcal{H})$. Denote by $\mathcal{L}_C = \mathcal{L} \otimes \mathcal{C} + \mathcal{C} \otimes \mathcal{L}$, then $\mathcal{L}_C$ is a $\mathcal{L}$-submodule of $\mathcal{H}$. The exact sequence $0 \to \mathcal{L}_C \to \mathcal{H} \to \mathcal{H}/\mathcal{L}_C \to 0$ induces

$$\to H^0(\mathcal{L}, \mathcal{H}/\mathcal{L}_C) \longrightarrow H^1(\mathcal{L}, \mathcal{L}_C) \longrightarrow H^1(\mathcal{L}, \mathcal{H}) \longrightarrow H^1(\mathcal{L}, \mathcal{H}/\mathcal{L}_C) \to .$$

Clearly $H^0(\mathcal{L}, \mathcal{H}/\mathcal{L}_C) = (\mathcal{H}/\mathcal{L}_C)^{\mathcal{L}} = 0$. Now we shall prove that $H^1(\mathcal{L}, \mathcal{H}/\mathcal{L}_C) = 0$, then we have $H^1(\mathcal{L}, \mathcal{L}_C) \cong H^1(\mathcal{L}, \mathcal{H})$.

Proposition 3.3. $H^1(\mathcal{L}, \mathcal{H}/\mathcal{L}_C) = 0$.

Proof of Proposition 3.3. For any $\varphi \in \mathrm{Der}(\mathcal{L}, \mathcal{H}/\mathcal{L}_C)_0$, $0 \neq n \in \mathbb{Z}$, one can write $\varphi(L_n)$ and $\varphi(I_n)$ as follows

$$\varphi(L_n) = \sum_{i \in \mathbb{Z},\, i \neq n} b_{n,i} L_i \otimes I_{n-i} + \sum_{i \in \mathbb{Z}, i \neq 0} b_{n,i}^\dagger I_i \otimes L_{n-i} + \sum_{i \neq 0, n} a_{n,i} I_i \otimes I_{n-i},$$

where the sums are all finite, and $a_{n,i}, b_{n,i}, b_{n,i}^\dagger \in \mathbb{C}$ for all $i \in \mathbb{Z}$.

For any $n \in \mathbb{Z}$, the following identities hold,

$$L_1 \cdot (L_n \otimes I_{-n}) = (n-1)L_{n+1} \otimes I_{-n} - nL_n \otimes I_{1-n},$$
$$L_1 \cdot (I_n \otimes L_{-n}) = nI_{n+1} \otimes L_{-n} - (1+n)I_n \otimes L_{1-n},$$
$$L_1 \cdot (I_n \otimes I_{-n}) = nI_{n+1} \otimes I_{-n} - nI_n \otimes I_{1-n}.$$

Let $\triangle$ denote the set consisting of 3 symbols $a, b, b^\dagger$. For each $x \in \triangle$ we define $I_x = \max\{|p| \mid x_{1,p} \neq 0\}$. For $n = 1$, using the induction on $\sum_{x \in \triangle} I_x$ in the above identities, and replacing φ by $\varphi - u_{\mathrm{inn}}$, where u is a proper linear combination of $L_p \otimes I_{-p}$, $I_p \otimes L_{-p}$ and $I_p \otimes I_{-p}$ with $p \in \mathbb{Z}$, one can safely suppose

$$a_{1,k} = b_{1,i} = b^\dagger_{1,j} = 0 \qquad \text{for } i \neq 0, 2,\ j \neq \pm 1,\ k \in \mathbb{Z}. \tag{3.3}$$

Applying φ to $[L_1, L_{-1}] = -2L_0$ and using the fact that $\varphi(L_0) \in \mathcal{C} \otimes \mathcal{C}$, and comparing the coefficients of $L_p \otimes I_{-p}$, $I_p \otimes L_{-p}$ and $I_p \otimes I_{-p}$, we obtain

$$\sum_{p \in \mathbb{Z}} \big((p-2)b_{-1,p-1} - (1+p)b_{-1,p} + (p-1)b_{1,p} - (p+2)b_{1,1+p}\big) = 0,$$
$$\sum_{p \in \mathbb{Z}} \big((p-1)b^\dagger_{-1,p-1} - (p+2)b^\dagger_{-1,p} + (p-2)b^\dagger_{1,p} - (p+1)b^\dagger_{1,1+p}\big) = 0,$$
$$\sum_{p \in \mathbb{Z}} (p-1)c_{-1,p-1} - (p-1)c_{-1,p} + (p-1)c_{1,p} - (p+1)c_{1,p+1} = 0.$$

Then

$$b_{\pm 1,q} = b^\dagger_{\pm 1,q} = a_{-1,q} = 0, \quad \forall q \in \mathbb{Z}. \tag{3.4}$$

Applying φ to $[L_2, L_{-1}] = -3L_1$, and comparing the coefficient of $L_p \otimes I_{1-p}$, $I_p \otimes L_{1-p}$ and $I_p \otimes I_{1-p}$ one can obtain that

$$a_{2,p} = b_{2,p} = b^\dagger_{2,p} = 0. \tag{3.5}$$

Similarly by $[L_1, L_{-2}] = -3L_{-1}$, we have $a_{-2,p} = 0$ for all $p \in \mathbb{Z}$. It follows from this formula and (3.3)–(3.5) that $\varphi(L_{\pm 1}) = \varphi(L_{\pm 2}) = 0$. Thus for any $n \in \mathbb{Z}$, one can deduce $\varphi(L_n) = 0$, since $\mathfrak{v}$ can be generated by $L_{\pm 1}$ and $L_{\pm 2}$. While for I_n, by $[L_{-n}, I_n] = nI_0$ and $[L_m, I_n] = nI_{m+n}$ we have $\varphi(I_n) = 0$. $\square$

Remark 1. The Propositions 3.2, 3.3 hold for many Lie algebras related to the Virasoro algebra. For example, $W(a,b) = \mathbb{C}\{L_m, I_n \mid m, n \in \mathbb{Z}\}$, where L_m, I_m, $m \in \mathbb{Z}$ are the centerless Virasoro, Heisenberg operators, and the twisted action is given by $[L_m, I_n] = (a + bm + n)I_{m+n}$ for all $m, n \in \mathbb{Z}$ and for some $a, b \in \mathbb{C}$ see [7] for detail.

Proposition 3.4. *$H^1(\mathcal{L}, \mathcal{L}_C) = \mathcal{D}$, where $\mathcal{L}_C = \mathcal{C} \otimes \mathcal{L} + \mathcal{L} \otimes \mathcal{C}$.*

Proof of Proposition 3.4. For any $\varphi \in \mathrm{Der}(\mathcal{L}, \mathcal{L}_C)_0$, $n \neq 0$, from Lemmas 2.1, 2.2, we suppose that

$$\varphi(L_n) = (n\alpha + \gamma)z_1 \otimes I_n + \big(n\alpha^\dagger + \gamma^\dagger\big)I_n \otimes z^\dagger_1,$$
$$\varphi(I_n) = \beta w_1 \otimes I_n + \beta^\dagger I_n \otimes w^\dagger_1, \tag{3.6}$$

for some $\alpha, \alpha^\dagger, \beta, \beta^\dagger, \gamma, \gamma^\dagger \in \mathbb{C}$.

By $[L_1, L_{-1}] = -2L_0$ we have

$$\varphi(L_0) = \gamma z_1 \otimes I_0 + \gamma^\dagger I_0 \otimes z_1^\dagger + (\alpha + \gamma) z_1 \otimes C_{LI} + (\alpha^\dagger + \gamma^\dagger) C_{LI} \otimes z_1^\dagger. \tag{3.7}$$

By $[L_1, I_{-1}] = -I_0$ we have

$$\varphi(I_0) = \beta w_1 \otimes I_0 + \beta^\dagger I_0 \otimes w_1^\dagger + (\alpha + \gamma) z_1 \otimes C_I + (\alpha^\dagger + \gamma^\dagger) C_I \otimes z_1^\dagger. \tag{3.8}$$

Moreover C_L, $\varphi(C_{LI})$ and $\varphi(C_I)$ can be computed by $[L_2, L_{-2}] = -4L_0 + \frac{1}{2}C_L$, $[L_{-1}, I_1] = I_0 + 2C_{LI}$ and $[I_1, I_{-1}] = -C_I$. Then we get $\varphi \in \mathcal{D}$. □

Then we get Theorem 3.1(i).

To prove the second part of Theorem 3.1(ii), we need the following lemma.

Lemma 3.5. *Suppose $v \in \mathcal{V}$ such that $x \cdot v \in \mathrm{Im}(1-\tau)$ for all $x \in \mathcal{L}$. Then there exists $u \in \mathrm{Im}(1-\tau)$ such that $v - u \in \mathcal{C} \otimes \mathcal{C}$.*

Proof of Lemma 3.5. First note that $\mathcal{L} \cdot \mathrm{Im}(1-\tau) \subset \mathrm{Im}(1-\tau)$. We prove that after several steps, by replacing v with $v - u$ for some $u \in \mathrm{Im}(1-\tau)$, we get $v \in \mathcal{C} \otimes \mathcal{C}$. Write $v = \sum_{n\in\mathbb{Z}} v_n$, $v_n \in \mathcal{V}_n$. Obviously,

$$v \in \mathrm{Im}(1-\tau) \iff v_n \in \mathrm{Im}(1-\tau), \quad \forall n \in \mathbb{Z}. \tag{3.9}$$

Then $\sum_{n\in\mathbb{Z}} n v_n = L_0 \cdot v \in \mathrm{Im}(1-\tau)$. By (3.9), $n v_n \in \mathrm{Im}(1-\tau)$, in particular, $v_n \in \mathrm{Im}(1-\tau)$ if $n \neq 0$. Thus by replacing v by $v - \sum_{n\in\mathbb{Z}^*} v_n$, one can suppose $v = v_0 \in \mathcal{V}_0$. Write

$$\begin{aligned} v = {} & \sum_{i\in\mathbb{Z}} a_i L_i \otimes L_{-i} + \sum_{0\neq p\in\mathbb{Z}} (b_p L_p \otimes I_{-p} + c_p I_p \otimes L_{-p} + d_p I_p \otimes I_{-p}) \\ & + b L_0 \otimes z + b^\dagger z^\dagger \otimes L_0 \pmod{\mathcal{C} \otimes \mathcal{C}}, \end{aligned}$$

where all the coefficients are in $\mathbb{C}$, $z, z^\dagger \in \mathcal{Z}$ and the sums are all finite. Since the elements of the form $u_{1,p} := L_p \otimes L_{-p} - L_{-p} \otimes L_p$, $u_{2,p} := L_p \otimes I_{-p} - I_{-p} \otimes L_p$, $u_{3,p} := I_p \otimes I_{-p} - I_{-p} \otimes I_p$, $u = L_0 \otimes z - z \otimes L_0$ are all in $\mathrm{Im}(1-\tau)$, replacing v by $v - u$, where u is a combination of some $u_{1,p}$, $u_{2,p}$ and $u_{3,p}$, one can suppose

$$c_p = 0, \quad \forall p \in \mathbb{Z}; \qquad a_p, d_p \neq 0 \implies p > 0 \text{ or } p = 0. \tag{3.10}$$

Then v can be rewritten as

$$\begin{aligned} v = {} & \sum_{p\in\mathbb{Z}_+} a_p L_p \otimes L_{-p} + \sum_{0\neq p\in\mathbb{Z}} b_p L_p \otimes I_{-p} + \sum_{p\in\mathbb{Z}_+} d_p I_p \otimes I_{-p} \\ & + b L_0 \otimes z \pmod{\mathcal{C} \otimes \mathcal{C}}. \end{aligned} \tag{3.11}$$

Assume $a_p \neq 0$ for some $p > 0$. Choose $q > 0$ such that $q \neq p$. Then $L_{p+q} \otimes L_{-p}$ appears in $L_q \cdot v$, but (3.10) implies that the term $L_{-p} \otimes L_{p+q}$ does not appear in $L_q \cdot v$, a contradiction with the fact that $L_q \cdot v \in \mathrm{Im}(1-\tau)$. Then one can suppose $a_p = 0$, $\forall p \in \mathbb{Z}^*$. Similarly, one can also suppose $d_p = 0$, $\forall p \in \mathbb{Z}^*$ and $e_p = 0$, $\forall p \in \mathbb{Z}$. Then (3.11) becomes

$$v = \sum_{0\neq p\in\mathbb{Z}} b_p L_p \otimes I_{-p} + a_0 L_0 \otimes L_0 + b L_0 \otimes z \bmod \mathcal{C} \otimes \mathcal{C}. \tag{3.12}$$

Recall the fact $\mathrm{Im}(1-\tau) \subset \mathrm{Ker}(1+\tau)$ and our hypothesis $\mathcal{L} \cdot v \subset \mathrm{Im}(1-\tau)$, one has

$$\begin{aligned}0 &= (1+\tau)L_1 \cdot v \\ &= -2a_0(L_1 \otimes L_0 + L_0 \otimes L_1) - b(L_1 \otimes z + z \otimes L_1) \\ &\quad + \sum_{0 \neq p \in \mathbb{Z}} \big((p-1)b_p L_{p+1} \otimes I_{-p} - pb_p L_p \otimes I_{1-p}\big) + \big((p-1)b_p I_{-p} \otimes L_{p+1} - pb_p I_{1-p} \otimes L_p\big).\end{aligned}$$

Comparing the coefficients, and noting that the set $\{p \mid b_p \neq 0\}$ is finite, one gets

$$a_0 = b_p = 0, \quad p \neq 1.$$

Moreover $b_1 = b = 0$ if $z \neq kI_0$ for some $k \in \mathbb{C}$, $b + b_1 = 0$ if $z = I_0$.

Then (3.12) can be rewritten as

$$v = b_1(L_1 \otimes I_{-1} - L_0 \otimes I_0) \ (\bmod \mathcal{C} \otimes \mathcal{C}). \tag{3.13}$$

Observing $(1+\tau)L_2 \cdot v = 0$, one has $b_1 = 0$. Thus the lemma follows. □

Proof of Theorem 3.1(ii). Let $(\mathcal{L}, [\cdot,\cdot], \Delta)$ be a Lie bialgebra structure on $\mathcal{L}$. By (2.3), (2.6) and Theorem 3.1(i), $\Delta = \Delta_r + \sigma$, where $r \in \mathcal{V}$ (mod $\mathcal{C} \otimes \mathcal{C}$) and $\sigma \in \mathcal{D}$. By (2.5), $\mathrm{Im}\,\Delta \subset \mathrm{Im}(1-\tau)$, so $\Delta_r(L_n) + \sigma(L_n) \in \mathrm{Im}(1-\tau)$ for $n \in \mathbb{Z}$, which implies that $\alpha + \alpha^\dagger = \gamma + \gamma^\dagger = 0$. Similarly, $\beta + \beta^\dagger = 0$ by the fact that $\Delta_r(I_n) + \sigma(I_n) \in \mathrm{Im}(1-\tau)$ for $n \in \mathbb{Z}$. Thus, $\sigma(\mathcal{L}) \in \mathrm{Im}(1-\tau)$. So $\mathrm{Im}\,\Delta_r \in \mathrm{Im}(1-\tau)$. It follows immediately from Lemma 3.5 that $r \in \mathrm{Im}(1-\tau)$ (mod $\mathcal{C} \otimes \mathcal{C}$), proving the first statement of Theorem 3.1(ii). If $\sigma \in \mathcal{D}^0$, one can easily verify that $(1 + \xi + \xi^2) \cdot (1 \otimes \sigma) \cdot \sigma = 0$ by acting it on generators of $\mathcal{L}$, which shows $(\mathcal{L}, [\cdot,\cdot], \sigma)$ is a Lie bialgebra, and the proof of Theorem 3.1(ii) is completed. □

4. Lie bialgebra structures on some other Lie algebras

With the methods and results in Section 3, we can easily to obtain Lie bialgebra structures on some other Lie algebras related the twisted Heisenberg–Virasoro algebra. Although some of them were studied case by case in some papers (see [12,8,6,9], etc.), their calculations are very complicated.

4.1. Lie algebra of differential operators of order at most one

Let $\mathcal{G}_1$ be the Lie algebra of differential operators of order at most one. Then $\mathcal{G}_1$ is just the centerless twisted Heisenberg–Virasoro algebra.

As a vector space over $\mathbb{C}$, $\mathcal{G}_1$ has a basis $\{L_m, I_m,\ m \in \mathbb{Z}\}$, subject to the following relations:

$$\begin{aligned}[L_m, L_n] &= (n-m)L_{m+n}, \\ [I_m, I_n] &= 0, \\ [L_m, I_n] &= nI_{m+n}.\end{aligned}$$

Lemma 4.1. *For the Lie algebra $\mathcal{G}_1$ and $r \in \mathcal{G}_1 \otimes \mathcal{G}_1$, r satisfies CYBE in* (2.8) *if and only if it satisfies MCYBE in* (2.9).

Remark 2. Let L be a Lie algebra such that $H^0(L; \bigwedge^3 L) = 0$. Then any solution $r \in L \bigwedge L$ of the MCYBE is actually a solution of CYBE. All the following Lie algebras satisfy this condition, then Lemma 4.1 holds for all the following Lie algebras.

For any 6 elements $\alpha, \alpha^\dagger, \beta, \beta^\dagger, \gamma, \gamma^\dagger \in \mathbb{C}$, one can easily verify that the linear map $\varrho : \mathcal{L} \to \mathcal{V}$ defined below is a derivation:

$$\varrho(L_n) = (n\alpha + \gamma) I_0 \otimes I_n + \big(n\alpha^\dagger + \gamma^\dagger\big) I_n \otimes I_0,$$
$$\varrho(I_n) = \beta I_0 \otimes I_n + \beta^\dagger I_n \otimes I_0. \tag{4.1}$$

Denote $\mathcal{D}_1$ the vector space spanned by the such elements ϱ over $\mathbb{C}$. Let $\mathcal{D}_1^0$ be the subspace of $\mathcal{D}_1$ consisting of elements ϱ such that $D(\mathcal{L}) \subseteq \mathrm{Im}(1-\tau)$. Namely, $\mathcal{D}_1^0$ is a subspace of $\mathcal{D}_1$ consisting of elements ϱ with $\alpha = -\alpha^\dagger$, $\beta = -\beta^\dagger$, $\gamma = -\gamma^\dagger$.

According to the results in Section 3, we obtain the following result.

Theorem 4.2.

(i) $\mathrm{Der}(\mathcal{G}_1, \mathcal{G}_1 \otimes \mathcal{G}_1) = \mathrm{Inn}(\mathcal{G}_1, \mathcal{G}_1 \otimes \mathcal{G}_1) \oplus \mathcal{D}_1$ *and* $H^1(\mathcal{G}_1, \mathcal{G}_1 \otimes \mathcal{G}_1) = \mathrm{Der}(\mathcal{G}_1, \mathcal{G}_1 \otimes \mathcal{G}_1)/\mathrm{Inn}(\mathcal{G}_1, \mathcal{G}_1 \otimes \mathcal{G}_1) \cong \mathcal{D}_1$.

(ii) *Let* $(\mathcal{G}_1, [\cdot,\cdot], \Delta)$ *be a Lie bialgebra such that* Δ *has the decomposition* $\Delta_r + \sigma$ *with respect to* $\mathrm{Der}(\mathcal{G}_1, \mathcal{G}_1 \otimes \mathcal{G}_1) = \mathrm{Inn}(\mathcal{G}_1, \mathcal{G}_1 \otimes \mathcal{G}_1) \oplus \mathcal{D}_1$, *where* $r \in \mathcal{G}_1 \otimes \mathcal{G}_1 \pmod{\mathcal{C} \otimes \mathcal{C}}$ *and* $\sigma \in \mathcal{D}_1$. *Then,* $r \in \mathrm{Im}(1-\tau)$ *and* $\sigma \in \mathcal{D}_1^0$. *Furthermore,* $(\mathcal{L}, [\cdot,\cdot], \sigma)$ *is a Lie bialgebra provided* $\sigma \in \mathcal{D}_1^0$.

(iii) *A Lie bialgebra* $(\mathcal{G}_1, [\cdot,\cdot], \Delta)$ *is triangular coboundary if and only if* Δ *is an inner derivation* (*thus* $\Delta = \Delta_r$, *where* $r \in \mathrm{Im}(1-\tau)$ *is a solution of CYBE*).

Proof. Theorem 4.2(iii) follows immediately from (2.5), Definition 2.3 and Lemma 4.1. □

4.2. The Lie algebra of differential operators

Now we consider Lie bialgebra structures on the Lie algebra of differential operators defined in Section 2. Although such works for the Lie algebra of Weyl type (including the centerless Lie algebra of differential operators) was consider in [24], our calculation is very simple. Clearly $\mathcal{C}_d = \mathbb{C}\{1\}$ is the center of $\mathcal{G}$. Moreover, we have

Lemma 4.3. *$\mathcal{G}$ is generated by* $\{t, t^{-1}, D^2\}$ (*see* [25]).

For any 2 elements $\alpha, \alpha^\dagger \in \mathbb{C}$, one can easily verify that the linear map $\zeta_{\alpha,\alpha^\dagger} : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ defined below is a derivation:

$$\zeta_{\alpha,\alpha^\dagger}\big(t^m\big) = 0,$$
$$\zeta_{\alpha,\alpha^\dagger}\big(t^m D^n\big) = \alpha n t^m D^{n-1} \otimes 1 + \alpha^\dagger n \cdot 1 \otimes t^m D^{n-1}, \quad \forall m \in \mathbb{Z},\ n \in \mathbb{Z}_+. \tag{4.2}$$

Clearly $\zeta_{\alpha,\alpha^\dagger}$ is an outer derivation of $\mathrm{Der}(\mathcal{G}, \mathcal{G} \otimes \mathcal{G})$ if $\alpha, \alpha^\dagger$ are not zeros. Denote $\mathcal{D}_d$ the vector space spanned by the such elements $\zeta_{\alpha,\alpha^\dagger}$ over $\mathbb{C}$. Let $\mathcal{D}_d^0$ be the subspace of $\mathcal{D}_d$ consisting of elements $\zeta_{\alpha,\alpha^\dagger}$ such that $\zeta_{\alpha,\alpha^\dagger}(\mathcal{G}) \subseteq \mathrm{Im}(1-\tau)$. Namely, $\mathcal{D}_d^0$ is a subspace of $\mathcal{D}_d$ consisting of elements σ with $\alpha = -\alpha^\dagger$.

With the results in Section 3, we can obtain the main results of this paper as follows.

Theorem 4.4.

(i) $\mathrm{Der}(\mathcal{G}, \mathcal{G} \otimes \mathcal{G}) = \mathrm{Inn}(\mathcal{G}, \mathcal{G} \otimes \mathcal{G}) \oplus \mathcal{D}_d$ *and* $H^1(\mathcal{G}, \mathcal{G} \otimes \mathcal{G}) \cong \mathcal{D}_d$.

(ii) *Let* $(\mathcal{G}, [\cdot,\cdot], \sigma)$ *be a Lie bialgebra such that* σ *has the decomposition* $\Delta_r + \sigma$ *with respect to* $\mathrm{Der}(\mathcal{G}, \mathcal{G} \otimes \mathcal{G}) = \mathrm{Inn}(\mathcal{G}, \mathcal{G} \otimes \mathcal{G}) \oplus \mathcal{D}_d$, *where* $r \in \mathcal{G} \otimes \mathcal{G} \pmod{\mathcal{C}_d \otimes \mathcal{C}_d}$ *and* $\sigma \in \mathcal{D}_d$. *Then,* $r \in \mathrm{Im}(1-\tau)$ *and* $\sigma \in \mathcal{D}_d^0$. *Furthermore,* $(\mathcal{G}, [\cdot,\cdot], \sigma)$ *is a Lie bialgebra provided* $\sigma \in \mathcal{D}_d^0$.

(iii) *A Lie bialgebra* $(\mathcal{G}, [\cdot,\cdot], \Delta)$ *is triangular coboundary if and only if* Δ *is an inner derivation* (*thus* $\Delta = \Delta_r$, *where* $r \in \mathrm{Im}(1-\tau)$ *is some solutions of CYBE*).

Proof. For any $\sigma \in \mathrm{Der}(\mathcal{G}, \mathcal{G} \otimes \mathcal{G})$, we can suppose that $\sigma(t) = 0$ (see [24], or [25]). With $[t^m, t] = 0$ and $[t^m D, t] = t^{m+1}$, we have $\sigma(t^m), \sigma(t^m D) \in \mathcal{G}_1 \otimes \mathcal{G}_1$. Then we have

$$\mathrm{Der}(\mathcal{G}_1, \mathcal{G} \otimes \mathcal{G}) \subset \mathrm{Der}(\mathcal{G}_1, \mathcal{G}_1 \otimes \mathcal{G}_1) + \mathrm{Inn}(\mathcal{G}_1, \mathcal{G} \otimes \mathcal{G}). \tag{4.3}$$

By Theorem 4.2, and replaced σ by $\sigma - \alpha\zeta_{1,0} - \alpha^\dagger\zeta_{0,1}$, one can suppose that

$$\begin{aligned} \sigma\big(t^m D\big) &= \gamma 1 \otimes t^m + \gamma^\dagger t^m \otimes 1, \\ \sigma\big(t^m\big) &= 0. \end{aligned} \tag{4.4}$$

By $[D^2, t^m] = m^2 t^m + 2mt^m D, [D^2, D] = 0$ we have $[\sigma(D^2), t^m] = 2m\sigma(t^m D)$ for all $m \in \mathbb{Z}$, and $[\sigma(D^2), D] = 0$. Then can suppose that

$$\sigma\big(D^2\big) = 2\gamma 1 \otimes D + 2\gamma^\dagger D \otimes 1 + \sum f_i t^i \otimes t^{-i}. \tag{4.5}$$

From $[D^2, t^m D] = m^2 t^m D + 2mt^m D^2$, we have

$$\sigma\big(t^m D^2\big) = 2\gamma 1 \otimes t^m D + 2\gamma^\dagger t^m D \otimes 1 + \frac{1}{2m}\Bigg(\sum_{i \neq 0} f_i\big(it^i \otimes t^{m-i} - it^{m+i} \otimes t^i\big)\Bigg), \tag{4.6}$$

for $m \neq 0$.

Applying σ to $[t^n D^2, t^{-n} D] = -3nD^2 + n^2 D$, and comparing the coefficients of the term $t^i \otimes t^{n-i}$ with lowest first degree, we get $f_i = 0$ if $i \neq 0$. Moreover, $-3f_0 n + n^2(\gamma + \gamma^\dagger) = 0$ holds for all $n \neq 0$. Then $f_0 = 0$ and $\gamma + \gamma^\dagger = 0$. Therefore

$$\sigma\big(t^m D^2\big) = 2\gamma t^m D \otimes 1 + 2\gamma^\dagger 1 \otimes t^m D \tag{4.7}$$

for all $m \in \mathbb{Z}$.

Applying σ to $[t^m D^2, t^n D] = (2n - m)t^{m+n} D^2 + n^2 t^{m+n} D$, and comparing the coefficients of terms $t^{m+n} \otimes 1$ and $1 \otimes t^{m+n}$, we get $\gamma = \gamma^\dagger = 0$. Therefore $\sigma = 0$ by Lemma 4.3. Then we get Theorem 4.4(i). Theorem 4.4 (ii) and (iii) are easily proved. □

4.3. The Schrödinger–Virasoro Lie algebra

The Schrödinger–Virasoro algebra $\mathfrak{sv}$, playing important roles in mathematics and statistical physics, is a infinite-dimensional Lie algebra first introduced by M. Henkle in [6] by looking at the invariance of the free Schrödinger equation in $(1+1)$ dimensions: $(2\mathcal{M}\partial_t - \partial_r^2)\psi = 0$. The structure and the representation theory for it have been well studied by many authors (see [8,13], etc.).

The *Schrödinger–Virasoro Lie algebra* $\mathfrak{sv}$ is the infinite-dimensional Lie algebra $\mathfrak{sv}$ with $\mathbb{C}$-basis $\{L_n, I_n, Y_r \mid n \in \mathbb{Z},\ r \in \mathbb{Z} + \frac{1}{2}\}$ and the following relations

$$\begin{aligned} &[L_m, L_n] = (n - m)L_{m+n} + \delta_{m+n,0}\frac{1}{12}\big(m^3 - m\big)C, \\ &[L_n, Y_r] = \left(r - \frac{n}{2}\right)Y_{n+r}, \\ &[L_m, I_n] = nI_{m+n}, \end{aligned}$$

$$[Y_r, Y_s] = (s-r)I_{r+s},$$
$$[Y_r, I_p] = [I_n, I_p] = 0,$$

where $m, n \in \mathbb{Z}$, $r, s \in \mathbb{Z} + \frac{1}{2}$.

Clearly $\mathcal{C}_s = \mathbb{C}\{I_0, C\}$ is the center of $\mathfrak{sv}$. Denote $\mathcal{Z}_s$ by the set $\{I_0, C\}$.

Lie bialgebra structures over the centerless Schrödinger–Virasoro Lie algebra ($C = 0$) were determined in [8]. However, using the methods and results in Section 3, the original proof in [8] can be greatly simplified.

For any 6 elements $\alpha, a^\dagger, \beta, \beta^\dagger, \gamma, \gamma^\dagger \in \mathbb{C}$, $z_1, z_1^\dagger, w_1, w_1^\dagger \in \mathcal{Z}_s$, one can easily verify that the linear map $\rho : \mathfrak{sv} \to \mathfrak{sv} \otimes \mathfrak{sv}$ defined below is a derivation:

$$\begin{aligned}
\rho(L_n) &= (n\alpha + \gamma)z_1 \otimes I_n + \left(n\alpha^\dagger + \gamma^\dagger\right)I_n \otimes z_1^\dagger, \\
\rho(I_n) &= 2\left(\beta w_1 \otimes I_n + \beta^\dagger I_n \otimes w_1^\dagger\right), \\
\rho(Y_{n-\frac{1}{2}}) &= \beta w_1 \otimes Y_{n-\frac{1}{2}} + \beta^\dagger Y_{n-\frac{1}{2}} \otimes w_1^\dagger, \\
\rho(C) &= 0, \quad n \in \mathbb{Z}.
\end{aligned} \tag{4.8}$$

Denote $\mathcal{D}_s$ the vector space spanned by the such elements ρ. Let $\mathcal{D}_s^0$ be the subspace of $\mathcal{D}_s$ consisting of elements ρ such that $\rho(\mathfrak{sv}) \subseteq \operatorname{Im}(1-\tau)$. Namely, $\mathcal{D}_s^0$ is the subspace of $\mathcal{D}_s$ consisting of elements ρ with $\alpha = -\alpha^\dagger$, $\beta = -\beta^\dagger$, $\gamma = -\gamma^\dagger$ and $z_1 = z_1^\dagger$, $w_1 = w_1^\dagger$.

Theorem 4.5. *(The centerless case was given in [8].)*

(i) $\operatorname{Der}(\mathfrak{sv}, \mathfrak{sv} \otimes \mathfrak{sv}) = \operatorname{Inn}(\mathfrak{sv}, \mathfrak{sv} \otimes \mathfrak{sv}) \oplus \mathcal{D}_s$ *and* $H^1(\mathfrak{sv}, \mathfrak{sv} \otimes \mathfrak{sv}) = \operatorname{Der}(\mathfrak{sv}, \mathfrak{sv} \otimes \mathfrak{sv}) / \operatorname{Inn}(\mathfrak{sv}, \mathfrak{sv} \otimes \mathfrak{sv}) \cong \mathcal{D}_s$.

(ii) *Let* $(\mathfrak{sv}, [\cdot,\cdot], \Delta)$ *be a Lie bialgebra such that* Δ *has the decomposition* $\Delta_r + \sigma$ *with respect to* $\operatorname{Der}(\mathfrak{sv}, \mathfrak{sv} \otimes \mathfrak{sv}) = \operatorname{Inn}(\mathfrak{sv}, \mathfrak{sv} \otimes \mathfrak{sv}) \oplus \mathcal{D}_s$*, where* $r \in \mathfrak{sv} \otimes \mathfrak{sv}$ (mod $\mathcal{C} \otimes \mathcal{C}$) *and* $\sigma \in \mathcal{D}_s$*. Then,* $r \in \operatorname{Im}(1-\tau)$ *and* $D \in \mathcal{D}_s^0$*. Furthermore,* $(\mathfrak{sv}, [\cdot,\cdot], \sigma)$ *is a Lie bialgebra provided* $\sigma \in \mathcal{D}_s^0$.

(iii) *A Lie bialgebra* $(\mathfrak{sv}, [\cdot,\cdot], \Delta)$ *is triangular coboundary if and only if* Δ *is an inner derivation* (*thus* $\Delta = \Delta_r$*, where* $r \in \operatorname{Im}(1-\tau)$ *is a solution of CYBE*).

Proof. Set $\mathcal{L}$ be the subalgebra of $\mathfrak{sv}$ generated by $\{L_m, I_m, C\}$, then $\mathcal{L}$ is the twisted Heisenberg–Virasoro Lie algebra with $C_{LI} = C_I = 0$, $C_L = C$. From the proof of Theorem 3.1(i), we see that the most difficulty of the proof (i) is to determine the terms of $\varphi(L_n), \varphi(I_n)$ for $\varphi \in \operatorname{Der}(\mathfrak{sv}, \mathfrak{sv} \otimes \mathfrak{sv})_0$. It is easy to prove that $\varphi(L_0) \in \mathcal{C}_s \otimes \mathcal{C}_s$ (see the proof of Theorem 3.1(i)).

Now we suppose that

$$\varphi(L_n) = \sum_{i \in \mathbb{Z}} a_{n,i} Y_{i-\frac{1}{2}} \otimes Y_{n-i+\frac{1}{2}} \ (\text{mod } \mathcal{L} \otimes \mathcal{L}),$$
$$\varphi(I_n) = \sum_{i \in \mathbb{Z}} b_{n,i} Y_{i-\frac{1}{2}} \otimes Y_{n-i+\frac{1}{2}} \ (\text{mod } \mathcal{L} \otimes \mathcal{L}),$$

where the sums are all finite, and $a_{n,i}, b_{n,i} \in \mathbb{C}$ for all $i \in \mathbb{Z}$.

Remark 3. Although $\mathcal{L}$ is not an ideal of $\mathfrak{sv}$, the terms $Y_r \otimes Y_s$ are only obtained from $Y \otimes Y$, where $Y = \mathbb{C}\{Y_r \mid r \in \mathbb{Z} + \frac{1}{2}\}$ by adjoint actions. So we can use the notation $x = y$ (mod $\mathcal{L} \otimes \mathcal{L}$) if $x - y \in \mathcal{L} \otimes \mathcal{L}$.

With

$$L_1 \cdot (Y_{n-\frac{1}{2}} \otimes Y_{\frac{1}{2}-n}) = (n-1)Y_{n+\frac{1}{2}} \otimes Y_{\frac{1}{2}-n} - nY_{n-\frac{1}{2}} \otimes Y_{3/2-n},$$

we can replace φ by $\varphi - u_{inn}$ and suppose that

$$\varphi(L_1) = a_{1,0}Y_{-\frac{1}{2}} \otimes Y_{\frac{3}{2}} + a_{1,2}Y_{\frac{3}{2}} \otimes Y_{-\frac{1}{2}} \pmod{\mathcal{L} \otimes \mathcal{L}}. \tag{4.9}$$

Applying φ to $[L_{-1}, L_1] = 2L_0$, we have $a_{-1,i} = 0$ if $i \neq 0, \pm 1$, and $2a_{-1,-1} + a_{-1,0} + 2a_{1,0} = a_{-1,0} + 2a_{-1,1} + 2a_{1,2} = 0$.

Applying φ to $[L_2, L_{-1}] = -3L_1$, $[L_{-2}, L_1] = 3L_{-1}$ and $[L_2, L_{-2}] = -4L_0$, we obtain $a_{\pm 1,i} = 0$ for all $i \in \mathbb{Z}$, $a_{2,p} = 0$ if $p \neq 0, 1, 2, 3$ $a_{-2,q} = 0$ if $q \neq -2, -1, 0, 1$, and $a_{2,2} = -a_{2,1} = 3a_{2,0}$, $a_{2,3} = -a_{2,0} = a_{-2,-2} = -a_{-2,1}$, $a_{-2,-1} = -3a_{-2,-2} = -a_{-2,0}$.

Moreover, replacing φ by suitable multiples of v_{inn} for $v = Y_{\frac{1}{2}} \otimes Y_{-\frac{1}{2}} - Y_{-\frac{1}{2}} \otimes Y_{\frac{1}{2}}$ (here $v \cdot L_{\pm 1} = 0$), one can suppose that $a_{\pm 2,i} = 0$ for all $i \in \mathbb{Z}$.

Since all L_n, $n \in \mathbb{Z}$ are generated by $L_{\pm 1}, L_{\pm 2}$, then we have $a_{n,i} = 0$ for all $i \in \mathbb{Z}$. However, by $[L_{-n}, I_n] = nI_0 \in \mathcal{C}$, we have $[L_{-n}, \varphi(I_n)] = 0 \pmod{\mathcal{L} \otimes \mathcal{L}}$. Then we have $\varphi(I_n) = 0$ if n is even, and $\varphi(I_n) = b_n(Y_{-\frac{n}{2}} \otimes Y_{\frac{3n}{2}} - Y_{\frac{3n}{2}} \otimes Y_{-\frac{n}{2}})$ if n is odd.

Moreover, applying φ to $[L_m, I_n] = nI_{m+n}$ we have $\varphi(I_n) = 0 \pmod{\mathcal{L} \otimes \mathcal{L}}$.

Now, according to the proof of Theorem 3.1, for any $0 \neq n \in \mathbb{Z}$, we can replace φ by $\varphi - \sigma$ for some $\sigma \in \mathcal{D}_s$, we can suppose that

$$\varphi(L_n) = 0,$$
$$\varphi(I_n) = 0.$$

It is easily to prove that $\varphi(Y_r) = 0$.

The proofs of (ii) and (iii) are the same as that in [8]. □

Denote by $Y = \mathbb{C}\{Y_r \mid r \in \mathbb{Z} + \frac{1}{2}\}$, then Y is a $\mathcal{L}$-module. From the proof of the Theorem 4.4, we obtain the following result, which is very useful in determining Lie bialgebra (super-bialgebra) structures on some Lie (super)algebras, for example, the $N = 2$ superconformal Neveu–Schwarz algebra, etc.

Corollary 4.6. $H^1(\mathfrak{v}, Y \otimes Y) = H^1(\mathcal{L}, Y \otimes Y) = 0$. □

Remark 4. We can also use the above ideas to determine Lie bialgebra structures on some other Lie algebras and Lie superalgebras related to the twisted Heisenberg–Virasoro algebra, for example, the twisted Schrödinger–Virasoro Lie algebra (for which, the direct calculation is very complicated, see [6]), the $N = 2$ superconformal algebra [9], etc.

Acknowledgments

This work was partially supported by the NNSF (No. 11071068, No. 11101285), the ZJNSF (Nos. D7080080, Y6100148), the "New Century 151 Talent Project" (2008), the "Innovation Team Foundation of the Department of Education" (No. T200924) of Zhejiang Province, the Shanghai Natural Science Foundation (11ZR1425900) and the Innovation Program of Shanghai Municipal Education Commission (11YZ85).

The part of the work was done during the authors visit in Kavli Institute for Theoretical Physics China (KITPC). The authors would like to express their special gratitude to KITPC for financial support.

References

[1] E. Arbarello, C. De Concini, V.G. Kac, C. Procesi, Moduli spaces of curves and representation theory, Comm. Math. Phys. 117 (1988) 1–36.
[2] Y. Billig, Representations of the twisted Heisenberg–Virasoro algebra at level zero, Canad. Math. Bull. 46 (4) (2003) 529–533.
[3] V.G. Drinfeld, Constant quasiclassical solutions of the Yang–Baxter quantum equation, Soviet Math. Dokl. 28 (3) (1983) 667–671.
[4] V.G. Drinfeld, Quantum groups, in: Proceeding of the International Congress of Mathematicians, Vols. 1, 2, Berkeley, CA, 1986, Amer. Math. Soc., Providence, RI, 1987, pp. 798–820.
[5] P. Etingof, D. Kazhdan, Quantization of Lie bialgebras I, Selecta Math. (N. S.) 2 (1996) 1–41.
[6] H. Fa, Y. Li, J. Li, The Schrödinger–Virasoro type Lie bialgebra: A twisted case, Front. Math. China 6 (4) (2011) 641–657.
[7] S. Gao, C. Jiang, Y. Pei, Low-dimensional cohomology groups of the Lie algebras $W(a,b)$, Comm. Algebra 39 (2) (2011) 397–423.
[8] J. Han, J. Li, Y. Su, Lie bialgebras of the Schrödinger–Virasoro algebra, J. Math. Phys. 50 (8) (2009) 083504.
[9] D. Liu, L. Chen, L. Zhu, Lie superbialgebra structures on the $N=2$ superconformal algebra, J. Geom. Phys. 62 (2012) 826–831.
[10] D. Liu, C. Jiang, Harish–Chandra modules over the twisted Heisenberg–Virasoro algebra, J. Math. Phys. 49 (1) (2008) 012901.
[11] D. Liu, Y. Wu, L. Zhu, Whittaker modules over the twisted Heisenberg–Virasoro algebra, J. Math. Phys. 51 (2) (2010) 023524.
[12] J. Li, Y. Su, Lie bialgebra structures on the W-algebra $W(2,2)$, Algebra Colloq. 17 (2) (2010) 181–190.
[13] J. Li, Y. Su, Representations of the Schrödinger–Virasoro algebras, J. Math. Phys. 49 (5) (2008) 053512.
[14] R. Lu, K. Zhao, Classification of irreducible weight modules over the twisted Heisenberg–Virasoro algebra, Commun. Contemp. Math. 12 (2) (2010) 183–205.
[15] W. Michaelis, Lie coalgebras, Adv. Math. 38 (1980) 1–54.
[16] W. Michaelis, A class of infinite-dimensional Lie bialgebras containing the Virasoro algebras, Adv. Math. 107 (1994) 365–392.
[17] S.H. Ng, E.J. Taft, Classification of the Lie bialgebra structures on the Witt and Virasoro algebras, J. Pure Appl. Alg. 151 (2000) 67–88.
[18] R. Shen, C. Jiang, The derivation algebra and automorphism group of the twisted Heisenberg–Virasoro algebra, Comm. Algebra 34 (2006) 2547–2558.
[19] G. Song, Y. Su, Lie bialgebras of generalized Witt type, Sci. China Ser. A 49 (2006) 533–544.
[20] E.J. Taft, Witt and Virasoro algebras as Lie bialgebras, J. Pure Appl. Alg. 87 (1993) 301–312.
[21] Y. Wu, G. Song, Y. Su, Lie bialgebras of generalized Witt type II, Comm. Algebra 35 (6) (2007) 1992–2007.
[22] Y. Wu, G. Song, Y. Su, Lie bialgebras of generalized Virasoro-like type, Acta Math. Sin. (Engl. Ser.) 22 (2006) 1915–1922.
[23] H. Yang, Y. Su, Lie bialgebras over the Ramond $N=2$ super-Virasoro algebras, Chaos Solitons Fractals 40 (2) (2009) 661–671.
[24] X. Yue, Y. Su, Lie bialgebra structures on Lie algebras of generalized Weyl type, Comm. Algebra 36 (4) (2008) 1537–1549.
[25] K. Zhao, Derivation algebras of the Lie algebra of differential operators, Chinese Sci. Bull. 38 (2) (1993) 100–104.