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Incidence and strong edge colorings of graphs

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Abstract

We define the incidence coloring number of a graph and bound it in terms of the maximum degree. The incidence coloring number turns out to be the strong chromatic index of an associated bipartite graph. We improve a bound for the strong chromatic index of bipartite graphs all of whose cycle lengths are divisible by 4.

1. Introduction

Let $G=(V,E)$ be a multigraph of order n and of size m. Let

$$I=\{(v,e)\colon v\in V,\ e\in E,\ v \text{ is incident with } e\}$$

be the set of *incidences* of G. We say that two incidences (v,e) and (w,f) are *neighborly* provided one of the following holds:

(i) $v=w$,
(ii) $e=f$,
(iii) the edge $\{v,w\}$ equals e *or* f.

The configurations associated with (i)–(iii) are pictured in Fig. 1.

We define an *incidence coloring* of G to be a coloring of its incidences in which neighborly incidences are assigned different colors. The *incidence coloring number* of G, denoted by $\iota(G)$, is the smallest number of colors in an incidence coloring. An *edge coloring* of G is a coloring of the edges of G in which edges of the same color form a matching. The *chromatic index* $q(G)$ of G equals the smallest number of colors in an

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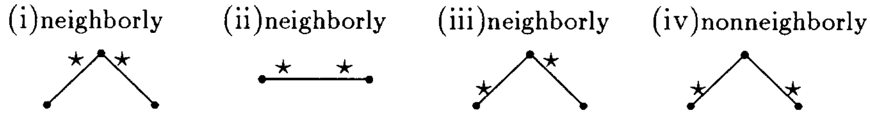

Fig. 1. Examples of neighborly and nonneighborly incidences. A $*$ above edge e closest to vertex v represents incidence (v, e).

edge coloring. A *strong edge coloring* of G is a coloring of the edges of G in which edges of the same color form an induced matching.[1] The *strong chromatic index* $\mathrm{sq}(G)$ equals the smallest number of colors in a strong edge coloring.

Let $V=\{v_1, \ldots, v_n\}$ be the vertices of G and let $E=\{e_1, \ldots, e_m\}$ be the edges. The vertex-edge incidence matrix of G is the n by m (0, 1)-matrix $B=[b_{ij}]$ with $b_{ij}=1$ if and only if v_i is incident with e_j. Thus the 1's of B correspond to the incidences of G. A set of k mutually nonneighborly incidences of G corresponds to a permutation submatrix of B of order k. The incidence coloring number of G equals the smallest number of permutation submatrices of B which partition its 1's. Let $H=H(G)$ be the bipartite graph of order $n+m$ with bipartition V, E in which v_i is adjacent to e_j if and only if v_i is incident with e_j in G. An incidence coloring of G corresponds to a partition of the edges of H into induced matchings. Thus $\iota(G)$ equals the strong chromatic index $\mathrm{sq}(H)$ of H.

It has been conjectured by Erdős and Nešetřil [2] that the strong chromatic index of a multigraph of maximum degree Δ is at most

$$\begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even,} \\ \frac{5}{4}\Delta^2-\frac{1}{2}\Delta+\frac{1}{4} & \text{if } \Delta \text{ is odd.} \end{cases}$$

Horák et al. [6] and Andersen [1] have proved this conjecture if $\Delta=3$. It has also been conjectured by Faudree et al. [3, 4] that the strong chromatic index of a bipartite multigraph of maximum degree Δ is at most Δ^2. They proved this conjecture under the assumption that all cycle lengths are divisible by 4. Let H be a bipartite multigraph with bipartition X, Y in which the maximum degree of a vertex of X is α and of Y is β. We *conjecture* that $\mathrm{sq}(H)\leqslant\alpha\beta$. We prove this conjecture if $H=H(G)$ for some graph G and for graphs H all of whose cycle lengths are divisible by 4.

We may generalize the notion of incidence coloring to any incidence structure $\mathscr{I}$. The incidence coloring number of $\mathscr{I}$ is the strong chromatic index of the associated incidence matrix (bipartite multigraph). We give upper bounds for the incidence coloring numbers of projective and affine planes verifying the above conjecture in these instances.

[1] That is, a matching which is an induced subgraph.

2. Main results

Let G be a graph with maximum degree Δ. A simple lower bound for its incidence coloring number is

$$\iota(G) \geqslant \Delta + 1. \tag{1}$$

To see this, let v be a vertex of degree Δ. Let $e_1, \ldots, e_\Delta$ be the edges incident with v, and let $e_1 = \{v, w\}$. Then each pair of the $\Delta + 1$ incidences $(v, e_1), \ldots, (v, e_\Delta)$, (w, e_1) is neighborly and hence cannot be colored the same. We show that equality holds in (1) for complete graphs and trees.

For each vertex v of a graph the set of incidences of the form (v, e) is denoted by I_v. For an incidence coloring the set of colors assigned to the incidences in I_v is denoted by C_v.

Theorem 2.1. *For each $n \geqslant 2$, $\iota(K_n) = n$.*

Proof. Let the vertices of K_n be $\{1, \ldots, n\}$. We prove by induction on n that there is an incidence coloring of K_n with the n colors $\{1, 2, \ldots, n\}$ having the property that $C_k = \{1, \ldots, k-1, k+1, \ldots, n\}$, $(k = 1, \ldots, n)$. If $n = 2$ this is obvious. Let $n \geqslant 3$ and take such a coloring for K_{n-1}. We color the incidence $(n, \{n, i\})$ with the color i and the incidence $(i, \{n, i\})$ with the color n $(i = 1, \ldots, n-1)$ and obtain the desired coloring for K_n. $\square$

Theorem 2.2. *Let T be a tree of order $n \geqslant 2$ with maximum degree Δ. Then $\iota(T) = \Delta + 1$.*

Proof. We prove the theorem by induction on n. Let v be a pendant vertex with pendant edge $e = \{v, w\}$, and let T' be the tree obtained from T by deleting v. Let $w_1, \ldots, w_k$ be the vertices adjacent to w in T'. Let T'_i be the subtree of T' rooted at w containing w_i (the trees $T'_1, \ldots, T'_k$ have only the vertex w in common). By induction there is an incidence coloring of T' with at most $\Delta + 1$ colors. If two of the incidences $(w_i, \{w, w_i\})$ $(w_j, \{w, w_j\})$ are colored differently, an interchange of their colors in T'_i gives another incidence coloring of T'. Hence we may assume that each of the incidences $(w_i, \{w, w_i\})$ is colored the same. There are now at most Δ colors affecting the incidence (w, e) and at most $\Delta - 1$ colors affecting (v, e). Hence at least two of the $\Delta + 1$ colors are available to color these two incidences giving an incidence coloring of T. $\square$

Theorem 2.3. *For all $m \geqslant n \geqslant 2$, $\iota(K_{m,n}) = m + 2$.*

Proof. Let the vertices of degree n be $w_1, \ldots, w_m$ and let the vertices of degree m be $u_1, \ldots, u_n$. By (1) $\iota(K_{m,n}) \geqslant m + 1$. Suppose $K_{m,n}$ has an $m + 1$ incidence coloring using the colors $1, \ldots, m+1$. Then each of the incidences $(w_i, \{w_i, u_1\})$ $(1 \leqslant i \leqslant m)$ is colored the same, as are the incidences $(w_i, \{w_i, u_2\})$. This easily gives a contradiction. Hence $\iota(K_{m,n}) \geqslant m + 2$.

To complete the proof, it suffices to incidence color $K_{m,m}$ with $m+2$ colors $1,2,\ldots,m+2$. We color the incidences $(w_i,\{w_i,u_j\})$ as follows. The incidences $(w_i,\{w_i,u_1\}),\ldots,(w_i,\{w_i,u_m\})$ are colored with the colors $1,\ldots,i-1,i+1,\ldots,m+1$, respectively. For each j, the m incidences I_{u_j} are affected by at most two colors, and hence there are at least m colors available with which to color the incidences in I_{u_j}. Since different I_{u_j}'s can be colored independently, we may complete the coloring. $\square$

An upper bound for the incidence coloring number of a graph can be obtained from Vizing's theorem (see e.g. [5]). Let G be a graph with maximum degree Δ. By Vizing's theorem $q(G)=\Delta$ or $\Delta+1$, and hence the edges of G can be partitioned into Δ or $\Delta+1$ matchings. The incidences of a matching can be colored with two colors. Hence the incidences of G can always be colored with $2(\Delta+1)$ colors. If the chromatic index of G is Δ, then 'doubling' an edge coloring gives a 2Δ incidence coloring. We now show that 2Δ colors suffice also for graphs with chromatic index $\Delta+1$.

Theorem 2.4. *For each graph G we have $\iota(G)\leqslant 2\Delta$.*

Proof. Choose an edge coloring of G with colors $\{1,\ldots,\Delta+1\}$ which minimizes the number of edges of color $\Delta+1$. We double each of the colors $1,\ldots,\Delta$ and show how to incidence color G with the colors $\{1,1',\ldots,\Delta,\Delta'\}$. The edges of color $\Delta+1$ form a matching M. We arbitrarily call one of the vertices of each edge of M left and the other right. This enables us to refer to the left incidence and right incidence of an edge of M. Every other edge $e=\{x,y\}$ of G has two colors i and i' assigned to it, and these colors will be assigned to the incidences (x,e) and (y,e).

Let $f=\{l_f,r_f\}$ be an edge of M. Since the degree of a vertex is at most Δ, there is some color $s\in\{1,\ldots,\Delta\}$ which is not a color of any edge incident with l_f. We assign the color s to the incidence (l_f,f). If some edge $g=\{r_f,x\}$ incident with r_f has color s, then we assign s to the incidence (x,g). Similarly, there is some color $t'\in\{1',\ldots,\Delta'\}$ which is not a color of any edge incident with r_f. We assign the color t' to the incidence (r_f,f). If some edge $h=\{l_f,y\}$ incident with l_f has color t', then we assign t' to the incidence (y,h). The two incidences of each edge of M are now assigned different colors. For each edge not belonging to M we claim that (i) the same color is not assigned to both of its incidences and (ii) no incidence is assigned more than one color.

(i) Suppose to the contrary that a color s is assigned to both of the incidences of an edge $\{x,y\}$. Then x and y are right vertices of edges e_1 and e_2, respectively, of M. We now change the color of e_1 and e_2 from $\Delta+1$ to s and the color of $\{x,y\}$ from s to $\Delta+1$ and obtain an edge coloring of G in which the number of edges of color $\Delta+1$ is smaller, contradicting the minimality assumption. Similarly no color s' is assigned to both of the incidences of an edge.

(ii) Suppose to the contrary that some incidence $(x,\{x,y\})$ is assigned both of the colors s and s' of the edge $\{x,y\}$. Then y is both a left vertex of an edge of M and a right vertex, contradicting the fact that M is a matching.

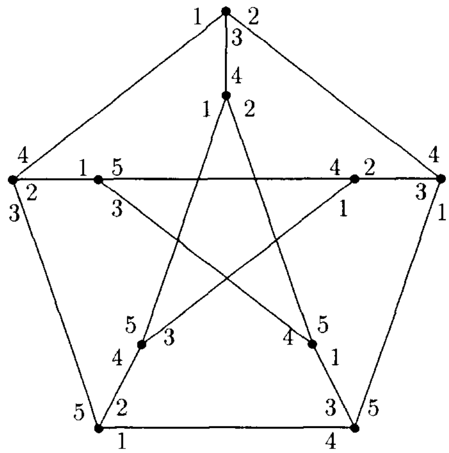

Fig 2. An incidence coloring of the Peterson graph using 5 colors. A four coloring of the incidences is not possible.

The colors of an edge not yet assigned to one of its incidences can now be used to complete the incidence coloring of G in 2Δ colors. □

The incidence coloring number of an n-cycle ($\Delta=2$) is 4 provided n is not divisible by 3 and hence the bound 2Δ in Theorem 2.4 can be attained. But we believe that for $\Delta>2$ the bound can never be attained. In Fig. 2 we exhibit an incidence coloring of the Petersen graph ($\Delta=3$) with $\Delta+2=5$ colors. We *conjecture* that every graph can be incidence colored with $\Delta+2$ colors.

Corollary 2.5. *Let H be a bipartite graph with bipartition X, Y with no cycles of length 4. Let the maximum degree of a vertex of X be 2 and the maximum degree of a vertex of Y be Δ. Then the strong chromatic index of H satisfies $sq(H)\leqslant 2\Delta$.*

Proof. First suppose that each vertex of X has degree 2. Since H has no cycles of length 4, there is a graph G such that $H=H(G)$ Since $\imath(G)=\mathrm{sq}(H)$, the corollary follows from Theorem 2.4. Now suppose that some vertices of X have degree 1. Let H' be the graph obtained from H by removing the vertices of X of degree 1. By the above $\mathrm{sq}(H')\leqslant 2\Delta$ and it is easy to extend a 2Δ strong edge coloring of H' to a 2Δ strong edge coloring of H. □

We now prove our conjecture about strong chromatic index for bipartite graphs whose cycle lengths are divisible by 4. An important property of such graphs is that all cycles are chordless. The approach in the proof is similar to that of Theorem 6 of [4].

Theorem 2.6. *Let H be a bipartite graph with bipartition X, Y. Let the maximum degree of a vertex in X be α and that in Y be β. Assume that all cycle lengths are divisible by* 4. *Then*

$$\mathrm{sq}(H) \leqslant \alpha\beta.$$

Proof. We prove the theorem by induction on the number of edges. First suppose that H has a pendant edge e. Since at most $\max\{\alpha\beta-\beta, \alpha\beta-\alpha\}$ edges affect the color of e, an $\alpha\beta$ strong edge coloring of $H-e$ can be extended to H. Thus we may assume that the degree of each vertex is at least 2. Let

$$\gamma\colon v_1, v_2, \ldots, v_t, u$$

be a path of maximum length. Without loss of generality we assume that v_t is in X, and hence the degree of v_t is at most α. The vertex u can only be adjacent to vertices of γ. If u is adjacent to two vertices different from v_t of γ, then there is a cycle of H with a chord. Hence each such u has degree 2, and for each u there is a unique v_k with $k<t$ such that u is adjacent to v_k. The vertex v_t cannot be adjacent to any of the vertices $v_k, \ldots, v_{t-2}$ and is adjacent to at most one of the vertices $v_1, \ldots, v_{k-1}$.

Case 1: *v_t is not adjacent to any of $v_1, \ldots, v_{k-1}$.*

By induction $H-v_t$ has an $\alpha\beta$ strong edge coloring μ. We extend μ to a strong edge coloring of H sequentially as follows. The number of edges in $H-v_t$ which affect the color of $\{v_{t-1}, v_t\}$ is at most

$$(\alpha-1)+(\beta-1)+(\beta-1)(\alpha-1)=\alpha\beta-1$$

and hence there is a color available for the edge $\{v_{t-1}, v_t\}$. For an edge $\{v_t, u\}$, the maximum number of edges in $H-v_t$ which affect its color equals $2\alpha+\beta-3$. Including the edge $\{v_{t-1}, v_t\}$ we get $2\alpha+\beta-2$. If $\beta \geqslant 3$ there are at least $\alpha-1$ colors available for each edge $\{v_t, u\}$ and hence we can extend μ to a strong edge coloring of H. If $\beta=2$, then each vertex of Y has degree equal 2 and $H=H(G)$ for some multigraph G. This multigraph G has maximum degree α and its cycle lengths are all divisible by 2. Hence G is bipartite and by the well-known theorem of König, there is an edge coloring of G using α colors. This edge coloring may be doubled to get a 2α incidence coloring of G. Thus $\mathrm{sq}(H)=\iota(G)\leqslant 2\alpha$.

Case 2: *v_t is adjacent to v_p for some p with $1 \leqslant p \leqslant k-1$.*

First suppose that for some u the cycle

$$u, v_k, \ldots, v_t, u \tag{2}$$

has length greater than 4. The path

$$\gamma'\colon v_1, \ldots, v_k, u, v_t, \ldots, v_{k+2}, v_{k+1}.$$

also has maximum length. There cannot be an edge joining v_{k+2} to any of the vertices $v_1, \ldots, v_{k-1}$, since otherwise H contains a cycle of length 2 mod 4. This puts us back in Case 1 with γ' replacing γ. Now suppose that for each u the length of the cycle (2) is 4. By induction $H-v_t$ has an $\alpha\beta$ strong edge coloring μ. The number of edges in $H-v_t$

affecting the color of $\{v_t, v_p\}$ is at most $\alpha\beta-1$, and hence there is a color available for $\{v_t, v_p\}$. Each of the edges $\{v_t, u\}$ and $\{v_t, v_{t-1}\}$ is affected by at most $\alpha+\beta$ colors, and since we may assume that $\beta \geqslant 3$, there are at least $\alpha-1$ colors available for each of these edges. Hence μ can be extended to a strong edge coloring of H. □

Let A be an m by n (0, 1)-matrix and let $H \subseteq K_{m,n}$ be its associated bipartite graph. The matrix A is called *restricted unimodular* [8] provided all cycle lengths of H are divisible by 4. The matrix A is *k-totally unimodular* [7] provided replacing as many as k 1's by 0's always gives a totally unimodular matrix.[2] It is proved in [7] that A is restricted unimodular if and only if it is 3-totally unimodular. In addition, A is 2-totally unimodular if and only if the submatrices of A corresponding to the blocks of H are either restricted unimodular or all 1's matrices (complete bipartite graphs). Using Theorem 2.6 and induction on the number of blocks, one can easily prove the following result.

Theorem 2.7. *If H is the bipartite graph associated with a 2-totally unimodular* (0, 1)*-matrix with maximum row sum α and maximum column sum β, then* $\mathrm{sq}(H) \leqslant \alpha\beta$.

We conclude with upper bounds for the incidence coloring number of finite projective and affine planes.

Theorem 2.8. *The incidence coloring number of a projective plane of order n is at most n^2+2n and of an affine plane of order n is at most n^2+n.*

Proof. Let P be a projective plane of order n. Let $p_1, \ldots, p_{n+1}$ be the $n+1$ points on some line ℓ. Let L_i be the set of the other n lines containing p_i and assume these lines have been numbered from 1 to n $(i=1, \ldots, n+1)$. Assume also that the points different from p_i on each line of L_i have been numbered from 1 to n. Color the incidences corresponding to the kth point on each line of L_i with the color (i, k) $(1 \leqslant k \leqslant n$, $1 \leqslant i \leqslant n+1)$. We also color the incidence (p_i, ℓ) with the color $(i+1, 1)$ $(i+1$ is taken mod $n+1)$. Finally, for each k between 1 and n we color the incidences corresponding to p_i and the kth line of L_i with the color $(n+2, k)$. It is now easy to check that we have defined an incidence coloring in which the number of colors used is $n(n+1)+n=n^2+2n$.

Now consider an affine plane of order n. The lines are partitioned into $n+1$ parallel classes of n lines each and the incidences for each parallel class can be colored with n colors, giving the bound in the theorem. □

It can be shown that the incidence coloring number of the projective plane of order 2 is 7, which is 1 better than the bound in the theorem. In fact, it is possible in general to improve the bounds in Theorem 2.8 using more of the geometrical structure.

[2] One all of whose subdeterminants equal 0, ± 1.

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