Discrete Mathematics 122 (1993) 1–6
North-Holland

On the honesty of graph complements*

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Received 15 December 1989
Revised 10 May 1990

Abstract

A graph is called honest if its edge-integrity equals its order. It is shown in this paper that except for the path of length 3, every graph that is not honest has an honest complement. This result is extended to complements of products and applied to the Nordhaus–Gaddum theory for edge-integrity.

The *edge-integrity* of a graph was introduced by Barefoot, Entringer, and Swart [3] (who also introduced its analogue, the vertex-integrity). It is defined as

$$I'(G) := \min_{S \subseteq E(G)} \{|S| + m(G-S)\}$$

where $m(G-S)$ denotes the order of a largest component of $G-S$. For a survey of results on this topic see [2].

A graph G of order p is called *honest* if $I'(G)=p$. Some elementary examples are complete graphs, stars, and cubes, so it can be seen that honest graphs take a variety of forms. A general and important class of honest graphs is provided by the following result [1].

Theorem 1. *If G has diameter 2, then G is honest.*

Since almost all graphs have diameter 2, it follows that almost all graphs are honest. This does not mean that almost all interesting graphs are, however. Many are of

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course, and Laskar, Stueckle, and Piazza [4] observed that Theorem 1 implies that the following graphs are honest:

(a) the complement $\overline{G \times H}$, for any graphs G and H of order at least 3;

(b) the complement $\overline{Q_n}$ of the n-cube, for $n \leqslant 3$;

(c) any regular self-complementary graph.

They also raised the following questions:

(1) For which graphs G, other than P_4, is neither G nor $\bar{G}$ honest?

(2) What is the edge-integrity of an arbitray self-complementary graph?

(3) What are the Nordhaus–Gaddum bounds for edge-integrity?

In this paper we answer these questions and also extend the family (a) to include all proper products except $K_2 \times K_3$. Our main result is the following.

Theorem 2. *If $G \neq P_4$, then G or $\bar{G}$ is honest.*

Before proving this, we introduce some terminology and prove some lemmas that are interesting in their own right. We say that S is an *I'-set* of G if $|S| + m(G-S) = I'(G)$. Note that if G is connected, then G is honest if and only if $\emptyset$ is an I'-set.

Lemma 1. *Let G be a subgraph of $K_{a,b}$ where $a \leqslant b$ and $a+b \geqslant 5$, and suppose that G has at least $ab-a+1$ edges. Then G is honest.*

Proof. Let D be the set of edges of $K_{a,b}$ that are not in G, and let S be an I'-set of G of minimum order. If $S=\emptyset$ we are done, so we assume $S \neq \emptyset$. Denote $|S|$ and $m(G-S)$ by s and m respectively. It is enough to show that $s+m \geqslant a+b$.

We first observe that $m>1$. For, if $m=1$, then $S=E(G)$. Hence, for any edge e, $E(G)-\{e\}$ is a smaller I'-set, which is a contradiction.

If $m=2$, then $G-S$ has at most a edges, so

$$s+m \geqslant |E(G)|-a+2 \geqslant ab-2a+3=(a-1)(b-3)+a+b.$$

Since the conditions on a and b imply $b \geqslant 3$, we have $s+m \geqslant a+b$ as desired.

So we may assume $m \geqslant 3$. Let C be a component of $G-S$ of order m, and assume that C contains i vertices from the partite set of order a and j from that of order b. Note that i and j are both positive. Also, let $x=a-i$ and $y=b-j$. It follows that $|S|+|D| \geqslant xj+yi$, so $s \geqslant xj+yi-(x+i-1)$. Consequently, $s+m \geqslant xj+yi-x+j+1$, or

$$s+m \geqslant (x+1)(j-1)+yi+2. \qquad (*)$$

We consider three cases that depend on the value of j.

Case 1: $j=1$.

Then $y=b-1$ and (since $m \geqslant 3$) $i \geqslant 2$, so by (*) we have $s+m \geqslant 2b \geqslant a+b$, and we are done.

Case 2: $1<j<b$. (Here we will use the elementary fact that if h and k are positive integers, then $hk \geqslant h+k-1$.)

In this case $x+1, j-1, i$, and y are all positive, so $(*)$ and the above observation imply that

$$\begin{aligned} s+m &\geqslant (x+1)+(j-1)-1+y+i-1+2 \\ &= x+j+i+y=a+b, \end{aligned}$$

and we are done.

Case 3: $j=b$.

Then $y=0$, so $x>0$ since $S \neq \emptyset$, and hence from $(*)$ we have

$$s+m \geqslant 2(b-1)+2=2b \geqslant a+b,$$

and again we are done. □

The next result is related to Lemma 1, and its proof is similar. In general, the number of edges in G in Lemma 1 cannot be lowered without destroying honesty, since there are subgraphs of $K_{a,b}$ with $ab-a$ edges that are disconnected. Lemma 2 deals with a case in which there are $ab-a$ edges, but the 'missing' edges are chosen carefully. It will be used later to extend the results of Laskar, Stueckle, and Piazza, and essentially to complete the Nordhaus–Gaddum theory for edge-integrity.

Let $L_{a,b}$ denote the graph obtained by removing a maximum matching from $K_{a,b}$. When $a=b$, this graph is sometimes known as the 'cocktail party' graph because it represents conversations at a gathering of b couples when each person talks with everyone of the opposite sex who is not their spouse (and only those). (We note that some authors call the octahedral graph K_{2n} less a 1-factor by this name.)

Lemma 2. *If* $2 \leqslant a \leqslant b$ *and* $a+b \geqslant 7$, *then* $L_{a,b}$ *is honest. In particular, the cocktail party graph is honest*

Proof. Following the notation of Lemma 1, it is again sufficient to show that $s+m \geqslant a+b$. Let $H=L_{a,b}-S$.

As before, the minimality of s implies that $m>1$. If $m=2$, then H has at most a edges, so $s+m \geqslant ab-2a+2$. It is straightforward to verify that under our conditions on a and b this is never less than $a+b$.

So we may assume $m \geqslant 3$. Let C be a largest component of H, and let i and j be the number of vertices of C in the partite sets of orders a and b respectively (so $m=i+j$). Also, let $x=a-i$ and $y=b-j$ (as before). Now if an edge e of the 'missing matching' of edges determining $L_{a,b}$ joins two vertices of the same component of H, then $L_{a,b} \cup \{e\}$ has the same edge-integrity as $L_{a,b}$ and satisfies the hypotheses of Lemma 1. This implies that $L_{a,b}$ is honest and so we may assume that there is no such edge. Furthermore, if H has at least three components, then some edge of S does not join C to the rest of the graph and $(*)$ in Lemma 1 must hold. Consequently $L_{a,b}$ must be honest.

Therefore, we may assume that H consists of just two components and they must be $K_{i,j}$ and $K_{x,y}$, with $i+j \geqslant x+y \geqslant a$. Hence

$$s+m=(xj+yi-a)+(i+j)=xj+yi-x+j.$$

Suppose that this is less than $a+b$. Then

$$xj+yi-x+j \leqslant x+i+y+j-1$$

or

$$x(j-2)+(y-1)(i-1) \leqslant 0.$$

Since x, y, i, and j are all positive, $j \leqslant 2$. But if $j=2$, then our hypotheses and other conditions imply that $y \geqslant 2$ and $i \geqslant 2$, a contradiction. If $j=1$, then $y=b-1$ and $i \geqslant a-1$, so $c=1$, and this is also impossible. Consequently, $s+m \geqslant a+b$, which concludes the proof. □

The next lemma returns to the theme of Lemma 1 in that we allow the 'missing' edges to be arbitrary. This is the second lemma needed for the proof of Theorem 2.

Lemma 3. *Let K be a complete multipartite graph of order p with at least three partite sets, the largest of which has order a. Let $H=K-D$, where D is a set of fewer than $p-a$ edges of K. Then H is honest.*

Proof. Suppose H is not honest, and suppose further that H is a counterexample of minimum order p. Then $D \neq \emptyset$ since otherwise H has diameter 2. Let S be an I'-set of H, let v be a vertex incident with an edge of D in K, and let $K'=K-\{v\}$. Also let $D'=D \cap E(K')$, $H'=K'-D'$, and $S'=S \cap E(H')$. We claim that H' is honest. If it is at least tripartite, then this follows from the minimality of p, since $|D'| \leqslant |D|-1 < p-1-a$. On the other hand, if H' is bipartite, then it is honest by Lemma 1. Therefore

$$p-1=I'(H') \leqslant |S'|+m(H'-S') \leqslant |S|+m(H-S) < p,$$

from which it follows that $S'=S$; that is, no vertex of H can be incident with both an edge in D and an edge in S.

Now let uw be an edge in S and let x be a vertex whose partite set is different from those of u and w. Since S is an I'-set, u and w must lie in different components of $H-S$, and so one of them, say w, is not in the same component of $H-S$ as x. It follows that edge wx is not in $H-S$. Since w is incident with an edge of S, it cannot be incident with an edge of D; hence wx is also an edge of S. Therefore x is not incident with an edge of D, and so every edge in D joins the partite sets of u and w. But this implies that H has diameter 2 and hence is honest, and the proof is complete. □

Proof of Theorem 2. For $p \leqslant 4$, it is easy to check that every graph satisfies the theorem. Assume $p>4$. If G is honest, we are done. Otherwise, we can assume that G is maximally non-honest. Let S be an I'-set for G, so $|S|+m(G-S)=p-1$. Suppose that

$G-S$ has t components. Each such component must be complete since the addition of a missing edge inside a component would not make G honest. It follows that G consists of t complete subgraphs together with the $p-1-m(G-S)$ edges of S. Hence G is the result of removing $p-1-m(G-S)$ edges from a complete multipartite graph K with largest partite set of order $m(G-S)$. That G is honest follows at once from Lemmas 1 and 3, and the proof of Theorem 2 is complete. □

The following corollary is immediate.

Corollary 1. *Every self-complementary graph with at least five vertices is honest.*

We now consider two other results of Laskar, Stueckle, and Piazza concerning the honesty of complements of products and Nordhaus–Gaddum bounds for edge-integrity. Our goal is to extend those results, essentially to completion. To that end, we established Lemma 2, and we use it here. (We plan to develop further results of this type in a future paper.)

Theorem 3. *If G and H are graphs of orders at least 2 and 3 respectively, then $\overline{G \times H}$ is honest unless $G=K_2$ and $H=K_3$.*

Proof. If G and H both have order at least 3, the result was established by Laskar, Stueckle, and Piazza. If G and H are any graphs of orders 2 and n, with $n \geqslant 4$, then $\overline{G \times H}$ must be honest since it contains the cocktail party graph $L_{n,n}$ as a spanning subgraph. Other than $K_2 \times K_3$ the complement of any product of graphs of order 2 and 3 has $P_2 \times P_3$ as a spanning subgraph and it is easy to check that $P_2 \times P_3$ is honest. □

As another consequence of Lemma 2, we have the following.

Lemma 4. *If $2 \leqslant a \leqslant b$ and $a+b \geqslant 7$, then both $L_{a,b}$ and its complement are honest.*

Proof. We need only show that $\overline{L_{a,b}}$ is honest. This graph consists of K_a and K_b joined by a independent edges and this graph is clearly of diameter 2. □

This in turn essentially completes the Nordhaus–Gaddum theory for edge-integrity. The bounds were formally stated by Laskar, Stueckle, and Piazza, who also observed that the complete-null pairs are the only ones satisfying the lower bounds. It was also noted that regular self-complementary graphs as well as n-cubes and their complements satisfy the upper bounds. The preceding lemma shows the sharpness of the upper bound for all $p \geqslant 5$. The complete set of graphs for which equality holds is not known.

Theorem 4. *For every graph G of order p,*

(a) $p+1 \leqslant I'(G)+I'(\bar{G}) \leqslant 2p$.

(b) $p \leqslant I'(G) \cdot I'(\bar{G}) \leqslant p^2$.

Furthermore, these bounds are sharp for $p \geqslant 5$.

We conclude by noting that for $p=2$, 3, and 4, the sharp upper bounds for the sum and the product are $2p-1$ and $p(p-1)$ respectively.

Acknowledgement

The authors would like to thank Sam Stueckle for his very helpful discussion of the ideas in this paper.

References

[1] K.S. Bagga, L.W. Beineke, M.J. Lipman and R.E. Pippert, On the edge-integrity of graphs, Congr. Numer. 60 (1987) 141–144.

[2] K.S. Bagga, L.W. Beineke, M.J. Lipman and R.E. Pippert, A survey of edge-integrity, Discrete Math., to appear.

[3] C.A. Barefoot, R. Entringer and H. Swart, Vulnerability in graphs — a comparative survey, J. Combin. Math. Combin. Comput. 1 (1987) 13–22.

[4] R. Laskar, S. Stueckle, and B. Piazza, On the edge-integrity of some graphs and their complements, Discrete Appl. Math., to appear.