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A Series Solution for Certain Ricatti Equations with Applications to Sturm–Liouville Problems

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1. INTRODUCTION

We consider here two related equations

$$\lambda - q(t) + r'(t) + r(t)^2 = 0 \tag{1.1}$$

and

$$y'' + (\lambda - q(t))\, y(t) = 0 \tag{1.2}$$

for $t \in (a, b)$ where $b \leqslant \infty$. We suppose that λ is a real and positive parameter and $q \in L(a, b)$.

Our object is to derive a series expansion for a solution of (1.1). This series gives rise to an exact solution set of the equation (1.2). That is to say, we derive a fundamental solution pair for (1.2). The members of this fundamental pair are of interest in their own right, but we are particularly interested in the use of these solutions to solve Sturm–Liouville problems for large values of λ.

Approximate solutions to (1.2) have been obtained before; we mention in particular [1, 3, 5, 6] and, under very general circumstances, [2]. These approximations are valid for large values of λ and share the feature that the approximations are dependent on the smoothness of q. Results are obtained, for example, which give an approximation with error $O(\lambda^{-N})$ as $\lambda \to \infty$ where N depends on the number of derivatives possessed by q.

Our results differ from those mentioned above in two respects. First, like the results of [4], we make minimal assumptions on the smoothness of q, and second we derive exact solutions of (1.2). Approximations may then be obtained by truncating the series involved which, at least in the case of finite b, are convergent rather than asymptotic.

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2. The Results

We set

$$r_1(x, \lambda) := -\int_x^b \exp(2i\lambda^{1/2}(t-x))\, q(t)\, dt \tag{2.1}$$

and, for $j = 1, 2, \ldots$,

$$r_{j+1}(x, \lambda) := \int_x^b \exp\left(2 \int_x^t i\lambda^{1/2} + \sum_{n=1}^{j} r_n(s, \lambda)\, ds\right) r_j(t, \lambda)^2\, dt.$$

THEOREM 1. *If there exists a non-increasing function $\rho(\cdot, \lambda)$ such that for all $\lambda > \lambda_0$*

(i) $|r_i(x, \lambda)| \leqslant \rho(x, \lambda) \leqslant \lambda^{1/2}$ *for all* $x \in (a, b)$,

(ii) $\int_a^b \rho(x, \lambda)\, dx < \frac{1}{8}$,

then for all $\lambda > \lambda_0$ and $x \in (a, b)$ the series

$$r(x, \lambda) = i\lambda^{1/2} + \sum_{n=1}^{\infty} r_n(x, \lambda)$$

is uniformly convergent and is a solution of (1.1).

THEOREM 2. *Under the same conditions as Theorem* 1 *there exist two linearly independent solutions y_1 and y_2 of* (1.2) *with*

$$y_1(x, \lambda) = \exp\left\{\int_a^x \sum_{n=1}^{\infty} \operatorname{Re}(r_n(t, \lambda))\, dt\right\} \cos\left\{\int_a^x \lambda^{1/2} + \sum_{n=1}^{\infty} \operatorname{Im}(r_n(t, \lambda))\, dt\right\},$$

$$y_2(x, \lambda) = \exp\left\{\int_a^x \sum_{n=1}^{\infty} \operatorname{Re}(r_n(t, \lambda))\, dt\right\} \sin\left\{\int_a^x \lambda^{1/2} + \sum_{n=1}^{\infty} \operatorname{Im}(r_n(t, \lambda))\, dt\right\}.$$

3. Corollaries

We consider the circumstances under which Theorems 1 and 2 apply and we derive estimates for the error introduced by a truncation of the series.

COROLLARY 1. *If $b < \infty$ and $q \in L(a, b)$ then the conditions of Theorems* 1 *and* 2 *are satisfied for some* λ_0.

Proof. By the Riemann–Lebesgue Lemma there exist λ_0 such that for $x \in (a, b)$ and $\lambda > \lambda_0$

$$|r_1(x, \lambda)| < \tfrac{1}{8}(b-a)^{-1} =: \rho(x, \lambda).$$

The truth of (i)–(iii) now follows.

The conditions of Corollary 1 are too general to give information about the truncation error. The obtain this we are forced to impose stronger conditions on q.

COROLLARY 2. *If $\rho(x, \lambda) \leqslant a(x)\, b(\lambda)$ for $x \in (a, b)$ and $\lambda > \lambda_0$ where*

(i) *$a(\cdot)$ is nonincreasing,*

(ii) *$a(\cdot) \in L(a, b)$,*

(iii) *$b(\lambda) \to 0$ as $\lambda \to \infty$,*

then

$$|r_j(x, \lambda)| < C a(x)\, b(\lambda)^{2^{j-1}}$$

for $j = 2, \ldots, x \in (a, b)$ if λ is sufficiently large.

COROLLARY 3. *If $q \in AC(a, b)$ and there exists a non-increasing function σ with*

(i) *$|q(x)| + \int_x^b |q'(t)|\, dt \leqslant \sigma(x)$ for $x \in (a, b)$,*

(ii) *$\sigma \in L(a, b)$,*

then $|r_j(x, \lambda)| < C\sigma(x)\, \lambda^{-2^{j-2}}$ for $x \in (a, b)$, λ sufficiently large and $j = 2, 3, \ldots$.

COROLLARY 4. *If $b < \infty$ and $q \in C^N(a, b)$, then*

$$|r_j(x, \lambda)| < C\lambda^{-(1/2)(2^j - 1)} \qquad j = 1, \ldots, N$$
$$|r_j(x, \lambda)| < C\lambda^{-(2^N - 1)\, 2^{N-j-1}} \qquad j > N$$

for $x \in (a, b)$ and λ sufficiently large.

We defer the proof of Corollaries 2, 3, 4 to Section 6 below.

4. PROOF OF THEOREM 1

We write

$$r(x, \lambda) := i\lambda^{1/2} + \sum_{n=1}^{\infty} r_n(x, \lambda) \tag{4.1}$$

and

$$Q(x, \lambda) := \lambda - q(x) + r(x, \lambda)^2 + r'(x, \lambda). \tag{4.2}$$

We choose the r_n in such a way that $Q(x, \lambda) = 0$ for all $x \in (a, b)$.

We may differentiate the series (4.1) term by term and substitute into (4.2) to obtain

$$Q = -q + \sum_{n=1}^{\infty} r_n' + 2i\lambda^{1/2} \sum_{n=1}^{\infty} r_n + \sum_{n=1}^{\infty} r_n \sum_{s=1}^{\infty} r_s \tag{4.3}$$

$$= -q + r_1' + 2i\lambda^{1/2} r_1 + r_1^2$$

$$+ \sum_{n=2}^{\infty} r_n' + 2i\lambda^{1/2} \sum_{n=2}^{\infty} r_n + \sum_{n=2}^{\infty} r_n \sum_{s=1}^{\infty} r_s + r_1 \sum_{s=2}^{\infty} r_s. \tag{4.4}$$

We now choose r_1 to satisfy

$$-q + r_1' + 2i\lambda^{1/2} r_1 = 0. \tag{4.5}$$

That is to say,

$$r_1(x, \lambda) := -\int_x^b e^{2i\lambda^{1/2}(t-x)} q(t)\, dt. \tag{4.6}$$

Equation (4.4) now beomes

$$Q = r_1^2 + 2(i\lambda^{1/2} + r_1)\, r_2 + r_2' + r_2^2$$

$$+ \sum_{n=3}^{\infty} r_n' + 2i\lambda^{1/2} \sum_{n=3}^{\infty} r_n + \sum_{n=3}^{\infty} r_n \sum_{s=1}^{\infty} r_s \sum_{n=1}^{2} r_n \sum_{s=3}^{\infty} r_s. \tag{4.7}$$

We choose r_2 to satisfy

$$r_2' + 2(i\lambda^{1/2} + r_1)\, r_2 = -r_1^2. \tag{4.8}$$

This process may be repeated and we find that after r_j has been defined

$$Q = r_j^2 + 2\left(i\lambda^{1/2} + \sum_{n=1}^{j} r_n\right) r_{j+1} + r_{j+1}' + r_{j+1}^2$$

$$+ \sum_{n=j+2}^{\infty} r_n' + 2i\lambda^{1/2} \sum_{n=j+2}^{\infty} r_n + \sum_{n=j+2}^{\infty} r_n \sum_{s=1}^{\infty} r_s + \sum_{n=1}^{j+1} r_n \sum_{s=j+2}^{\infty} r_s. \tag{4.9}$$

We choose r_{j+1} to satisfy

$$r_{j+1}' + 2\left(i\lambda^{1/2} + \sum_{n=1}^{j} r_n\right) r_{j+1} = -r_j^2 \tag{4.10}$$

so that

$$r_{j+1}(x, \lambda) := \int_x^b \exp\left(2 \int_x^t i\lambda^{1/2} + \sum_{n=1}^{j} r_n(s, \lambda)\, ds\right) r_j(t, \lambda)^2\, dt.$$

Then

$$\begin{aligned} Q &= r_{j+1}^2 + \sum_{n=j+2}^{\infty} r_n' + 2i\lambda^{1/2} \sum_{n=j+2}^{\infty} r_n + \sum_{n=j+2}^{\infty} r_n \sum_{s=1}^{\infty} r_s + \sum_{n=1}^{j+1} r_n \sum_{s=j+2}^{\infty} r_s \\ &= r_{j+1}^2 + r_{j+2}' + 2\left(i\lambda^{1/2} + \sum_{n=1}^{j+1} r_n\right) r_{j+2} + r_{j+2}^2 \\ &\quad + \sum_{n=j+3}^{\infty} r_n' + 2i\lambda^{1/2} \sum_{n=j+3}^{\infty} r_n + \sum_{n=j+3}^{\infty} r_n \sum_{s=1}^{\infty} r_s + \sum_{n=1}^{j+2} r_n \sum_{s=j+3}^{\infty} r_s. \end{aligned} \tag{4.11}$$

Observe that (4.11) has the form of (4.9) with $j+1$ replaced by $j+2$. The process of defining r_{j+1} by mean of (4.10) may thus be continued indefinitely.

In order to discuss the convergence of the series $\sum_{n=1}^{\infty} r_n(x, \lambda)$ we require a bound for the r_n.

LEMMA 1. *Let $\rho(x, \lambda)$ be a non-increasing function of x with*

(i) $|r_1(x, \lambda)| \leqslant \rho(x, \lambda)$ *for* $x \in (a, b)$, $\lambda > \lambda_0$,

(ii) $\int_a^b \rho(x, \lambda)\, dx < \frac{1}{8}$ *for* $\lambda > \lambda_0$. *Then*

$$|r_n(x, \lambda)| \leqslant 2^{-2^{n-2}} \rho(x, \lambda) \qquad \text{for} \quad x \in (a, b),\ \lambda > \lambda_0,\ n = 2, 3, \ldots.$$

Proof. Consider first the case $n = 2$:

$$r_2(x, \lambda) := \int_x^b \exp\left\{2 \int_x^t i\lambda^{1/2} + r_1(s, \lambda)\, ds\right\} r_1(t, \lambda)^2\, dt$$

so that

$$\begin{aligned} |r_2(x, \lambda)| &\leqslant \int_x^b \exp\left\{2 \int_a^b \rho(s, \lambda)\, ds\right\} \rho(t, \lambda)^2\, dt \\ &\leqslant e^{1/2} \int_x^b \rho(t, \lambda)^2\, dt \\ &\leqslant 4\rho(x, \lambda) \int_a^b \rho(t, \lambda)\, dt \\ &\leqslant 2^{-1} \rho(x, \lambda). \end{aligned}$$

Suppose that the result has been proved for r_n with $n = 1, \ldots, j$. Now,

$$r_{j+1}(x, \lambda) := \int_x^b \exp\left\{2 \int_x^t i\lambda^{1/2} + \sum_{n=1}^{j} r_n(s, \lambda)\, ds\right\} r_j(t, \lambda)^2\, dt \tag{4.12}$$

so that

$$\begin{aligned}
|r_{j+1}(x, \lambda)| &\leqslant \int_x^b \exp\left\{2 \sum_{n=1}^{j} \int_a^b |r_n(s, \lambda)|\, ds\right\} |r_j(t, \lambda)|^2\, dt \\
&\leqslant \int_x^b \exp\left\{2\left(1 + \sum_{n=2}^{j} 2^{-2^{n-2}}\right) \int_a^b \rho(s, \lambda)\, ds\right\} \rho(t, \lambda)^2\, 2^{-2^{j-1}}\, dt \\
&\leqslant 2^{-2^{j-1}} \int_x^b \exp\left\{4 \cdot \frac{1}{8}\right\} \rho(t, \lambda)^2\, dt \\
&\leqslant 2^{-2^{j-1}} \rho(x, \lambda) \cdot 4 \cdot \int_a^b \rho(t, \lambda)\, dt \\
&\leqslant 2^{-2^{j-1}} \rho(x, \lambda).
\end{aligned} \tag{4.13}$$

The result now follows by induction from (4.12) and (4.13).

It follows from Lemma 1 that the function

$$r(x, \lambda) = i\lambda^{1/2} + \sum_{n=1}^{\infty} r_n(x, \lambda)$$

is well defined for $x \in (a, b)$. Moreover, the series is uniformly convergent in this region and the term by term differentiation is justified.

We derive now a bound for the derivatives of r_n with respect to x.

We have chosen r_{j+1} so that

$$r'_{j+1} = -r_j^2 - 2\left\{i\lambda^{1/2} + \sum_{n=1}^{j} r_n\right\} r_{j+1}. \tag{4.14}$$

Thus, by Lemma 1

$$\begin{aligned}
|r'_{j+1}(x, \lambda)| &\leqslant 2^{-2^{j-1}} \rho^2 + 2\{\lambda^{1/2} + 4\rho\}\, 2^{-2^{j-1}} \rho, \\
&\leqslant 2^{-2^{j-1}} \rho(x, \lambda) \cdot 11 \cdot \max(\lambda^{1/2}, \rho(x, \lambda)).
\end{aligned} \tag{4.15}$$

If we add to the requirements of Lemma 1 the stipulation that

$$\lambda^{1/2} \geqslant \rho(x, \lambda) \qquad \text{for} \quad x \in (a, b), \lambda > \lambda_0 \tag{4.16}$$

then

$$|r'_{j+1}(x, \lambda)| \leqslant 11 \cdot 2^{-2^{j-2}} \lambda^{1/2} \rho(x, \lambda). \tag{4.17}$$

In order to prove Theorem 1 we show that for the r chosen above the Q which was defined in (4.2) by $Q := \lambda - q + r^2 + r'$ is zero.

For any integer $k \geqslant 2$ we have from (4.9) that

$$Q(x, \lambda) = r_k^2 + 2i\lambda^{1/2} \sum_{n=k+1}^{\infty} r_n + \sum_{n=k+1}^{\infty} r_n' + \sum_{n=k+1}^{\infty} r_s \sum_{s=1}^{\infty} r_s + \sum_{n=1}^{k} r_n \sum_{s=k+1}^{\infty} r_s.$$

It follows from Lemma 1 and (4.17) that

$$\begin{aligned} |Q(x, \lambda)| &\leqslant 2^{-2^{k-1}}\rho^2 + 2\lambda^{1/2}\rho \sum_{n=k+1}^{\infty} 2^{-2^{n-2}} + 11 \cdot \lambda^{1/2} \cdot \rho \sum_{n=k+1}^{\infty} 2^{-2^{n-2}} \\ &\quad + \rho^2 \sum_{n=k+1}^{\infty} 2^{-2^{n-2}} \left(1 + \sum_{s=2}^{\infty} 2^{-2^{s-2}}\right) \\ &\quad + \rho^2 \left(1 + \sum_{n=2}^{k} 2^{-2^{n-2}}\right) \sum_{s=k+1}^{\infty} 2^{-2^{s-2}}. \\ &\leqslant 2^{-2^{k-1}}\rho^2 + 4\lambda^{1/2}\rho 2^{-2^{k-1}} \\ &\quad + 11\lambda^{1/2}\rho \cdot 2^{-2^{k-1}} \cdot 2 + 8\rho^2 2^{-2^{k-1}} \\ &\leqslant 35 \cdot 2^{-2^{k-1}} \lambda^{1/2} \rho(x, \lambda) \qquad \text{for} \quad x \in (a, b), \lambda > \lambda_0. \end{aligned} \tag{4.18}$$

Sice k is arbitrary it follows from (4.18) that $Q(x, \lambda) = 0$ for $x \in (a, b)$, $\lambda > \lambda_0$. The proof of Theorem 1 is now complete.

5. Proof of Theorem 2

We consider now the equation

$$y'' + (\lambda - q)\, y = 0 \qquad \text{on} \quad (a, b). \tag{5.1}$$

Let y_p denote the particular solution of (5.1) which satisfies the initial conditions

$$y_p(a, \lambda) = 1, \qquad y_p'(a, \lambda) = r(a, \lambda), \tag{5.2}$$

where r is the function defined above. Let

$$v := \frac{-y_p'}{y_p} + r, \tag{5.3}$$

then v satisfies the equation

$$v' = \{\lambda - q + r' + r^2\} - 2rv + v^2$$

and, by Theorem 1,

$$v' = -2rv + v^2 \qquad \text{for} \quad x \in (a, b), \lambda > \lambda_0. \tag{5.4}$$

In consequence of (5.2) we also have

$$v(a, \lambda) = 0. \tag{5.5}$$

It follows from (5.4) and (5.5) by uniqueness of solutions that

$$v(x, \lambda) = 0 \qquad \text{for} \quad x \in (a, b), \lambda > \lambda_0. \tag{5.6}$$

Thus, for the particular solution of (5.1) which satisfies (5.2) we have

$$\begin{aligned} y_p(x, \lambda) &= \exp\left\{\int_a^x r(t, \lambda)\, dt\right\} \qquad \text{for} \quad x \in (a, b), \lambda > \lambda_0, \\ &= \exp\left\{\int_a^x i\lambda^{1/2} + \sum_{n=1}^{\infty} r_n(t, \lambda)\, dt\right\}. \end{aligned} \tag{5.7}$$

We recall that q and $\lambda^{1/2}$ are supposed real so if $y_p = y_1 + iy_2$ where y_1 and y_2 are real-valued then y_1 and y_2 are also solutions of (5.1). It follows from (5.7) that

$$y_1(x, \lambda) := \exp\left\{\int_a^x \sum_{n=1}^{\infty} \operatorname{Re}(r_n(t, \lambda))\, dt\right\} \cos\left\{\int_a^x \lambda^{1/2} + \sum_{n=1}^{\infty} \operatorname{Im}(r_n(t, \lambda))\, dt\right\}$$

$$y_2(x, \lambda) := \exp\left\{\int_a^x \sum_{n=1}^{\infty} \operatorname{Re}(r_n(t, \lambda))\, dt\right\} \sin\left\{\int_a^x \lambda^{1/2} + \sum_{n=1}^{\infty} \operatorname{Im}(r_n(t, \lambda))\, dt\right\}.$$

A calculation shows that the Wronskian $W(y_1, y_2) = y_1 y_2' - y_1' y_2$ satisfies

$$W(y_1, y_2) = \left\{\lambda^{1/2} + \sum_{n=1}^{\infty} \operatorname{Im}(r_n(x, \lambda)\right\} \exp\left\{2\int_a^x \operatorname{Re}\left(\sum_{n=1}^{\infty} r_n(t, \lambda)\right) dt\right\}.$$

It follows that y_1 and y_2 are a fundamental pair of solutions for (1.2). The proof is now complete.

6. Proof of Corollaries

Proof of Corollary 2. It is clear that ρ satisfies the condition of the theorems for $x \in (a, b)$ if λ is sufficiently large. Also, from (4.13)

$$|r_{j+1}(x, \lambda)| \leqslant C \int_x^b |r_j(t, \lambda)|^2\, dt. \tag{6.1}$$

The result now follows from (6.1) by induction.

Proof of Corollary 3. An integration by parts yields

$$r_1(x, \lambda) = \frac{1}{2i\lambda^{1/2}} \left(q(t)\, e^{2i\lambda^{1/2}(t-x)} \Big|_x^b - \int_x^b e^{2i\lambda^{1/2}(t-z)} q'(t)\, dt \right). \tag{6.2}$$

In the case $b = \infty$ we take in Corollary 1

$$a(x) := \frac{1}{2} \left(|q(x)| + \int_x^b |q'(t)|\, dt \right), \qquad b(\lambda) := \lambda^{-1/2},$$

while in the case $b < \infty$ we take

$$a(x) := c, \qquad b(\lambda) := \lambda^{-1/2}.$$

Proof of Corollary 4. It follows by successive integration by parts that for $j \leqslant N$

$$|r_j(x, \lambda)| < C\lambda^{-1/2(2^j - 1)},$$

while for $j > N$ we use (6.1) inductively and the result follows.

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