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Quantum Hamiltonians with quasi-ballistic dynamics and point spectrum

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Abstract

Consider the family of Schrödinger operators (and also its Dirac version) on $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{N})$

$$H^W_{\omega,S} = \Delta + \lambda F\big(S^n\omega\big) + W, \quad \omega \in \Omega,$$

where S is a transformation on (compact metric) Ω, F is a real Lipschitz function and W is a (sufficiently fast) power-decaying perturbation. Under certain conditions it is shown that $H^W_{\omega,S}$ presents quasi-ballistic dynamics for ω in a dense G_δ set. Applications include potentials generated by rotations of the torus with analytic condition on F, doubling map, Axiom A dynamical systems and the Anderson model. If W is a rank one perturbation, examples of $H^W_{\omega,S}$ with quasi-ballistic dynamics and point spectrum are also presented.

MSC: 81Q10; 47B99

1. Introduction

Quantum Hamiltonians, i.e., Schrödinger and Dirac, with potentials along dynamical systems is a very interesting subject that has been considered in the mathematics and physics literature, mainly one-dimensional discrete versions. Although not explicitly stated, it is natural to expect that the more "chaotic" the underlining dynamical system, the more singular the corresponding

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spectrum; the extreme cases could be represented by periodic potentials on one hand, which impose absolutely continuous spectrum and ballistic dynamics (see Definition 1), and random potentials on the other hand, that lead to point spectrum and absence of transport (bounded moments of the position operator). We mention the papers [9,12–14,16,17,22,25,37] for references and additional comments on important recent results on quantum dynamics for Dirac and Schrödinger operators.

Exceptions of the above picture are known, since there are examples of one-dimensional quantum models with pure point spectrum and transport. Here we refer to the random dimer model [25] for the Schrödinger case and the random Bernoulli–Dirac operator [16,17] (with no potential correlation). The first example of (Schrödinger) operators with such "unexpected behavior" has appeared in [18, Appendix 2], what the authors have called "A Pathological Example;" the potential was the almost-Mathieu (see Application 5.5.1 ahead), which is built along irrational rotations of the circle, with a combination of suitable rational approximations for the rotation angle and a rank one perturbation.

Rotations of the circle are by far the most considered dynamical systems to generate quantum potentials [6,23,30]; their finite-valued versions [3,11,19], together with substitution dynamical system potentials (see [28,29] and references therein) are mathematical models of one-dimensional quasi-crystals with predominance of singular continuous spectrum. These dynamical systems are not "chaotic," which could be characterized by positive entropy [26] or via a more dynamical definition gathered in [20]; the paradigms of chaotic systems are the Anosov and, more generally, Axiom A systems.

Since chaotic motion mimics randomness, it is natural to conjecture that for quantum operators with suitable potentials built along Axiom A (and other chaotic) systems there is a predominance of point spectrum and absence of transport. A small step in this direction are the results of [7] about Anderson localization for potentials related to the doubling map $\theta \mapsto 2\theta$ on the circle and also hyperbolic toral automorphisms—both systems have positive entropy.

The main goal of this paper is to have a close inspection on the construction of the above mentioned "unexpected example" in [18], together with the related analysis in [22], in order to get a different view of them and so provide new examples of quantum operators with quasi-ballistic dynamics, some of them with pure point spectrum. In spite of the above conjecture, as applications we can prove that for a generic (i.e., dense G_δ) set of initial conditions of Axiom A systems, as well as of chaotic dynamical systems as defined in Devaney [20], the associated quantum operators present quasi-ballistic dynamics. We will also have something to say about the random Anderson model, that is, there is a dense G_δ set of initial conditions so that the quantum operators present quasi-ballistic dynamics; see Section 5 for details and other examples. The applications are the principal contributions of this paper. From now on we shall formulate more precisely the context we work at.

Let (Ω, d) be a compact metric space. Consider the family of bounded Schrödinger operators $H^W_{\omega,S}$ given by

$$\left(H^W_{\omega,S}\psi\right)(n) = (\Delta\psi)(n) + \lambda F\left(S^n\omega\right)\psi(n) + W(n)\psi(n), \quad \omega \in \Omega, \tag{1}$$

acting on $\psi \in \ell^2(\mathbb{N})$ (with a Dirichlet, or any other, boundary condition) or the whole lattice case $\ell^2(\mathbb{Z})$, where the Laplacian Δ is the finite difference operator

$$(\Delta\psi)(n) = \psi(n+1) + \psi(n-1),$$

S is a transformation on Ω (invertible in the whole lattice case), $F:\Omega\to\mathbb{R}$ satisfies a Lipschitz condition, i.e., there exists $L>0$ such that

$$\left|F(\theta)-F(\omega)\right|\leqslant Ld(\theta,\omega),\quad \forall\theta,\omega\in\Omega, \tag{2}$$

and, for some $\eta>0$ and $0<\tilde{C}<\infty$, the perturbation W satisfies

$$\left|W(n)\right|\leqslant\tilde{C}\big(1+|n|\big)^{-1-\eta},\quad \forall n\in\mathbb{Z}. \tag{3}$$

The coupling constant λ is a positive real number. Throughout W is supposed to satisfy (3). We shall denote by ℓ the Lebesgue measure (normalized, when necessary) and by $\sigma(H)$ the spectrum of a self-adjoint operator H.

We are interested in situations where nontrivial quantum transport for systems governed by the above Hamiltonians can be established. To this end consider the time averaged moments of order $p>0$ associated to the initial state δ_1 (a member of the canonical basis of ℓ^2), defined by

$$M^W_{\omega,S}(p,T):=\frac{2}{T}\int_0^\infty e^{-2t/T}\sum_n\big(1+n^2\big)^{p/2}\big|\big\langle\delta_n,e^{-itH^W_{\omega,S}}\delta_1\big\rangle\big|^2\,dt. \tag{4}$$

The presence of quantum transport will be probed through the upper diffusion exponents

$$\beta^+_{\omega,S,W}(p):=\limsup_{T\to\infty}\frac{\log M^W_{\omega,S}(p,T)}{p\log T}. \tag{5}$$

The lower diffusion exponents will be denoted by

$$\beta^-_{\omega,S,W}(p):=\liminf_{T\to\infty}\frac{\log M^W_{\omega,S}(p,T)}{p\log T}. \tag{6}$$

Definition 1. If $\beta^-_{\omega,S,W}(p)=1$ for all $p>0$, the operator $H^W_{\omega,S}$ is said to present ballistic dynamics. If $\beta^+_{\omega,S,W}(p)=1$ for all $p>0$, the operator $H^W_{\omega,S}$ is said to present quasi-ballistic dynamics.

Although point spectrum has been associated with localized dynamics, as already mentioned the first example of a Schrödinger operator with quasi-ballistic dynamics and point spectrum was the half lattice almost Mathieu operator under rank one perturbation [18]. The random dimer model [15] and the Bernoulli–Dirac model [16,17] (zero mass case) are other examples of operators with nontrivial quantum transport (due to existence of critical energies) and pure point spectrum. In [22] a new method was developed to obtain dynamical lower bounds with application for random decaying potentials.

Here we are confined to quasi-ballistic transport; the ideas in [18] for the almost Mathieu operator, and then revisited in [22], is presented from a rather different viewpoint in order to provide new examples of quantum operators with quasi-ballistic dynamics, some of them also with pure point spectrum (see ahead).

The abstract result we shall present can be summarized as (see Theorem 1 for a precise statement): If there exists a dense set of initial conditions in Ω for which the transfer matrices are bounded from above in energy intervals with positive Lebesgue measure, and if the iterations of S

satisfies a suitable continuity-like condition, then one obtains a dense G_δ set $\tilde{\Omega} \subset \Omega$ such that for any $\omega \in \tilde{\Omega}$, $H^W_{\omega,S}$ defined by (1) presents quasi-ballistic dynamics. With respect to the spectral type, we shall highlight a known result (see Theorem 2) that will be used in some applications: If the Lyapunov exponent corresponding the $H^{W=0}_{\omega,S}$ is strictly positive for energies in the spectrum, then under the rank one perturbation $W = \kappa\langle\delta_1, \cdot\rangle\delta_1$ the Schrödinger operator $H^W_{\omega,S}$ on $\ell^2(\mathbb{N})$ has pure point spectrum for a.e. ω (with respect to an ergodic measure) and a.e. κ (with respect to Lebesgue ℓ measure). There is a restricted version for the whole lattice case. That result will be so important for some applications here that a sketch of its proof will be provided. We shall apply the abstract result to several types of potential $V_\omega(n) = F(S^n\omega)$ (see Section 5): Rotations of $\mathbb{S}^1$ and of the torus with analytic condition on F, doubling map, Anderson model, Anosov and Axiom A, and chaotic dynamical systems. For the particular case of incommensurate rotations of the torus under rank one perturbations (see Sections 5.5 and 5.6), besides quasi-ballistic dynamics, it is also found the concomitant presence of pure point spectrum.

This paper is organized as follows: In Section 2 the results about quasi-ballistic dynamics (Theorem 1) and point spectrum (Theorem 2) for the model (1) are presented, whose proofs appear in Section 4. In Section 3 some preliminary results used in those proofs are collected. Section 5 is devoted to applications. In Section 6 the adaptation of the results for the discrete Dirac model is briefly mentioned.

2. Abstract results

In this section we will present our result about quasi-ballistic dynamics (Theorem 1) for the operators $H^W_{\omega,S}$ defined by (1) and also a spectral result (Theorem 2) that will be used in some applications. First of all, we recall the notion of transfer matrices. These matrices $\Phi^W_{\omega,S}$ are uniquely defined by the condition that

$$\begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} = \Phi^W_{\omega,S}(E,n,0)\begin{pmatrix} \psi(1) \\ \psi(0) \end{pmatrix}$$

for every solution ψ of the eigenvalue equation

$$H^W_{\omega,S}\psi = E\psi.$$

Hence,

$$\Phi^W_{\omega,S}(E,n,0) = \begin{cases} T^W_{\omega,S}(E,n)\cdots T^W_{\omega,S}(E,1), & n \geqslant 1, \\ \mathrm{Id}, & n = 0, \\ (T^W_{\omega,S}(E,n+1))^{-1}\cdots(T^W_{\omega,S}(E,0))^{-1}, & n \leqslant -1, \end{cases}$$

where

$$T^W_{\omega,S}(E,k) = \begin{pmatrix} E - \lambda F(S^k\omega) - W(k) & -1 \\ 1 & 0 \end{pmatrix}.$$

Now we are in position to state the main abstract result of this paper.

Theorem 1. *Let $H^W_{\omega,S}$ be the operator defined by* (1) *on* $\ell^2(\mathbb{N})$ *with* S *and* W (*as in* (3)) *fixed. Suppose that there exists a dense set* A *of initial conditions in* Ω *such that, for each* $\omega \in A$, *there are a closed interval* $J^\omega_S \subset \sigma(H^0_{\omega,S})$ (*i.e., the spectrum in the case* $W \equiv 0$) *with* $\ell(J^\omega_S) > 0$ *and* $0 < C_\omega(S) < \infty$ *so that*

$$\left\| \Phi^0_{\omega,S}(E,n,0) \right\| \leqslant C_\omega(S), \quad \forall E \in J^\omega_S \text{ and } \forall n \in \mathbb{N}. \tag{7}$$

Assume that there are $0 < C < \infty$ *and a nonnegative function* $h_S : \mathbb{N} \to \mathbb{R}$ *satisfying*

$$d\left(S^n\theta, S^n\omega\right) \leqslant C d(\theta,\omega) h_S(n), \quad \forall \theta, \omega \in \Omega \text{ and } \forall n \in \mathbb{N}.$$

Then there exists a dense G_δ *set* $\tilde{\Omega} \subset \Omega$ *such that, for each* $\omega \in \tilde{\Omega}$, *the operator* $H^W_{\omega,S}$ *presents quasi-ballistic dynamics.*

Let ν be an ergodic probability measure on Ω with respect to S. By Furstenberg and Kesten Theorem [4], for ν-a.s. ω the Lyapunov exponent

$$\Gamma^W_S(E) = \lim_{n\to\infty} \frac{1}{|n|} \log\left\| \Phi^W_{\omega,S}(E,n,0) \right\|$$

exists and is independent of ω. The next result is a consequence of the Simon–Wolff criterion [35] (see Lemma 5 ahead). Recall that the cyclic subspace generated by $\phi \in \ell^2$ for a self-adjoint operator H is the closure of $\{(H-z)^{-1}\phi\colon z \in \mathbb{C}\}$; the vector ϕ is cyclic for H if such subspace is the whole ℓ^2.

Theorem 2. *(See [34].) Let* $H^W_{\omega,S}$ *be the operator defined by* (1) *on* $\ell^2(\mathbb{Z})$ *under rank one perturbations* $W = \kappa\langle\delta_1,\cdot\rangle\delta_1$, $\kappa \in \mathbb{R}$. *Fix an interval* $[a,b]$. *If* $\Gamma^0_S(E) > 0$ *for* ℓ*-a.s.* $E \in [a,b]$, *then restricted to the cyclic subspace generated by* δ_1, *the operator* $H^W_{\omega,S}$ *has pure point spectrum in* $[a,b]$ *for* ℓ*-a.s.* κ *and* ν*-a.s.* ω.

Remarks.

(i) Theorem 1 can be readily adapted to the whole lattice case.
(ii) The set $\tilde{\Omega}$ in Theorem 1 does not depend on the perturbation W (including $W \equiv 0$).
(iii) Although both theorems above have half and whole lattice versions, the proofs of such versions are quite similar; so Theorem 1 will be proven for the half lattice case while Theorem 2 for the whole lattice one.
(iv) Since in the half lattice case the vector δ_1 is cyclic for $H^0_{\omega,S}$, then in this case the conclusions of Theorem 2 hold on $\ell^2(\mathbb{N})$.
(v) In this paper (and perhaps in most future applications) the set A in Theorem 1 is composed of periodic orbits of the map S.

3. Preliminaries

In this section we collect some results that will be used in the proofs of Theorems 1 and 2. Most of them are known results whose proofs are easily found in the references. Denote by $\mu^W_{\omega,S}$

the spectral measure associated to the pair $(H^W_{\omega,S}, \delta_1)$ and introduce the "local spectral moments" [22]

$$K_{\mu^W_{\omega,S}}(q,\epsilon) := \frac{1}{\epsilon}\int_{\mathbb{R}} \big(\mu^W_{\omega,S}(x-\epsilon, x+\epsilon)\big)^q \, dx \tag{8}$$

defined for $q > 0$ and $\epsilon > 0$. A key point for the proof of Theorem 1 will be the following lower bound for the diffusion exponents $\beta^+_{\omega,S,W}(p)$.

Lemma 1. *For all $p > 0$ and $q = (1+p)^{-1}$, one has*

$$\beta^+_{\omega,S,W}(p) \geqslant \limsup_{\epsilon\to 0} \frac{\log K_{\mu^W_{\omega,S}}(q,\epsilon)}{(q-1)\log\epsilon}.$$

The proof of Lemma 1 follows directly from Theorem 2.1 of [1] and Lemmas 2.1 and 2.3 of [2]. The next result converts an upper bound on the norm of transfer matrices into a lower bound on the spectral measure; for its proof see Proposition 2.1 of [22].

Lemma 2. *Let $H^W_{\omega,S}$ be the operator defined by* (1) *on $\ell^2(\mathbb{N})$ and let I be a compact interval. There exist a universal constant C_1 and, for all $M > 0$ and $\tau > 0$, a constant $C_2 = C_2(I, M, \tau)$ such that for all $\epsilon \in (0,1)$ and all $x \in I$, one has*

$$\mu^W_{\omega,S}(x-\epsilon, x+\epsilon) \geqslant C_1 \int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} \frac{dE}{\|\Phi^W_{\omega,S}(E,N,0)\|^2} - C_2\epsilon^M,$$

with $N = [\epsilon^{-1-\tau}]$ (integer part).

In order to establish relations between the transfer matrices with different initial conditions, the next result will be used.

Lemma 3. *Let $E \in \mathbb{R}$, $N > 0$ and set*

$$L^\omega_S(N) := \sup_{1\leqslant n\leqslant N} \big\|\Phi^W_{\omega,S}(E,n,0)\big\|.$$

Then, for $1 \leqslant n \leqslant N$ and $\theta \in \Omega$,

$$\big\|\Phi^W_{\theta,S}(E,n,0)\big\| \leqslant L^\omega_S(N) e^{L^\omega_S(N)\lambda|F(S^n\theta)-F(S^n\omega)|n}.$$

Proof. An inductive argument shows that, for $\theta, \omega \in \Omega$ and $n \geqslant 1$, one can write the identity

$$\Phi^W_{\theta,S}(E,n,0) = \Phi^W_{\omega,S}(E,n,0) + \lambda\sum_{j=1}^n \Phi^W_{\omega,S}(E,n,j)\, B^{\theta,\omega}_S(n)\Phi^W_{\theta,S}(E,j,1),$$

where

$$B_S^{\theta,\omega}(n)=\begin{pmatrix} F(S^n\omega)-F(S^n\theta) & 0\\ 0 & 0\end{pmatrix}.$$

By iteration, using the fact that $\|\Phi^W_{\omega,S}(E,n,0)\|\leqslant L^\omega_S(N)$ for all $1\leqslant n\leqslant N$, one obtains

$$\begin{aligned}\big\|\Phi^W_{\theta,S}(E,n,0)\big\| &\leqslant L^\omega_S(N)\big[1+\lambda\big|F\big(S^n\theta\big)-F\big(S^n\omega\big)\big|L^\omega_S(N)\big]^{n-1}\\ &\leqslant L^\omega_S(N)e^{L^\omega_S(N)\lambda|F(S^n\theta)-F(S^n\omega)|n},\end{aligned}$$

for $1\leqslant n\leqslant N$. □

Now we describe two results that will be used in the proof of Theorem 2. Details will be presented only for the whole lattice case. Consider the function

$$G_{\theta,S}(E)=\int\frac{d\mu^0_{\theta,S}(x)}{(E-x)^2}$$

which is defined for $E\in(-\infty,\infty)$ and takes values in $(0,\infty]$. The first result relates $G_{\theta,S}(E)$ with the solutions of the eigenvalue equation

$$H^0_{\theta,S}\psi=E\psi. \tag{9}$$

See Theorem 2.4 of [34] for its proof.

Lemma 4. *Let $H^0_{\theta,S}$ be the operator defined by* (1) *on $\ell^2(\mathbb{Z})$, with $W\equiv 0$. Then one has $G_{\theta,S}(E)<\infty$ if and only if*

(i) *E is not an eigenvalue of $H^0_{\theta,S}$*;
(ii) *One of the following holds*:
  (ii.1) *Eq.* (9) *has an ℓ^2 solution on $(0,\infty)$ with $\psi(0)=0$*;
  (ii.2) (9) *has an ℓ^2 solution on $(-\infty,0)$ with $\psi(0)=0$*;
  (ii.3) (9) *has an ℓ^2 solutions $\psi_\pm$ on both $(0,\infty)$ and $(-\infty,0)$ with both $\psi_+(0)\neq 0$ and $\psi_-(0)\neq 0$.*

Finally we remind of Simon–Wolff criterion [35]:

Lemma 5. *Let $H^W_{\omega,S}$ be the operator defined by* (1) *on $\ell^2(\mathbb{Z})$ with $W=\kappa\langle\delta_1,\cdot\rangle\delta_1$, $\kappa\in\mathbb{R}$. Fix an interval $[a,b]$. Then the following assertions are equivalent*:

(i) *$G_{\omega,S}(E)<\infty$ for ℓ-a.s. $E\in[a,b]$*;
(ii) *restricted to the cyclic subspace generated by δ_1, the operator $H^W_{\omega,S}$ has only pure point spectrum in $[a,b]$ for ℓ-a.s. κ.*

4. Proofs

In this section the proofs of Theorems 1 and 2 are presented. In order to prove Theorem 1, the following technical result will be used.

Lemma 6. *Let A be the set described in Theorem* 1 *and fix $\omega \in A$. Then there exists $\epsilon(\omega, S) > 0$ such that for every $0 < \epsilon < \epsilon(\omega, S)$ it is possible to choose $\delta(\epsilon, \omega, S) > 0$ such that if $d(\theta, \omega) < \delta(\epsilon, \omega, S)$, then for any $q \in (0, 1)$ there exists $0 < C_q < \infty$ so that*

$$K_{\mu^W_{\theta,S}}(q, \epsilon) \geqslant C_q \frac{\epsilon^{-1+q}}{\log(\epsilon^{-1})}.$$

Proof. For each $\omega \in A$ fixed, there exists a closed interval $J^\omega_S \subset \sigma(H^0_{\omega,S})$ with $\ell(J^\omega_S) \geqslant L_\omega(S) > 0$ and $0 < C_\omega(S) < \infty$ such that

$$\left\| \Phi^0_{\omega,S}(E, n, 0) \right\| \leqslant C_\omega(S), \quad \forall E \in J^\omega_S \text{ and } \forall n \in \mathbb{N}.$$

Since W satisfies (3), it is found that

$$\left\| \Phi^W_{\omega,S}(E, n, 0) \right\|^2 \leqslant \tilde{C}_\omega(S), \quad \forall E \in J^\omega_S \text{ and } \forall n \in \mathbb{N}. \tag{10}$$

We remark that inequality (10) is closely related to discrete versions of the Levinson's theorem (see, e.g., [24,33] and references therein), but in Theorem 2 of [13] a detailed and ad hoc proof is presented.

Pick $\epsilon(\omega, S) > 0$ such that if $\epsilon < \epsilon(\omega, S)$, then for any $q \in (0, 1)$,

$$\max\left\{\tilde{C}_\omega(S), L_\omega(S)^{-1}\right\} \leqslant \left(\log\left(\epsilon^{-1}\right)\right)^{1/(1+q)}. \tag{11}$$

Now note that, by (2) and the hypotheses of Theorem 1, one has

$$\left|F\left(S^n\theta\right) - F\left(S^n\omega\right)\right| \leqslant Ld\left(S^n\theta, S^n\omega\right) \leqslant LCd(\theta, \omega)h_S(n), \tag{12}$$

for every $\theta, \omega \in \Omega$ and for all $n \in \mathbb{N}$. Note that by using the new function $H_S(n) := \max_{0 \leqslant j \leqslant n} h_S(j)$, one may assume that h_S is nondecreasing; this will be done in what follows.

Pick $\tau > 0$. As a consequence of (10), (12) and Lemma 3, it is found that for $\omega \in A$ fixed and for any $\epsilon < \epsilon(\omega, S)$,

$$\begin{aligned} \left\| \Phi^W_{\theta,S}(E, n, 0) \right\|^2 &\leqslant \tilde{C}_\omega(S) e^{2\lambda \tilde{C}_\omega(S) LCd(\theta,\omega) h_S([\epsilon^{-1-\tau}])\epsilon^{-1-\tau}} \\ &\leqslant 2\lambda LC \tilde{C}_\omega(S), \end{aligned} \tag{13}$$

for every $E \in J^\omega_S$ and for all $1 \leqslant n \leqslant [\epsilon^{-1-\tau}]$, where we required that $d(\theta, \omega)$ is small enough (which determines $\delta(\epsilon, \omega, S) > 0$) so that

$$2\lambda \log\left(\epsilon^{-1}\right) LCd(\theta, \omega) h_S\left(\left[\epsilon^{-1-\tau}\right]\right)\epsilon^{-1-\tau} \leqslant \log(2\lambda LC).$$

Thus, by Lemma 2 with $M = 2$ and by (11) and (13), it follows that for ϵ small enough,

$$\mu^W_{\theta,S}(E-\epsilon, E+\epsilon) \geqslant C_1\bigl(2\lambda LC\tilde{C}_\omega(S)\bigr)^{-1}\epsilon - C_2\epsilon^2$$
$$\geqslant C_3\frac{\epsilon}{(\log(\epsilon^{-1}))^{1/(1+q)}},$$

for every $E \in J^\omega_S$. Therefore, for any $q \in (0,1)$ and $\epsilon < \epsilon(\omega, S)$, it follows from (8), (11) and the above inequality that

$$K_{\mu^W_{\theta,S}}(q,\epsilon) \geqslant C_q\,\frac{\epsilon^{-1+q}}{(\log(\epsilon^{-1}))^{q/(1+q)}}\ell\bigl(J^\omega_S\bigr) \geqslant C_q\frac{\epsilon^{-1+q}}{\log(\epsilon^{-1})}. \qquad \square$$

Remarks. Both Lemma 6 and the proof of Theorem 1 hold if the logarithm function is replaced by any $g : \mathbb{R} \to \mathbb{R}$ with $\lim_{t\to\infty} g(t) = \infty$ and $\lim_{t\to\infty} g(t)/t = 0$.

Proof of Theorem 1. For each $n \in \mathbb{N} \setminus \{0\}$ define the sets

$$B_n = \left\{\theta \in \Omega \;\middle|\; \exists\epsilon < \frac{1}{n}\colon\ K_{\mu^W_{\theta,S}}(q,\epsilon) \geqslant C_q\frac{\epsilon^{-1+q}}{\log(\epsilon^{-1})}\right\}.$$

Since A is dense, by Lemma 6 each of the sets B_n contains a dense open set. Therefore, by Baire theorem, $\bigcap_{n=1}^\infty B_n$ contains a dense G_δ set $\tilde{\Omega}$. Note that for each $\theta \in \tilde{\Omega}$ there exists a sequence $\epsilon_n \to 0$ such that

$$K_{\mu^W_{\theta,S}}(q,\epsilon_n) \geqslant C_q\frac{\epsilon_n^{-1+q}}{\log(\epsilon_n^{-1})},$$

for any $q \in (0,1)$. Choosing $q = (1+p)^{-1}$, it follows by Lemma 1 that for any $\theta \in \tilde{\Omega}$ and for all $p > 0$, $\beta^+_{\theta,S,W}(p) = 1$, i.e., the operator $H^W_{\theta,S}$ presents quasi-ballistic dynamics. $\square$

Proof of Theorem 2. By hypothesis, $\Gamma^0_S(E) > 0$ for ℓ-a.s. $E \in [a,b]$. The theorem of Ruelle–Oseledec [31] implies that there exist solutions $\psi_\pm$ of Eq. (9), for ℓ-a.s. $E \in [a,b]$, that are ℓ^2 at $\pm\infty$ (they decay exponentially). Hence by Lemma 4, either E is an eigenvalue of $H^0_{\theta,S}$ or $G_{\theta,S}(E) < \infty$. Since $H^0_{\theta,S}$ has only countably many eigenvalues, it is possible to conclude that $G_{\theta,S}(E) < \infty$ for ℓ-a.s. $E \in [a,b]$. Therefore, it follows by Lemma 5 that, restricted to the cyclic subspace generated by δ_1, the operator $H^W_{\theta,S}$ has pure point spectrum in $[a,b]$ for ν-a.s. θ and ℓ-a.s. κ. $\square$

5. Applications

This section is devoted to applications of Theorems 1 and 2. Some of them provide examples of quantum operators with quasi-ballistic dynamics and point spectrum (pure point in the half lattice case).

5.1. *Anosov and Axiom A*

Let M be a differentiable compact manifold. Recall that a diffeomorphism $S : M \to M$ satisfies Axiom A of Smale [26] if its nonwandering set $\Omega = \Omega(S)$ is hyperbolic with respect to S

and the set of periodic points of S is dense in Ω. Recall also that the nonwandering set of a diffeomorphism is closed and invariant under S. It is known that for Axiom A dynamical systems the set Ω is a finite (disjoint) union of closed, invariant and transitive sets (i.e., there is a dense orbit); each of these sets is called a basic set for S.

By the continuity of the derivative of S and compactness (or hyperbolicity), there are $C > 0$ and $\gamma > 1$ so that

$$d\big(S^n\theta, S^n\omega\big) \leqslant C\gamma^{|n|}d(\theta,\omega), \quad \forall\theta,\omega \in \Omega \text{ and } \forall n \in \mathbb{Z} \text{ (or } \mathbb{N}).$$

Now by taking $F : M \to \mathbb{R}$ continuously differentiable, the Lipschitz condition (2) is immediately satisfied. So Theorem 1 is applicable with A being the set of initial conditions giving rise to periodic orbits of S. Therefore, there is a dense G_δ set $\tilde{\Omega} \subset \Omega$ so that for each initial condition $\omega \in \tilde{\Omega}$ the Schrödinger operator $H^W_{\omega,S}$ presents quasi-ballistic dynamics.

It is interesting to note that due to hyperbolicity of Ω the set of periodic points of S is at most countable, so that for "chaotic" Axiom A systems the dense G_δ set with quasi-ballistic dynamics is actually a nontrivial one.

Recall also that if M is hyperbolic with respect to S, then S is said to be an Anosov diffeomorphism. These systems satisfy Axiom A and so the above conclusion about quasi-ballistic dynamics holds. It seems to be an open question if for Anosov diffeomorphisms the nonwandering sets Ω always coincide with M; in the case of Anosov diffeomorphism on the torus $\mathbb{T}^2$ it is known that there is just one basic set and it coincides with the whole torus.

To the best of our knowledge, the only spectral specification related to such systems are a.s. purely point spectrum (at the border of the spectrum) for hyperbolic toral automorphisms S on $\mathbb{T}^2$ (i.e., a particular class of Anosov maps and so the torus $\mathbb{T}^2$ is a basic set) and $F \in C^1(\mathbb{T}^2)$ with zero average studied in [7]. Such spectral results are similar to those mentioned for the doubling map in Section 5.2. Of course the periodic orbits of S generate absolutely continuous spectrum.

5.2. Doubling map

Consider the operator $H^W_{\theta,S}$ defined by (1) on $\ell^2(\mathbb{N})$ where S is the transformation on $\Omega = [0,1]$ given by $S\theta = 2\theta \pmod 1$ and $F = \cos : \Omega \to \mathbb{R}$. Note that F satisfies the Lipschitz condition. The set $A = \{\theta$ whose expansion in the basis 2 is periodic$\}$ is dense in Ω. Since each element of A corresponds to a periodic orbit of S, it follows from [18,27] that A satisfies the hypotheses of Theorem 1. Furthermore, for every $\theta,\omega \in \Omega = [0,1]$ and $n \in \mathbb{N}$, one has

$$d\big(S^n\theta, S^n\omega\big) = \big|2^n\theta - 2^n\omega\big| = 2^n d(\theta,\omega).$$

Therefore, by Theorem 1, there exists a dense G_δ set $\tilde{\Omega} \subset [0,1]$ such that for any $\theta \in \tilde{\Omega}$ and for every $p > 0$, $\beta^+_{\theta,S,W}(p) = 1$—note that indeed this result holds for any (nonconstant) periodic continuously differentiable F.

Now fix (small) $\delta > 0$ and $\lambda > 0$ sufficiently small. Bourgain and Schlag [7] have proven that for ℓ-a.s. $\theta \in [0,1]$, the operator $H^0_{\theta,S}$ has pure point spectrum in $[-2+\delta,-\delta] \cup [\delta, 2-\delta]$ with eigenfunctions decaying exponentially. In particular, $\Gamma^0_S(E) > 0$ for $E \in [-2+\delta,-\delta] \cup [\delta, 2-\delta]$. Since in principle $\tilde{\Omega}$ can have null measure, we cannot conclude that there are elements of $\tilde{\Omega}$ whose corresponding operator has a point spectrum component.

5.3. Chaotic one-dimensional maps

Let I be a compact interval in $\mathbb{R}$ and $S: I \to I$ a continuously differentiable map (for simplicity we restrict ourselves to one-dimensional maps). Suppose that restricted to $\Lambda \subset I$ the map S is chaotic as defined by Devaney [20, p. 50], i.e., it is sensitive on initial conditions, topologically transitive, and the periodic points are dense in Λ. The potential for the Schrödinger operator (1) on $\ell^2(\mathbb{N})$ will be the own orbits of S, so that F is the identity map (or any other Lipschitz function). The hypotheses on F and S in Theorem 1 are clearly satisfied, as well as the existence of the set A.

Specific examples are the Tchebycheff polynomials [20]; for instance, $x \mapsto 4x^3 - 3x$ and $x \mapsto 8x^4 - 8x^2 + 1$ are chaotic on $[-1, 1]$. For $r > 2 + \sqrt{5}$, the map $S_r(x) = rx(1-x)$ is chaotic on the set $\Lambda \subset [0, 1]$ of points which never escape from $[0, 1]$ upon iterates of S_r; for $r = 4$ the map S_4 is chaotic on $\Lambda = [0, 1]$.

Therefore, for the family of operators $H^W_{x,S}$, with chaotic S as above, there is a dense G_δ set $\tilde{\Omega} \subset \Lambda$ such that for any $x \in \tilde{\Omega}$ the corresponding operator presents quasi-ballistic dynamics. It is a very interesting open problem to say something about the spectra of such operators. Are they "in general" pure point as the intuition says? What about for $x \in \tilde{\Omega}$?

5.4. Anderson model

Consider the operator $H^W_{\omega,S}$ defined by (1) on $\ell^2(\mathbb{Z})$ where S is the shift on $\Omega = [-1, 1]^{\mathbb{Z}}$ given by $(S\omega)_j = \omega_{j+1}$ and $F : \Omega \to [-1, 1]$ defined by $F(\omega) = \omega_0$. Note that $F(S^n\omega) = (S^n\omega)_0 = \omega_n$. It is assumed that ω_n, $n \in \mathbb{Z}$, are independent identically distributed random variables with common probability measure σ not concentrated on a single point and $\int |\omega_n|^\alpha \, d\sigma(\omega_n) < \infty$ for some $\alpha > 0$. Denote by $\nu = \prod_{n\in\mathbb{Z}} \sigma$ the probability measure on Ω. The metric on Ω is given by

$$d(\omega, \theta) = \sum_{j\in\mathbb{Z}} \frac{d_0(\omega_j, \theta_j)}{2^{|j|}},$$

where d_0 is the discrete metric. For every $\omega, \theta \in \Omega$, one has

$$\bigl|F(\omega) - F(\theta)\bigr| = |\omega_0 - \theta_0| \leqslant 2d(\omega, \theta),$$

and so F is Lipschitz. The set A of periodic sequences in Ω is dense in Ω. Since each periodic sequence determines a periodic orbit of S, it follows from [18,27] that A satisfies the hypotheses of Theorem 1. Furthermore, for every $\omega, \theta \in \Omega$ and for all $n \in \mathbb{Z}$, one has

$$d\bigl(S^n\omega, S^n\theta\bigr) \leqslant 2^{|n|} \sum_{j\in\mathbb{Z}} \frac{d_0(\omega_{j+n}, \theta_{j+n})}{2^{|j+n|}} = d(\omega, \theta)2^{|n|}.$$

Therefore, by Theorem 1, there exists a dense G_δ set $\tilde{\Omega} \subset \Omega$ such that for any $\omega \in \tilde{\Omega}$ and for every $p > 0$, $\beta^+_{\omega,S,W}(p) = 1$.

In [8,32,38] it was proven that for ν-a.s. ω, $H^0_{\omega,S}$ has pure point spectrum with eigenfunctions decaying exponentially. In particular, $\Gamma^0_S(E) > 0$ for every E. As in Application 5.2, we cannot conclude that there are elements of $\tilde{\Omega}$ whose corresponding operators present point spectrum.

Such quasi-ballistic dynamics should be contrasted with the dynamical localization proven ν-a.s. [21] for this model. In the sequel an important particular case is selected.

5.4.1. Bernoulli–Anderson model

Take $H^W_{\omega,S}$ as in Application 5.4 with $\Omega = \{a_1, \dots, a_k\}^{\mathbb{Z}}$, $a_i \in \mathbb{R}$, and for each $n \in \mathbb{Z}$, $\sigma(\omega_n = a_i) = p_i$, $0 < p_i < 1$, and $\sum_{i=1}^k p_i = 1$. The same conclusions of Application 5.4 hold.

5.5. Rotations of $\mathbb{S}^1$ with analytic condition on F

Consider the operator $H^W_{(\theta,\alpha),S}$ defined by (1) with S the transformation on $\Omega_a := \mathbb{S}^1 \times [-a,a]$, $a > 0$, fixed, given by $S(\theta,\alpha) = (\theta + \pi\alpha, \alpha)$, and $F = g \circ \pi_1$, with $g : \mathbb{S}^1 \to \mathbb{R}$ nonconstant analytic of period 1 and $\pi_1 : \Omega_a \to \mathbb{S}^1$ the projection $\pi_1(\theta,\alpha) = \theta$. For every $(\theta,\alpha), (\omega,\beta) \in \Omega_a$, it follows by the Mean Value Theorem that

$$\begin{aligned}
\bigl|F(\theta,\alpha) - F(\omega,\beta)\bigr| &= \bigl|g(\theta) - g(\omega)\bigr| \\
&\leqslant \Bigl(\sup_{z\in\mathbb{S}^1}\bigl|g'(z)\bigr|\Bigr)|\theta - \omega| \\
&\leqslant Ld\bigl((\theta,\alpha),(\omega,\beta)\bigr),
\end{aligned}$$

where $L = \sup_{z\in\mathbb{S}^1} |g'(z)|$ and $d((\theta,\alpha),(\omega,\beta)) = \sqrt{(\theta-\omega)^2 + (\alpha-\beta)^2}$; in other words, F satisfies the Lipschitz condition. The set

$$A = \bigl\{(\theta,\alpha_0)\colon \theta \in \mathbb{S}^1,\ \alpha_0 \in \mathbb{Q} \cap [-a,a]\bigr\}$$

is dense in Ω_a and for each $(\theta,\alpha_0) \in A$, $S^n(\theta,\alpha_0)$ describes a periodic orbit at the "height" α_0. Therefore the potential $\lambda F(S^n(\theta,\alpha_0))$ is periodic and, due to [18,27], A satisfies the hypothesis (7) of Theorem 1. Now note that for every $(\theta,\alpha), (\omega,\beta) \in \Omega_a$ and $n \in \mathbb{Z} \setminus \{0\}$ ($n = 0$ is trivial), one has

$$\begin{aligned}
d\bigl(S^n(\theta,\alpha), S^n(\omega,\beta)\bigr) &\leqslant d\bigl(S^n(\theta,\alpha), S^n(\theta,\beta)\bigr) + d\bigl(S^n(\theta,\beta), S^n(\omega,\beta)\bigr) \\
&= \sqrt{n^2\pi^2(\alpha-\beta)^2 + (\alpha-\beta)^2} + \sqrt{(\theta-\omega)^2} \\
&\leqslant \bigl(\sqrt{\pi^2+1}\,|\alpha-\beta| + |\theta-\omega|\bigr)|n| \\
&\leqslant \bigl(\sqrt{\pi^2+1}+1\bigr)d\bigl((\theta,\alpha),(\omega,\beta)\bigr)|n|.
\end{aligned}$$

Therefore, by Theorem 1, there exists a dense G_δ set $\tilde{\Omega}_a \subset \Omega_a$ such that for any $(\theta,\alpha) \in \tilde{\Omega}_a$ and for every $p > 0$, $\beta^+_{(\theta,\alpha),S,W}(p) = 1$, with W satisfying (3). Observe that $F(S^n(\theta,\alpha)) = g(\theta + n\pi\alpha)$.

It follows by Sorets and Spencer [36] that there exists a number $\lambda_0(F) > 0$ such that for $\lambda > \lambda_0$, $\Gamma^0_S(E) > 0$ for every E, every irrational α and ℓ-a.s. θ (ℓ on $\mathbb{S}^1$ is ergodic with respect to $\pi_1 \circ S$). Since the generic set $\tilde{\Omega}_a$ can have zero measure, we are not assured to be able to apply Theorem 2 to elements of $\tilde{\Omega}_a$ in order to obtain quasi-ballistic dynamics with pure point spectrum. Nevertheless, the original view in [18,22] (i.e., to consider for each α a different map) for the cosine function, implies uniformity in θ also in our case, so that we get new examples of

Schrödinger operators with pure point spectrum and quasi-ballistic dynamics. Let us reconsider the construction, since it will also be employed in Application 5.6.

Begin by replacing (rewriting, in fact) $H^W_{(\theta,\alpha),S}$ with H^W_{θ,S_α}, where $S_\alpha(\theta)=\theta+\pi\alpha$, $\theta\in\mathbb{S}^1$, and $F=g$ (take π_1 as the identity). For each $\alpha_0\in\mathbb{Q}$ and $\theta\in\mathbb{S}^1$, the potential $\lambda F(S^n_{\alpha_0}(\theta))$ is periodic and there is $J^\theta_{\alpha_0}\subset\sigma(H^0_{\theta,S_{\alpha_0}})$ with $\ell(J^\theta_{\alpha_0})>0$ so that uniformly in θ

$$\left\|\Phi^0_{\theta,S_{\alpha_0}}(E,n,0)\right\|\leqslant C_{\alpha_0},\quad \forall E\in J^\theta_{\alpha_0},\ n\in\mathbb{Z}.$$

Furthermore, for all θ and n

$$d\big(S^n_\alpha(\theta),S^n_{\alpha_0}(\omega)\big)=d(\theta+n\pi\alpha,\theta+n\pi\alpha_0)=\pi|\alpha-\alpha_0||n|.$$

By repeating the arguments of Lemma 6 and Theorem 1, but now with

$$\tilde{B}_n=\left\{\alpha\in[-a,a]\;\middle|\;\exists\epsilon<\frac{1}{n}\colon\ \forall\theta\in\mathbb{S}^1,\ K_{\mu^W_{\theta,S_\alpha}}(q,\epsilon)\geqslant C_q\,\frac{\epsilon^{-1+q}}{\log(\epsilon^{-1})}\right\}$$

instead of B_n, one concludes that there exists a dense G_δ set of irrational numbers $\mathcal{G}\subset[-a,a]$, so that for each fixed $\alpha\in\mathcal{G}$ and every $\theta\in\mathbb{S}^1$, the operator $H^W_{(\theta,\alpha),S}=H^W_{\theta,S_\alpha}$ presents quasi-ballistic dynamics.

Therefore, by Theorem 2, for $\alpha\in\mathcal{G}$ and ℓ-a.s. θ the operator $H^W_{(\theta,\alpha),S}$ with nonconstant analytic F on the half lattice $\ell^2(\mathbb{N})$ case, under the rank one perturbation $W=\kappa\langle\delta_1,\cdot\rangle\delta_1$, $\lambda>\lambda_0$ and α irrational, has pure point spectrum for ℓ-a.s. κ, and also presents quasi-ballistic dynamics.

Now some interesting particular cases of potentials generated by this dynamical system will be described.

5.5.1. Almost Mathieu

This is just a reconsideration of the "pathological example" of [18]. $H^W_{(\theta,\alpha),S}$ is defined by (1), where S is the transformation on Ω_a given by $S(\theta,\alpha)=(\theta+\pi\alpha,\alpha)$, $F=\cos\circ\pi_1:\Omega_a\to\mathbb{R}$, $W=\kappa\langle\delta_1,\cdot\rangle\delta_1$, α is irrational and $\lambda>\lambda_0=2$. Under such conditions both Theorems 1 and 2 hold for proper sets, as discussed in Application 5.5.

5.5.2. Circular billiards [10]

The potential now is along the orbits of a particle under specular reflections on a circular billiard. $H^W_{(r,\phi),S}$ is defined by (1), where S is the transformation on $\Omega_{\frac{\pi}{2}}$ given by $S(r,\phi)=(r+\pi-2\phi,\phi)$ and $F=g\circ\pi_1$ with $g:\mathbb{S}^1\to\mathbb{R}$ nonconstant analytic of period 1. Again the conclusions of Application 5.5 hold.

5.5.3. Twist map [26]

$H^W_{(\theta,r),S}$ is defined by (1), with S the transformation on $\Omega=\overline{D(0,1)}$ (closed disk of center 0 and radius 1 in $\mathbb{R}^2$) given by $S(\theta,r)=(\theta+\rho(r),r)$, with $\rho:[0,1]\to[0,2\pi]$ continuous, $\rho(0)=0$, $\rho'(r)>0$, and $F=g\circ\pi_1$ with g a nonconstant real-analytic function of period 1. So, the potential is defined along orbits of an integrable twist map, and Theorems 1 and 2 hold concomitant for proper sets.

5.6. Rotations of the torus with analytic condition on F

Consider the operator $H^W_{(\theta,\alpha),S}$ defined by (1), where S is the transformation on $\Omega^k_a := \mathbb{T}^k \times [-a,a]^k$ ($\mathbb{T}^k$ is the k-dimensional torus; $a > 0$ fixed) given by $S(\theta,\alpha) = (\theta + \pi\alpha, \alpha)$, with $\theta = (\theta_1,\dots,\theta_k)$, $\alpha = (\alpha_1,\dots,\alpha_k)$, $F = g \circ \pi_k$ with $g : \mathbb{T}^k \to \mathbb{R}$ nonconstant analytic of period 1 in each component, and $\pi_k : \Omega^k_a \to \mathbb{T}^k$ the projection $\pi_k(\theta,\alpha) = \theta$. Observe that for $k = 1$, we are in the case of Application 5.5 above. Similarly to Application 5.5, one obtains a dense G_δ set $\tilde{\Omega}^k_a \subset \Omega^k_a$ such that for any $(\theta,\alpha) \in \tilde{\Omega}^k_a$ and for every $p > 0$, $\beta^+_{(\theta,\alpha),S,W}(p) = 1$. Note that $F(S^n(\theta_1,\dots,\theta_k,\alpha_1,\dots,\alpha_k)) = g(\theta_1 + n\pi\alpha_1,\dots,\theta_k + n\pi\alpha_k)$.

The construction of $\mathcal{G}$ in Application 5.5 has a direct counterpart here, so that $\beta^+_{(\theta,\alpha),S,W}(p) = 1$ for α in a dense G_δ set $\mathcal{G}_k \subset [-a,a]^k$ and every θ. The above mentioned result of Sorets and Spencer is still valid in this case [5]: there is $\lambda_0 > 0$ so that if $\lambda > \lambda_0$, then $\Gamma^0_S(E) > 0$ for every E, every incommensurate vector α (i.e., $\alpha \cdot j \neq 0$ for all $j \in \mathbb{Z}^k \setminus \{0\}$) and ℓ-a.s. $\theta = (\theta_1,\dots,\theta_k)$ (ℓ on $\mathbb{T}^k$ is ergodic with respect to $\pi_k \circ S$). Therefore, by Theorem 2, for such α's the operator $H^W_{(\theta,\alpha),S}$ on $\ell^2(\mathbb{N})$, with $W = \kappa\langle\delta_1,\cdot\rangle\delta_1$ and λ large enough, has pure point spectrum for ℓ-a.s. θ and κ. Since necessarily a G_δ set in $[-a,a]^k$ contains incommensurate vectors α (in particular for $\mathcal{G}_k$), again we have got new examples of Schrödinger operators with pure point spectrum and quasi-ballistic dynamics. We stress once more that, in fact, our arguments come from a (simple) closer inspection of the original arguments of [18,22] for the almost-Mathieu operator.

6. The discrete Dirac model

The single particle one-dimensional discrete Dirac operator was studied in [16,17] and is described by ($\omega \in \Omega$)

$$\mathbf{D}_\omega(m,c) := \begin{pmatrix} mc^2 & cD^* \\ cD & -mc^2 \end{pmatrix} + V_\omega \,\mathrm{Id}_2,$$

acting on $\ell^2(\mathbb{Z},\mathbb{C}^2)$ or $\ell^2(\mathbb{N},\mathbb{C}^2)$, where Id_2 is the 2×2 identity matrix, $c > 0$ represents the speed of light, $m \geqslant 0$ is the mass of the particle, D is the finite difference operator defined by $(D\psi)(n) = \psi(n+1) - \psi(n)$ and D^* is the adjoint of D.

Besides being a physical model, it was of interest because for the massless case and two-valued Bernoulli potentials, its behavior is similar to the corresponding Schrödinger case after dimerization, with presence of the so-called critical energies [17,25]. So, under certain conditions it is possible to get pure point spectrum and nontrivial transport a.s. with $\beta^-(p) \geqslant (1 - \frac{1}{2p})$.

By considering the potential $V_\omega(n) = \lambda F(S^n\omega) + W(n)$, with F and W satisfying (2) and (3), respectively, Theorems 1 and 2 hold, and so all applications in Section 5 have a counterpart for this model; this follows after a huge set of technical details (not presented) are checked and adapted by following the lines of [17]. Hence this discrete Dirac version, with suitable potentials along some dynamical systems, provides examples of relativistic quantum operators with quasi-ballistic dynamics, some also with point spectrum.

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