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Independence of Boolean algebras and forcing

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Abstract

If $\kappa \geqslant \omega$ is a cardinal, a complete Boolean algebra B is called κ-dependent if for each sequence $\langle b_\beta\colon \beta < \kappa\rangle$ of elements of B there exists a partition of the unity, P, such that each $p \in P$ extends b_β or b'_β, for κ-many $\beta \in \kappa$. The connection of this property with cardinal functions, distributivity laws, forcing and collapsing of cardinals is considered.

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1. Introduction

The notation used in this paper is mainly standard. So, if $\langle \mathsf{B}, \wedge, \vee, ', 0, 1\rangle$ is a Boolean algebra, then B^+ denotes the set of all positive elements of B. A subset $P \subset \mathsf{B}^+$ is an antichain if $p \wedge q = 0$ for each different $p, q \in P$. If, in addition $\bigvee P = 1$, then P is called a partition of the unity. The cardinal $\mathrm{c}(\mathsf{B}) = \sup\{|P|\colon\ P$ is an antichain in $\mathsf{B}\}$ is the cellularity of B. A subset $D \subset \mathsf{B}^+$ is said to be dense if for each $p \in \mathsf{B}^+$ there exists $q \in D$ such that $q \leqslant p$. The algebraic density of B is the cardinal $\pi(\mathsf{B}) = \min\{|D|\colon\ D$ is dense in $\mathsf{B}\}$. A set $D \subset \mathsf{B}$ is called open if for each $p \in D$ and $q \leqslant p$ there holds $q \in D$. If $\kappa \geqslant \omega$ and $\lambda \geqslant 2$ are cardinals, by ${}^{<\kappa}\lambda$ we denote the set $\bigcup_{\xi<\kappa} {}^{\xi}\lambda$ ordered by the reversed inclusion and by $\mathrm{Col}(\kappa, \lambda)$ the Boolean completion of this partial order, the (κ, λ)-collapsing algebra.

In order to simplify notation, for $p \in \mathsf{B}$ and $B \subset \mathsf{B}$ we write $p \prec B$ if $p \leqslant b$ for some $b \in B$. Also, if $p, b \in \mathsf{B}^+$, we say that b splits p (p is splitted by b) if $p \wedge b > 0$ and

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$p \wedge b' > 0$, that is if $p \not\prec \{b, b'\}$. Specially, a set X splits a set A if the sets $A \cap X$ and $A \setminus X$ are non-empty. Finally, if κ is a cardinal, we say that a property $P(\beta)$ holds for almost all $\beta \in \kappa$ if $|\{\beta \in \kappa\colon \neg P(\beta)\}| < \kappa$.

The property of complete Boolean algebras investigated in this paper can be introduced as a modification of the $(\kappa, 2)$-distributive law (see [4,6,7]). Namely, a complete Boolean algebra B is said to be $(\kappa, 2)$-distributive if and only if the equality $\bigwedge_{\beta<\kappa} \bigvee_{n<2} p_{\beta n} = \bigvee_{f:\kappa\to 2} \bigwedge_{\beta<\kappa} p_{\beta f(\beta)}$ holds for each double sequence $\langle p_{\beta n}\colon \langle \beta, n\rangle \in \kappa \times 2\rangle$ of elements of B, if and only if in each generic extension $V_{\mathsf{B}}[G]$ every subset of κ belongs to the ground model V and, finally, if and only if

> for each sequence $\langle b_\beta\colon \beta < \kappa\rangle \in {}^\kappa\mathsf{B}$ there exists a partition of the unity, P, such that each $p \in P$ satisfies $p \prec \{b_\beta, b'_\beta\}$ for all $\beta \in \kappa$.

So, a complete Boolean algebra B will be called κ-dependent if and only if

> for each sequence $\langle b_\beta\colon \beta < \kappa\rangle \in {}^\kappa\mathsf{B}$ there exists a partition of the unity, P, such that each $p \in P$ satisfies $p \prec \{b_\beta, b'_\beta\}$ for κ-many $\beta \in \kappa$.

Otherwise, B will be called κ-independent. The algebra B will be called strongly κ-independent, if and only if

> there exists a sequence $\langle b_\beta\colon \beta < \kappa\rangle \in {}^\kappa\mathsf{B}$ such that each positive $p \in \mathsf{B}$ is splitted by b_β for almost all $\beta \in \kappa$.

In this paper we investigate what can be said about κ-independence of complete Boolean algebras in general. So, in Sections 2 and 3, after establishing some algebraic and forcing equivalents of the property, we restrict our attention firstly to atomless Boolean algebras (since atomic algebras are κ-dependent for all infinite cardinals κ) and secondly, considering an atomless algebra B, to cardinals which are either regular and between $\mathfrak{h}_2(\mathsf{B}) = \min\{\kappa\colon \mathsf{B}$ is not $(\kappa, 2)$-distributive$\}$ and $\pi(\mathsf{B})$, or singular of cofinality $\leqslant \pi(\mathsf{B})$ (since for all other cardinals B is κ-dependent). Regarding regular cardinals it turns out that "everything is possible" if, for example, the GCH holds.

In Section 4 we show that, under some reasonable conditions (specially, under the GCH), collapse of cardinals implies independence, and that (in ZFC) the algebras $\mathrm{Col}(\kappa, \lambda)$ are θ-independent for all possible values of θ.

In Section 5 singular cardinals are considered. It is shown that for a singular κ, $\mathrm{cf}(\kappa)$-independence implies κ-independence and investigated when dependence of B on an unbounded subset of a singular cardinal κ implies κ-dependence of B.

2. Algebraic and forcing equivalents

If B is a complete Boolean algebra in the universe (ground model) V and $G \subset \mathsf{B}$ a B-generic filter over V, then $V_{\mathsf{B}}[G]$ or briefly $V[G]$ will denote the corresponding generic extension. If κ is a cardinal in V, then by Old_κ we denote the set of all κ-sized subsets of κ belonging to V, that is $\mathrm{Old}_\kappa = ([\kappa]^\kappa)^V$. A subset X of κ belonging to $V[G]$ is called independent if it splits all $A \in \mathrm{Old}_\kappa$. Otherwise, if $A \subset X$ or $A \subset \kappa \setminus X$ for some $A \in \mathrm{Old}_\kappa$, the set X is called dependent.

Theorem 1. *For each complete Boolean algebra* B *and each infinite cardinal* κ *the following conditions are equivalent*:

(a) B *is* κ*-dependent, that is for each sequence* $\langle b_\beta \colon \beta < \kappa \rangle \in {}^\kappa\mathsf{B}$ *there exists a partition of the unity,* P*, such that for each* $p \in P$, $p \prec \{b_\beta, b'_\beta\}$ *for* κ*-many* $\beta \in \kappa$.

(b) $\bigvee_{A \in [\kappa]^\kappa} (\bigwedge_{\beta \in A} b_\beta) \vee (\bigwedge_{\beta \in A} b'_\beta) = 1$, *for each sequence* $\langle b_\beta \colon \beta < \kappa \rangle \in {}^\kappa\mathsf{B}$.

(c) *In each generic extension* $V_{\mathsf{B}}[G]$ *each subset of* κ *is dependent*.

(d) *In each generic extension* $V_{\mathsf{B}}[G]$ *each unbounded subset of* κ *is dependent*.

If κ *is a regular cardinal, then each of these conditions is equivalent to the condition*

(e) *For each* $C \in [\mathsf{B}]^\kappa$ *the set* $D_C = \{p \in \mathsf{B}^+ \colon p \prec \{c, c'\}$ *for* κ*-many* $c \in C\}$ *is dense in* B.

Proof. (a $\Rightarrow$ b). Let (a) hold and $\langle b_\beta \colon \beta < \kappa \rangle \in {}^\kappa\mathsf{B}$. If P is the corresponding partition of the unity provided by (a) then each $p \in P$ extends b_β for κ-many $\beta \in \kappa$ or extends b'_β for κ-many $\beta \in \kappa$, so, there is $A \in [\kappa]^\kappa$ such that $p \leqslant \bigwedge_{\beta \in A} b_\beta$ or $p \leqslant \bigwedge_{\beta \in A} b'_\beta$. Hence $1 = \bigvee P \leqslant \bigvee_{A \in [\kappa]^\kappa} (\bigwedge_{\beta \in A} b_\beta) \vee (\bigwedge_{\beta \in A} b'_\beta)$.

(b $\Rightarrow$ c). Let condition (b) hold and let $V[G]$ be a generic extension containing $X \subset \kappa$. Then $X = \tau_G$ for some B-name τ. Applying (b) to the sequence $b_\beta = \|\check{\beta} \in \tau\|$, $\beta < \kappa$, we obtain $\|\exists A \in \mathrm{Old}_{\check{\kappa}}\, (A \subset \tau \vee A \subset \check{\kappa} \setminus \tau)\| = 1$, so there is $A \in \mathrm{Old}_\kappa$ such that $A \subset X$ or $A \subset \kappa \setminus X$ and (c) is true.

(c $\Rightarrow$ a). Let (c) hold and $\langle b_\beta \colon \beta < \kappa \rangle \in {}^\kappa\mathsf{B}$. Then $\tau = \{\langle \check{\beta}, b_\beta \rangle \colon \beta \in \kappa\}$ is a B-name and $1 \Vdash \tau \subset \check{\kappa}$ so by (c) $1 \Vdash \exists A \in \mathrm{Old}_{\check{\kappa}}\, (A \subset \tau \vee A \subset \check{\kappa} \setminus \tau)$ or equivalently $1 \Vdash \neg \forall A \in \mathrm{Old}_{\check{\kappa}}$ $(\neg A \subset \tau \wedge \neg A \subset \check{\kappa} \setminus \tau)$. The last condition is equivalent to the condition

$$\forall b \in \mathsf{B}^+ \exists p \leqslant b \exists A \in \mathrm{Old}_\kappa\ (\forall \beta \in A (p \leqslant b_\beta) \vee \forall \beta \in A\ (p \leqslant b'_\beta)).$$

So the set $D = \{p \in \mathsf{B}^+ \colon p \prec \{b_\beta, b'_\beta\}$ for κ-many $\beta \in \kappa\}$ is dense in B and open. Let $P \subset D$ be a maximal antichain of elements of D. Clearly P is a partition of the unity satisfying the condition from (a).

(c $\Leftrightarrow$ d). The direction "$\Rightarrow$" is trivial. Let (d) hold and $X \in V[G]$, where $X \subset \kappa$. If the set X is unbounded in κ then by (d) there exists $A \in \mathrm{Old}_\kappa$ such that $A \subset X$ or $A \subset \kappa \setminus X$. Otherwise, $X \subset \xi$ for some $\xi < \kappa$ and for $A = \kappa \setminus \xi$ we have $A \in \mathrm{Old}_\kappa$ and $A \subset \kappa \setminus X$.

(a $\Rightarrow$ e). Let condition (a) hold. If $\kappa > |\mathsf{B}|$ then (e) is vacuously true. Let $\kappa \leqslant |\mathsf{B}|$, $C \in [\mathsf{B}]^\kappa$ and let $C = \{c_\beta \colon \beta < \kappa\}$ be an 1-1 enumeration of C. By (a) there exists a partition of the unity, P, such that each $p \in P$ satisfies $p \prec \{c_\beta, c'_\beta\}$, for κ-many $\beta \in \kappa$. Now, if $b \in \mathsf{B}^+$ then there is $p \in P$ such that $p \wedge b = p_1 > 0$, thus $p_1 \in D_C$ and $p \leqslant b$, so the set D_C is dense in B.

(e $\Rightarrow$ a, for a regular κ). Let condition (e) hold and $\kappa \in \mathrm{Reg}$. For a sequence $\langle b_\beta \colon \beta < \kappa \rangle \in {}^\kappa\mathsf{B}$ we will prove that the set $D = \{p \in \mathsf{B}^+ \colon p \prec \{b_\beta, b'_\beta\}$ for κ-many $\beta \in \kappa\}$ is dense in B.

If $|\{b_\beta \colon \beta < \kappa\}| = \kappa$ and $C = \{b_\beta \colon \beta < \kappa\}$ then, clearly, $\kappa \leqslant |\mathsf{B}|$ and by (e) the set D_C is dense in B. For $p \in D_C$ if $p \leqslant c$ for κ-many $c \in C$ then $p \leqslant b_\beta$ for κ-many $\beta \in \kappa$, so $p \in D$. Otherwise $p \leqslant c'$ for κ-many $c \in C$ and $p \in D$ again. So $D_C \subset C$ and D is dense in B.

If $|\{b_\beta \colon \beta < \kappa\}| < \kappa$, then, by the regularity of κ, there exists $b \in \mathsf{B}$ such that $b_\beta = b$ for κ-many $\beta \in \kappa$. Let $q \in \mathsf{B}^+$. Firstly, if $p_1 = q \wedge b > 0$ then $p_1 \leqslant b_\beta$ for κ-many $\beta \in \kappa$ so $p \in D$. Otherwise, if $q \wedge b = 0$, then $q \leqslant b'_\beta$ for κ-many $\beta \in \kappa$ and $q \in D$. Thus D is dense in B.

Now, let $P \subset D$ be a maximal antichain in D. Then P is a partition of the unity satisfying (a). $\square$

Theorem 1 can be restated in the following way:

Theorem 2. *For each complete Boolean algebra* B *and each infinite cardinal* κ *the following conditions are equivalent*:

(a) B *is* κ*-independent, that is there exist a sequence* $\langle b_\beta \colon \beta < \kappa \rangle \in {}^\kappa\mathsf{B}$ *and* $q \in \mathsf{B}^+$ *such that each non-zero* $p \leqslant q$ *is splitted by* b_β *for almost all* $\beta \in \kappa$.

(b) $\bigvee_{A \in [\kappa]^\kappa} (\bigwedge_{\beta \in A} b_\beta) \vee (\bigwedge_{\beta \in A} b'_\beta) < 1$, *for some sequence* $\langle b_\beta \colon \beta < \kappa \rangle \in {}^\kappa\mathsf{B}$.

(c) *In some extension* $V_{\mathsf{B}}[G]$ *there exists an independent subset* $X \subset \kappa$.

Theorem 3. *For each complete Boolean algebra* B *and each infinite cardinal* κ *the following conditions are equivalent*:

(a) B *is strongly* κ*-independent, that is there exists a sequence* $\langle b_\beta \colon \beta < \kappa \rangle \in {}^\kappa\mathsf{B}$ *such that each positive* $p \in \mathsf{B}$ *is splitted by* b_β *for almost all* $\beta \in \kappa$.

(b) $\bigvee_{A \in [\kappa]^\kappa} (\bigwedge_{\beta \in A} b_\beta) \vee (\bigwedge_{\beta \in A} b'_\beta) = 0$, *for some sequence* $\langle b_\beta \colon \beta < \kappa \rangle \in {}^\kappa\mathsf{B}$.

(c) *In each extension* $V_{\mathsf{B}}[G]$ *there exists an independent subset* $X \subset \kappa$.

Proof. (a $\Rightarrow$ b). Let $\langle b_\beta \colon \beta < \kappa \rangle$ be a sequence provided by (a). Suppose $\bigwedge_{\beta \in A} b_\beta = p > 0$, for some $A \in [\kappa]^\kappa$. But then for some $\beta \in A$, b_β splits p, which is impossible. So, for each $A \in [\kappa]^\kappa$ we have $\bigwedge_{\beta \in A} b_\beta = 0$ and similarly $\bigwedge_{\beta \in A} b'_\beta = 0$ and (b) is proved.

(b $\Rightarrow$ c). Let $\langle b_\beta \colon \beta < \kappa \rangle$ be a sequence provided by (b). Then for $\tau = \{\langle \check{\beta}, b_\beta \rangle \colon \beta \in \kappa\}$ we have $1 \Vdash \tau \subset \check{\kappa}$ and (b) implies $\|\tau \text{ splits all } A \in \mathrm{Old}_{\check{\kappa}}\| = 1$.

(c $\Rightarrow$ a). Let (c) hold. Then, by the Maximum principle (see [4]) there exists a name τ such that: (i) $1 \Vdash \tau \subset \check{\kappa}$; (ii) $1 \Vdash \forall A \in \mathrm{Old}_{\check{\kappa}} (A \cap \tau \neq \emptyset)$; (iii) $1 \Vdash \forall A \in \mathrm{Old}_{\check{\kappa}} (A \setminus \tau \neq \emptyset)$. Putting $b_\beta = \|\beta \in \tau\|$, for $\beta < \kappa$ and using (ii) we easily conclude that $|\{\beta \in \kappa \colon p \wedge b_\beta = 0\}| < \kappa$, for each $p \in \mathsf{B}^+$. Similarly, by (iii) we have $|\{\beta \in \kappa \colon p \wedge b'_\beta = 0\}| < \kappa$ for each $p \in \mathsf{B}^+$ so, if $p \in \mathsf{B}^+$ then $p \prec \{b_\beta, b'_\beta\}$ for $< \kappa$-many $\beta \in \kappa$ and (a) is proved. $\square$

Remark 1. It is known (see [4, p. 65]) that if B is a weakly homogeneous c.B.a., $\varphi(v_1, v_2, \ldots, v_n)$ a formula of ZFC and $a_1, a_2, \ldots, a_n \in V$, then $\varphi(a_1, a_2, \ldots, a_n)$ holds in some iff it holds in all generic extensions of V by B. So considering parts (c) of the previous two theorems we conclude that a weakly homogeneous c.B.a. is κ-independent iff it is strongly κ-independent.

Theorem 4. *If a complete Boolean algebra* B *is atomic, then it is* κ*-dependent for every infinite cardinal* κ.

Proof. Although a proof by forcing arguments is evident, we will demonstrate a combinatorial one. Let $\langle b_\beta\colon \beta<\kappa\rangle \in {}^\kappa\mathsf{B}$. Since the algebra B is atomic, the set $\mathrm{At}(\mathsf{B})$ of all its atoms is a partition of the unity and (because atoms cannot be splitted) if $p\in \mathrm{At}(\mathsf{B})$, then $p\prec\{b_\beta, b'_\beta\}$ for all $\beta\in\kappa$. So, B is κ-dependent by definition. □

3. Dependence, supportedness and distributivity

In this section we compare κ-dependence with some other forcing related properties of complete Boolean algebras and determine the position of the cardinals κ for which a given algebra can be κ-independent.

Theorem 5. *A complete Boolean algebra* B *is* κ*-dependent for each cardinal* κ *satisfying* $\mathrm{cf}(\kappa)>\pi(\mathsf{B})$.

Proof. On the contrary, suppose $\mathrm{cf}(\kappa)>\pi(\mathsf{B})$ and B is κ-independent. Then by Theorem 2 there is a sequence $\langle b_\beta\colon \beta<\kappa\rangle\in{}^\kappa\mathsf{B}$ satisfying $\bigvee_{A\in[\kappa]^\kappa}(\bigwedge_{\beta\in A} b_\beta)\vee(\bigwedge_{\beta\in A} b'_\beta)=c<1$, thus we have: (i) $\bigwedge_{\beta\in A} b_\beta\leqslant c$, for each $A\in[\kappa]^\kappa$; and (ii) $0<c'\leqslant\bigvee_{\beta\in A} b_\beta$, for each $A\in[\kappa]^\kappa$.

By (ii), c' is compatible with b_β for almost all $\beta\in\kappa$, thus the set $A_{c'}=\{\beta\in\kappa\colon b_\beta\wedge c'>0\}$ is of size κ. Let $D\subset\mathsf{B}^+$ be a dense subset of B of size $\pi(\mathsf{B})$. Now, for each $\beta\in A_{c'}$ we pick $d_\beta\in D$ such that $d_\beta\leqslant b_\beta\wedge c'$, obtaining a function from $A_{c'}$ to D. Since $|D|<\mathrm{cf}(\kappa)$ there exists $d\in D$ such that $d_\beta=d$ for κ-many $\beta\in A_{c'}$. Thus the set $A_d=\{\beta\in A_{c'}\colon d\leqslant b_\beta\wedge c'\}$ is of cardinality κ and $\bigwedge_{\beta\in A_d} b_\beta\wedge c'\geqslant d>0$, which is impossible by (i). □

In [10] a complete Boolean algebra B is called κ-supported (for a cardinal $\kappa\geqslant\omega$) iff the equality $\bigwedge_{\alpha<\kappa}\bigvee_{\beta>\alpha} b_\beta=\bigvee_{A\in[\kappa]^\kappa}\bigwedge_{\beta\in A} b_\beta$ is satisfied for each sequence $\langle b_\beta\colon \beta<\kappa\rangle$ of elements of B. Otherwise, the algebra B is called κ-unsupported. In the sequel we will use the following facts proved in [10]:

Fact 1. *Let* B *be an arbitrary complete Boolean algebra. Then*
(a) B *is* κ*-unsupported for each singular cardinal* κ.
(b) B *is* κ*-supported if and only if in every generic extension* κ *is a regular cardinal and each new set* $X\in[\kappa]^\kappa$ *has an old subset of size* κ.
(c) $\mathrm{Unsupp}(\mathsf{B})=\{\kappa\in\mathrm{Reg}\colon \mathsf{B}$ *is* κ*-unsupported*$\}\subset[\mathfrak{h}_2(\mathsf{B}),\pi(\mathsf{B})]$.
(d) *If* $2^{<\mathfrak{h}_2(\mathsf{B})}=\mathfrak{h}_2(\mathsf{B})$, *specially, if* $\mathfrak{h}_2(\mathsf{B})=\aleph_0$, *then* B *is* $\mathfrak{h}_2(\mathsf{B})$*-unsupported. If* $0^\sharp\notin V$ *and forcing by* B *preserves* $\mathfrak{h}_2(\mathsf{B})^+$, *then* B *is* $\mathfrak{h}_2(\mathsf{B})$*-unsupported.*

Theorem 6. *Let* B *be a c.B.a. and* $\mathrm{Indep}(\mathsf{B})=\{\kappa\in\mathrm{Reg}\colon \mathsf{B}$ *is* κ*-independent*$\}$. *Then*
(a) *If* B *is* κ*-supported, it is* κ*-dependent.*
(b) $\mathrm{Indep}(\mathsf{B})\subset\mathrm{Unsupp}(\mathsf{B})\subset[\mathfrak{h}_2(\mathsf{B}),\pi(\mathsf{B})]$.

Proof. The assertion (a) follows from forcing characterizations given in Fact 1(b) and Theorem 1(d). The first inclusion in (b) is a consequence of (a), while the second is Fact 1(c). The inclusion $\mathrm{Indep}(\mathsf{B})\subset[\mathfrak{h}_2(\mathsf{B}),\pi(\mathsf{B})]$ also follows from Theorem 5 and the fact that $(\kappa,2)$-distributivity implies κ-dependence. □

Remark 2. There exist κ-dependent algebras which are not κ-supported. Firstly, if κ is a singular cardinal and $\mathrm{cf}(\kappa)>\pi(\mathsf{B})$, then B is κ-dependent by Theorem 5 and κ-unsupported by Fact 1(a). Also there are such examples for regular cardinals κ. Namely, Sacks' perfect set forcing (see [13,3]) and Miller's rational perfect set forcing (see [12]) produce new subsets of ω, but all of them are dependent. So, the corresponding Boolean algebras are ω-dependent by Theorem 1 and ω-unsupported by Fact 1(d). For uncountable regular cardinals we mention the forcing of Kanamori (see [8]) which has the observed property for κ strongly inaccessible.

Remark 3. κ-dependence and weak (κ,κ)-distributivity are unrelated properties. A complete Boolean algebra B is called weakly (κ,λ)-distributive if and only if the equality $\bigwedge_{\alpha<\kappa}\bigvee_{\beta<\lambda}b_{\alpha\beta}=\bigvee_{f:\kappa\to\lambda}\bigwedge_{\alpha<\kappa}\bigvee_{\beta<f(\alpha)}b_{\alpha\beta}$ holds for each double sequence $\langle b_{\alpha\beta}\colon\ \langle\alpha,\beta\rangle\in\kappa\times\lambda\rangle$ of elements of B, if and only if in each generic extension $V_{\mathsf{B}}[G]$ every function $f\colon\ \kappa\to\lambda$ is majorized by some function $g\colon\ \kappa\to\lambda$ belonging to V. Since both κ-dependence and weak (κ,κ)-distributivity are weakenings of $(\kappa,2)$-distributivity (and, moreover, of κ-supportedness) it is natural to ask whether these two properties are related. The answer is "No". It is easy to check that a c.B.a. B is weakly (ω,ω)-distributive iff forcing by B does not produce weak dominating functions from ω to ω ($f\in{}^{\omega}\omega\cap V[G]$ is a w.d.f. iff for each $g\in{}^{\omega}\omega\cap V$ the set $\{n\in\omega\colon\ g(n)<f(n)\}$ is infinite). Now, firstly, it is well-known that adding a random real to V produces independent subsets of ω, but does not produce w.d.f.'s. Secondly, Miller's rational perfect set forcing produces w.d.f.'s, but does not produce independent subsets of ω (see [12]).

According to Theorems 5 and 6, the question on κ-independence of a given Boolean algebra remains open for $\kappa\in\mathrm{Reg}\cap[\mathfrak{h}_2(\mathsf{B}),\pi(\mathsf{B})]$ and for singular κ satisfying $\mathrm{cf}(\kappa)\leqslant\pi(\mathsf{B})$. In the sequel we show that for regular cardinals everything is possible if, for example, the GCH is assumed. Singular cardinals will be considered later.

Theorem 7. *Let* B_i, $i\in I$, *be a family of complete Boolean algebras. Then* $\mathrm{Indep}(\prod_{i\in I}\mathsf{B}_i)=\bigcup_{i\in I}\mathrm{Indep}(\mathsf{B}_i)$.

Proof. Let $\mathsf{B}=\prod_{i\in I}\mathsf{B}_i$. It is known that if $V_{\mathsf{B}}[G]$ is a B-generic extension, then $V_{\mathsf{B}}[G]=V_{\mathsf{B}_i}[H]$ for some $i\in I$ and some B_i-generic filter H, and conversely, if $V_{\mathsf{B}_i}[H]$ is a B_i-generic extension, then $V_{\mathsf{B}_i}[H]=V_{\mathsf{B}}[G]$ for some B-generic filter G. Now, using characterization given in Theorem 2(c), we easily finish the proof. □

Theorem 8. *For each set S of regular cardinals κ satisfying $2^{<\kappa}=\kappa$ there exists a complete Boolean algebra B such that* $\mathrm{Indep}(\mathsf{B})=S$. *If* $|S|>1$, *then B is not strongly λ-independent for any regular λ. Specially, under the GCH, for each set* $S\subset\mathrm{Reg}$ *there is a complete Boolean algebra B satisfying* $\mathrm{Indep}(\mathsf{B})=S$.

Proof. It is easy to show that if κ is a regular cardinal, then $\mathfrak{h}_2(\mathrm{Col}(\kappa,2))=\kappa$ and $\pi(\mathrm{Col}(\kappa,2))=2^{<\kappa}$, so, under the assumptions, for each $\kappa\in S$ we have $\mathrm{Indep}(\mathrm{Col}(\kappa,2))$

$\subset\{\kappa\}$. On the other hand, if G is a ${}^{<\kappa}2$-generic filter over V, then a simple density argument shows that $f_G = \bigcup G\colon\ \kappa\to 2$ is the characteristic function of an independent subset of κ. Thus $\mathrm{Indep}(\mathrm{Col}(\kappa,2)) = \{\kappa\}$ and by the previous theorem $\mathsf{B} = \prod_{\kappa\in S}\mathrm{Col}(\kappa,2)$ satisfies $\mathrm{Indep}(\mathsf{B}) = S$. If $|S|>1$ and $\lambda\in\mathrm{Reg}$, then we choose $\kappa\in S\setminus\{\lambda\}$. In extensions by $\mathrm{Col}(\kappa,2)$ each subset of λ is dependent, so, by Theorem 3, B is not strongly λ-independent. Finally, the GCH implies $2^{<\kappa}=\kappa$ for each κ. □

4. Independence and collapsing

Theorem 9. *Let λ be a cardinal in V and let $V[G]$ be a generic extension of V. Then*

(a) *If $|(\lambda^+)^V|^{V[G]} = |\lambda|^{V[G]}$ and if λ obtains an independent subset in $V[G]$, then $(\lambda^+)^V$ obtains an independent subset too.*

(b) *If $|\lambda|^{V[G]} = \kappa$ and if $(\mu^\kappa)^V \leqslant \lambda$ for each V-cardinal $\mu<\lambda$, then each $\theta\in\mathrm{Card}^V$ satisfying $\kappa\leqslant\theta\leqslant\lambda$ obtains an independent subset in $V[G]$.*

(c) *If $|(2^\lambda)^V|^{V[G]} = |\lambda|^{V[G]}$, then each $\theta\in\mathrm{Card}^V$ satisfying $|\lambda|^{V[G]}\leqslant\theta\leqslant(2^\lambda)^V$ obtains an independent subset in $V[G]$.*

Proof. (a) *Let* $|\lambda^+|^{V[G]} = |\lambda|^{V[G]} = \kappa$. Then $\mathrm{cf}^{V[G]}(\lambda^+) = \rho\leqslant\kappa$ and in $V[G]$ there is an increasing sequence $\langle\alpha_\xi\colon\ \xi<\rho\rangle$ of elements of λ^+, unbounded in λ^+.

We will show that in $V[G]$ there exists a sequence $\langle\beta_\xi\colon\ \xi<\rho\rangle\in{}^\rho(\lambda^+)$ such that $\lambda^+ = \bigcup_{\xi<\rho}[\beta_\xi,\beta_{\xi+1})$ and $|[\beta_\xi,\beta_{\xi+1})|^V = \lambda$, for each $\xi<\rho$. Firstly, let $\rho>\omega$. Using recursion in $V[G]$ we define $\beta_\xi, \xi<\rho$, by: $\beta_0 = 0; \beta_{\xi+1} = \max\{\alpha_\xi, \beta_\xi+\lambda\}$ (where $\beta_\xi+\lambda$ is the ordinal addition) and $\beta_\gamma = \sup\{\beta_\xi\colon\ \xi<\gamma\}$, if γ is a limit ordinal. Since the ordinal addition is an absolute operation and since each subset of λ^+ of size $<\rho$ is bounded in λ^+, an easy induction shows that $\beta_\xi\in\lambda^+$, for each $\xi<\rho$. So $\bigcup_{\xi<\rho}[\beta_\xi,\beta_{\xi+1})\subset\lambda^+$ and we will prove the equality. Let $\delta<\lambda^+$. The sequence $\langle\beta_\xi\colon\ \xi<\rho\rangle$ is (clearly) unbounded in λ^+ so there exists $\xi_0 = \min\{\xi<\rho\colon\ \delta<\beta_\xi\}$. Now, ξ_0 is a successor ordinal (otherwise we would have $\xi_0\leqslant\delta$) say $\xi_0 = \xi'+1$. Thus $\delta\in[\beta_{\xi'},\beta_{\xi'+1})$ and the equality is proved. If $\rho=\omega$, then the sequence $\langle\beta_\xi\colon\ \xi<\omega\rangle$ defined by: $\beta_0=0$ and $\beta_{\xi+1} = \max\{\alpha_\xi, \beta_{\xi+1}+\lambda\}$, satisfies two desired properties.

In V, the sets $[\beta_\xi,\beta_{\xi+1})$ are of size λ, so, working in $V[G]$ we can pick bijections $f_\xi\colon\ \lambda\to[\beta_\xi,\beta_{\xi+1}), \xi<\rho$, belonging to V. Let $X\in V[G]$ be an independent subset of λ. We will prove that $Y = \bigcup_{\xi<\rho} f_\xi[X]$ is an independent subset of λ^+.

Let $A\in\mathrm{Old}_{\lambda^+}$. Suppose $|A\cap[\beta_\xi,\beta_{\xi+1})|^V<\lambda$, for every $\xi<\rho$. Then the ordinals $\delta_\xi = \mathrm{type}^V(A\cap[\beta_\xi,\beta_{\xi+1}))$ are less than λ and in $V[G]$ the well-ordered set A is isomorphic to $\sum_{\xi<\rho}\delta_\xi$. Clearly, if $\mathrm{type}^V(\rho\cdot\lambda)=\eta$, where $\rho\cdot\lambda$ denotes the ordinal product, then $|\eta|^V=\lambda<\lambda^+$. In $V[G]$ the set A is isomorphic to a subset of η, so $\mathrm{type}^{V[G]}(A)\leqslant\eta$ and, since type is an absolute notion, we have $\mathrm{type}^V(A)\leqslant\eta<\lambda^+$. But $A\in\mathrm{Old}_{\lambda^+}$ implies $\mathrm{type}^V(A)=\lambda^+$. A contradiction. Thus there exists $\xi_0<\rho$ such that $|A\cap[\beta_{\xi_0},\beta_{\xi_0+1})|^V=\lambda$ hence $A\cap[\beta_{\xi_0},\beta_{\xi_0+1})\cap f_{\xi_0}[X]\neq\emptyset$ and $A\cap[\beta_{\xi_0},\beta_{\xi_0+1})\setminus f_{\xi_0}[X]\neq\emptyset$ which implies $A\cap Y\neq\emptyset$ and $A\setminus Y\neq\emptyset$.

(b) In $V[G]$ λ is an ordinal of size κ, so $\mathrm{cf}^{V[G]}(\lambda)=\rho\leqslant\kappa$ and there exists an increasing sequence $\langle\alpha_\delta\colon\ \delta<\rho\rangle$ unbounded in λ. W.l.o.g. we suppose $\alpha_\delta\geqslant\kappa$. In V, each ordinal α_δ is of size $<\lambda$ so, by the assumption, the set $[\alpha_\delta]^\kappa$ is of size $\leqslant\lambda$ in V and of size κ in $V[G]$. Consequently in $V[G]$ the set $\bigcup_{\delta<\rho}([\alpha_\delta]^\kappa)^V$ is of size κ, hence there exists an enumeration $\bigcup_{\delta<\rho}([\alpha_\delta]^\kappa)^V=\{A_\xi\colon\ \xi<\kappa\}$. By recursion in $V[G]$ we define the sequences $\langle\alpha_\xi\colon\ \xi<\kappa\rangle$ and $\langle\beta_\xi\colon\ \xi<\kappa\rangle$ by

$$\alpha_\xi=\min(A_\xi\setminus(\{\alpha_\zeta\colon\ \zeta<\xi\}\cup\{\beta_\zeta\colon\ \zeta<\xi\})),$$

$$\beta_\xi=\min(A_\xi\setminus(\{\alpha_\zeta\colon\ \zeta\leqslant\xi\}\cup\{\beta_\zeta\colon\ \zeta<\xi\})).$$

Since $\xi<\kappa$ implies $|\xi|^V<\kappa$, the sequences are well-defined.

Let $Y=\{\alpha_\xi\colon\ \xi<\kappa\}$ and let θ be a cardinal in V, where $\kappa\leqslant\theta\leqslant\lambda$. We will prove that $Y_\theta=Y\cap\theta$ is an independent subset of θ.

If $A\in\mathrm{Old}_\theta$, then $\mathrm{type}^V(A)=\theta$ and in V there exists an isomorphism $f:\theta\to A$. If $\theta<\lambda$, then $f[\kappa]\subset\theta<\lambda$ and if $\theta=\lambda$ then $\kappa<\lambda$ implies $f[\kappa]\subset f(\kappa)<\lambda$. So, $f[\kappa]$ is a bounded subset of λ and there exists $\delta<\rho$ such that $f[\kappa]\subset\alpha_\delta$. Clearly, the set $f[\kappa]$ is of size κ in V so $f[\kappa]\in([\alpha_\delta]^\kappa)^V$ and consequently there exists $\xi_0<\kappa$ such that $f[\kappa]=A_{\xi_0}$. Now, $\alpha_{\xi_0}\in f[\kappa]\cap Y_\theta$ and $\beta_{\xi_0}\in f[\kappa]\setminus Y_\theta$, which implies $A\cap Y_\theta\neq\emptyset$ and $A\setminus Y_\theta\neq\emptyset$.

(c) Let $|(2^\lambda)^V|^{V[G]}=|\lambda|^{V[G]}=\kappa$. In V, for $\mu<2^\lambda$ we have $\mu^\kappa\leqslant 2^{\lambda\kappa}=2^\lambda$ (since $\kappa\leqslant\lambda$) and we apply (b). □

Corollary 1. (*GCH*) *If in some extension* $V_{\mathsf{B}}[G]$ *a cardinal* λ *is collapsed to* κ, *then each cardinal* θ *satisfying* $\kappa\leqslant\theta\leqslant\lambda$ *obtains an independent subset in* $V_{\mathsf{B}}[G]$ *and consequently the algebra* B *is* θ*-independent for all such* θ.

Proof. Under the assumptions, for each $\mu<\lambda$ there holds $\mu^\kappa\leqslant\max\{\kappa^\kappa,\mu^\mu\}=\max\{\kappa^+,\mu^+\}\leqslant\lambda$ and we apply (b) of the previous theorem. □

Problem 1. Is Corollary 1 a theorem of ZFC?

Example 1 (Independence of the algebras of Bukovský and Namba). Let $\kappa\geqslant\aleph_2$ be a regular cardinal such that $2^{<\kappa}<2^\kappa,\aleph_\kappa$ and that $\mu^\omega<\kappa$, for all $\mu<\kappa$. Let $\mathsf{B}=\mathrm{r.o.}(\mathrm{Nm}(\kappa))$ or $\mathsf{B}=\mathrm{r.o.}(\mathrm{Pf}(\kappa))$, where $\mathrm{Nm}(\kappa)$ is the generalized Namba forcing and $\mathrm{Pf}(\kappa)$ the generalized perfect forcing (see [5]). Since by Theorem 3.5 of [2] the condition $2^{<\kappa}<2^\kappa,\aleph_\kappa$ implies the existence of a 2^κ-sized mad family on κ, using Theorem 14 of [11] we conclude that if in a generic extension $V_{\mathsf{B}}[G]$ the cardinal κ is collapsed to κ_0, then each cardinal θ satisfying $\kappa_0\leqslant\theta\leqslant 2^\kappa$ is collapsed to κ_0 too and $V_{\mathsf{B}}[G]$ is a $|\theta|=\kappa_0$-minimal extension. Now, since $\mu<2^\kappa$ implies $\mu^{\kappa_0}\leqslant 2^\kappa$, using Theorem 9(b) we conclude that B is θ-independent for all such θ. We note that if $\kappa=\aleph_2$ or if $0^\sharp$ does not exist, then $\kappa_0=\aleph_1^V$ (see [11]).

Theorem 10. *If* $\kappa\geqslant\omega$ *and* $\lambda\geqslant 2$ *are cardinals, then the algebra* $\mathsf{B}=\mathrm{Col}(\kappa,\lambda)$ *is strongly* θ*-independent for each cardinal* $\theta\in[\mathrm{cf}(\kappa),\lambda^{<\kappa}]=[\mathfrak{h}_2(\mathsf{B}),\pi(\mathsf{B})]$.

Proof. We distinguish the cases κ is regular and κ is singular and firstly prove two auxiliary claims

Claim 1. *If κ is a regular cardinal and $\lambda \geqslant \kappa$, then for each cardinal μ satisfying $\kappa \leqslant \mu \leqslant \lambda$ the algebra* $\mathrm{Col}(\kappa, \lambda)$ *is strongly μ-independent.*

Proof of Claim 1. Let G be an arbitrary ${}^{<\kappa}\lambda$-generic filter. Then $f_G = \bigcup G : \kappa \to \lambda$ and we will show that the set

$$Y = \{\zeta \in \mu \cap f_G[\kappa]:\ \min f_G^{-1}[\{\zeta\}] \in \mathrm{Even}\}$$

(where Even is the class of even ordinals) is an independent subset of μ. Let $A \in ([\mu]^\mu)^V$. Working in V we prove that the set

$$D_A = \{\varphi \in {}^{<\kappa}\lambda:\ \exists \zeta \in A\ \exists \xi \in \kappa \cap \mathrm{Even}\ \varphi(\xi) = \zeta \notin \varphi[\xi]\}$$

is dense in ${}^{<\kappa}\lambda$. Let $\psi \in {}^{<\kappa}\lambda$ be arbitrary and let $\mathrm{dom}\,\psi = \alpha$. Clearly $\psi[\psi^{-1}[\mu]] \subset \mu$ and since $\alpha < \kappa$, we have

$$|\psi[\psi^{-1}[\mu]]| \leqslant |\psi^{-1}[\mu]| \leqslant |\alpha| < \kappa \leqslant \mu.$$

Now, since $|A| = \mu$, we can choose $\zeta \in A \setminus \psi[\psi^{-1}[\mu]]$. Also, we choose $\xi \in$ Even $\cap \kappa \setminus \alpha$ and $\zeta' \in \mu \setminus \{\zeta\}$ and define $\varphi : \xi + 1 \to \lambda$ by

$$\varphi(\beta) = \begin{cases} \psi(\beta) & \text{if } \beta \in \mathrm{dom}\,\psi, \\ \zeta' & \text{if } \beta \in \xi \setminus \mathrm{dom}\,\psi, \\ \zeta & \text{if } \beta = \xi. \end{cases}$$

Clearly $\varphi \leqslant \psi$ and for the proof that $\varphi \in D_A$ it remains to be shown $\zeta \notin \varphi[\xi]$. For $\gamma \in \xi$, if $\gamma \notin \mathrm{dom}\,\psi$ then $\varphi(\gamma) = \zeta' \neq \zeta$. Otherwise, if $\gamma \in \mathrm{dom}\,\psi$, then $\varphi(\gamma) = \psi(\gamma)$ and we have two possibilities. Firstly, if $\psi(\gamma) \notin \mu$, then $\varphi(\gamma) \neq \zeta$ since $\zeta \in \mu$. Secondly, if $\psi(\gamma) \in \mu$, then $\gamma \in \psi^{-1}[\mu]$ thus $\varphi(\gamma) \in \psi[\psi^{-1}[\mu]]$ so, by choice of ζ, we have $\varphi(\gamma) \neq \zeta$. The set D_A is dense.

Let $\varphi \in G \cap D_A$, $\zeta \in A$, $\xi \in \kappa \cap \mathrm{Even}$, $\varphi(\xi) = \zeta \notin \varphi[\xi]$. Since $\varphi \in G$ we have $\varphi \subset f_G$ so $f_G(\xi) = \zeta \notin f_G[\xi]$, and consequently $\min f_G^{-1}[\{\zeta\}] = \xi \in \mathrm{Even}$. Thus $\zeta \in A \cap Y$ and $A \cap Y \neq \emptyset$. The proof of $A \setminus Y \neq \emptyset$ is analogous and Y is an independent subset of μ.

Thus, in each generic extension by ${}^{<\kappa}\lambda$, or equivalently by $\mathrm{Col}(\kappa, \lambda)$, the cardinal μ obtains an independent set, so, by Theorem 3 the algebra $\mathrm{Col}(\kappa, \lambda)$ is strongly μ-independent and Claim 1 is proved.

Claim 2. *If κ is a singular cardinal and $\lambda \geqslant 2$, then in each generic extension by* $\mathrm{Col}(\kappa, \lambda)$ *the cardinal $\lambda^{<\kappa}$ is collapsed to* $\mathrm{cf}(\kappa)$.

Proof of Claim 2. In V, let $\mathrm{cf}(\kappa) = \rho$ and let $\langle \kappa_\xi : \xi < \rho \rangle$ be an increasing sequence of cardinals less than κ, unbounded in κ. We prove that $|(\lambda^{\kappa_\xi})^V|^{V[G]} = \rho$, for each $\xi < \rho$. In V let the bijections $f_{\xi,\zeta} : \kappa_\xi \to [\kappa_\zeta, \kappa_\zeta + \kappa_\xi), \zeta \in [\xi, \rho)$, be defined by $f_{\xi,\zeta}(\alpha) = \kappa_\zeta + \alpha$ (here $+$ denotes the ordinal addition). If G is a $\mathrm{Col}(\kappa, \lambda)$-generic filter over V and

$f_G = \bigcup G : \kappa \to \lambda$, we prove that

$$({}^{\kappa_\xi}\lambda)^V \subset \{f_G \circ f_{\xi,\zeta}:\ \zeta \in [\xi, \rho)\}.$$

If $F \in ({}^{\kappa_\xi}\lambda)^V$ then it is easy to show that the set $D_F = \{\varphi \in ({}^{<\kappa}\lambda)^V:\ \exists \zeta \geqslant \xi(\kappa_\zeta + \kappa_\xi \subset \mathrm{dom}\varphi \wedge \varphi \circ f_{\xi,\zeta} = F)\}$ is dense in $({}^{<\kappa}\lambda)^V$. So, if $\varphi \in G \cap D_F$ then $\varphi \circ f_{\xi,\zeta} = F$ for a $\zeta \geqslant \xi$, and $f_G \circ f_{\xi,\zeta} = F \in \{f_G \circ f_{\xi,\zeta}:\ \zeta \in [\xi, \rho)\}$.

Thus, in $V[G]$ the sets $({}^{\kappa_\xi}\lambda)^V$ are of size ρ and $(\lambda^{<\kappa})^V$ is a supremum of ρ many ordinals of cardinality ρ, which implies $|(\lambda^{<\kappa})^V|^{V[G]} = \rho$. Claim 2 is proved.

Now, if κ is a regular cardinal, then the algebras $\mathrm{Col}(\kappa, \lambda)$ and $\mathrm{Col}(\kappa, \lambda^{<\kappa})$ are isomorphic (see [1, p. 342]). In V, clearly, $\kappa \leqslant \lambda^{<\kappa}$ and we apply Claim 1.

If κ is a singular cardinal, then by Claim 2 we have $|(\lambda^{<\kappa})^V|^{V[G]} = \mathrm{cf}^V(\kappa) = \rho < \kappa$ and in order to apply Theorem 9(b) we prove that in V, for each $\mu < \lambda^{<\kappa}$ there holds $\mu^\rho \leqslant \lambda^{<\kappa}$. So, if $\mu < \lambda^{<\kappa}$, then $\mu \leqslant \lambda^\nu$, for some cardinal $\nu < \kappa$, hence $\mu^\rho = \lambda^{\nu\rho} \leqslant \lambda^{<\kappa}$, and (b) of Theorem 9 can be applied. □

5. Independence at singular cardinals

Theorem 11. *Let* B *be a complete Boolean algebra and* κ *a singular cardinal. If* B *is* (*strongly*) $\mathrm{cf}(\kappa)$-*independent*, *it is* (*strongly*) κ-*independent too*.

Proof. Let $\mathrm{cf}^V(\kappa) = \rho$. Working in V we choose an increasing unbounded sequence $\langle \xi_\alpha:\ \alpha \in \rho \rangle \in {}^\rho\kappa$ and using recursion define a sequence of cardinals $\langle \kappa_\alpha:\ \alpha < \rho \rangle$ by: $\kappa_0 = 0$; $\kappa_{\alpha+1} = \min\{\lambda \in \mathrm{Card}:\ \lambda > \max\{\kappa_\alpha, \xi_\alpha\}\}$ and $\kappa_\gamma = \sup\{\kappa_\alpha:\ \alpha < \gamma\}$, for a limit $\gamma < \rho$. It is easy to show that $\kappa_\alpha < \kappa$ for all $\alpha < \rho$ and that this sequence is increasing, unbounded in κ and continuous. Consequently, $\kappa = \bigcup_{\alpha < \rho} [\kappa_\alpha, \kappa_{\alpha+1})$ is a partition of κ.

Let $V[G]$ be a generic extension containing an independent set $X \subset \rho$. We will prove that $Y = \bigcup_{\alpha \in X} [\kappa_\alpha, \kappa_{\alpha+1})$ is an independent subset of κ.

Suppose $B \subset Y$ for some $B \in \mathrm{Old}_\kappa$. Since B is an unbounded subset of κ, the set $A = \{\alpha \in \rho:\ B \cap [\kappa_\alpha, \kappa_{\alpha+1}) \neq \emptyset\}$ is an unbounded subset of ρ and, clearly, belongs to V. So, $A \in \mathrm{Old}_\rho$ and $A \subset X$, which is impossible by the independence of X. Thus $B \setminus Y \neq \emptyset$ and analogously $B \cap Y \neq \emptyset$, for each $B \in \mathrm{Old}_\kappa$, so Y is an independent subset of κ and the algebra B is κ-independent by Theorem 2. □

Example 2 (The converse of the previous theorem does not hold). The algebra Col $(\aleph_1, \aleph_{\omega+1})$ is strongly $\aleph_\omega$-independent (Theorem 10) but $\aleph_0$-dependent, since it is $(\aleph_0, 2)$-distributive.

Theorem 12. *In* V, *let* κ *be a singular cardinal and* B *a complete Boolean algebra and let in each generic extension* $V[G]$ *the following conditions hold*:

(i) *The set* D *of all* $\lambda \in \kappa \cap \mathrm{Card}^V$ *such that each subset of* λ *is dependent*, *is unbounded in* κ.

(ii) *Each* $Y \subset (2^{<\kappa})^V$ *of size* $\mathrm{cf}^{V[G]}(\kappa)$ *has a subset* $A \in V$ *such that* $|A|^{V[G]} = \mathrm{cf}^{V[G]}(\kappa)$.

Then the algebra B *is* κ-*dependent*.

Proof. Let $V[G]$ be a generic extension and $V[G] \ni X \subset \kappa$. Let $\mathrm{cf}^{V[G]}(\kappa) = \rho$ and let $f : \rho \to \kappa$ be an increasing cofinal mapping belonging to $V[G]$. In $V[G]$ we define the sequence $\langle \lambda_\alpha \colon \alpha < \rho \rangle$ of elements of D by $\lambda_\alpha = \min(D \setminus (\bigcup_{\beta<\alpha} \lambda_\beta \cup f(\alpha)) + 1), \alpha < \rho$. Clearly, the sequence is increasing and unbounded in κ. Now, using (i), for each $\alpha < \rho$ we choose an $A_\alpha \in ([\lambda_\alpha]^{\lambda_\alpha})^V$ such that $A_\alpha \subset \lambda_\alpha \cap X$ or $A_\alpha \subset \lambda_\alpha \setminus X$. Since each A_α is unbounded in λ_α and since $\alpha < \beta$ implies $\lambda_\alpha < \lambda_\beta$, the set $\{A_\alpha \colon \alpha < \rho\}$, belonging to $V[G]$, is of size ρ. Obviously $\{A_\alpha \colon \alpha < \rho\} \subset S = (\bigcup_{\lambda \in \kappa \cap \mathrm{Card}} [\lambda]^\lambda)^V$ and $|S|^V = (2^{<\kappa})^V$.

If the set $\mathscr{Y} = \{A_\alpha \colon \alpha < \rho \wedge A_\alpha \subset \lambda_\alpha \cap X\}$ is of size ρ, then $\mathscr{Y} \subset S$ and using (ii) we easily show that there exists a subset $\mathscr{A} = \{A_\alpha \colon \alpha \in I\} \subset \mathscr{Y}$ belonging to V such that $|\mathscr{A}|^{V[G]} = \rho$. So, the set $A = \bigcup_{\alpha \in I} A_\alpha \subset X$ belongs to V too. Clearly I is an unbounded subset of ρ, hence for each $\lambda \in D$ we have $|A|^V \geqslant \lambda$, and consequently $|A|^V = \kappa$.

Otherwise, if $|\mathscr{Y}|^{V[G]} < \rho$, then the set $\mathscr{Z} = \{A_\alpha \colon \alpha < \rho \wedge A_\alpha \subset \lambda_\alpha \setminus X\}$ is of cardinality ρ and, proceeding as above, we obtain a set $A \subset \kappa \setminus X$ such that $A \in V$ and $|A|^V = \kappa$. □

We note that the assumptions of the previous theorem imply $1 \Vdash \mathrm{cf}(\check{\kappa}) = \mathrm{cf}^V(\kappa)\check{}$ and B is $\mathrm{cf}^V(\kappa)$-supported.

Example 3. (Condition (ii) in the previous theorem cannot be replaced by the weaker condition (ii′): In each generic extension $V[G]$ each $Y \subset \mathrm{cf}^{V[G]}(\kappa)$ of size $\mathrm{cf}^{V[G]}(\kappa)$ has a subset $A \in V$ of the same size). Let the GCH holds in V, let B be the Boolean completion of the Namba forcing, $\mathrm{Nm}(\omega_2)$, and $\kappa = \aleph_{\omega_2}$. Since $\pi(\mathsf{B}) = \aleph_3$, the algebra B is λ-dependent for all regular $\lambda < \aleph_{\omega_2}$ bigger than $\aleph_3$ (Theorem 5) so condition (i) is satisfied. Condition (ii′) is also satisfied, since $1 \Vdash \mathrm{cf}(\check{\kappa}) = \check{\omega}$ and the algebra B is $(\omega, 2)$-distributive, so forcing by B does not produce new subsets of ω. But, since $\aleph_2 = 2^{\aleph_1}$ is collapsed to $\aleph_1^V$, by Theorem 9(c) the algebra B is $\aleph_2$-independent and, by Theorem 11, B is $\aleph_{\omega_2} = \kappa$-independent.

Example 4 (B is $\aleph_n$-independent for each $n > 0$ but $\aleph_\omega$-dependent). Let in V the GCH holds and let $\mathsf{B} = \prod_{n>0} \mathrm{Col}(\aleph_n, 2)$. Then like in the proof of Theorem 8 we conclude B is $\aleph_n$-independent for all $n > 0$. But B is $\aleph_\omega$-dependent, since each generic extension $V_B[G]$ is equal to a generic extension $V_{\mathrm{Col}(\aleph_n,2)}[H]$ which, clearly, satisfies conditions (i) and (ii) of the previous theorem.

Theorem 13. *Suppose κ is a singular cardinal of cofinality ρ, the algebra B is ρ-supported and the set $D = \{\lambda \in \mathrm{Card} \cap \kappa \colon \mathsf{B}$ is λ-dependent$\}$ is unbounded in κ. Then each of the conditions given below implies B is* κ-dependent.
(a) $\rho < \mathfrak{h}(\mathsf{B})$;
(b) $\rho \geqslant \mathfrak{c}(\mathsf{B})$;
(c) $0^\sharp$ *does not exist in V and forcing by B preserves* $(\rho + \aleph_1)^+$.

Proof. Firstly we note that, since the algebra B is ρ-supported, ρ is a regular cardinal in each generic extension $V[G]$, so $\mathrm{cf}^{V[G]}(\kappa) = \mathrm{cf}^{V[G]}(\rho) = \rho$. In order to apply Theorem 12 we show that each extension $V[G]$ satisfies conditions (i) and (ii). Clearly, since the set D is unbounded in κ, condition (i) holds. For the proof of (ii) we assume $Y \in V[G]$ is a subset of $B=(2^{<\kappa})^V$ of size ρ.

If $\rho<\mathfrak{h}(\mathsf{B})$ then $Y \in V$, by the ρ-distributivity of B.

Let $\rho \geqslant \mathrm{c}(\mathsf{B})$ and let $f:\ \rho \to Y$ be a bijection belonging to $V[G]$. Since B is ρ^+-cc applying Lemma 6.8 of [9] we obtain $F \in V$, where $F:\rho \to P^V(B)$, such that $f(\alpha)\in F(\alpha)$ and $|F(\alpha)|^V \leqslant \rho$ for every $\alpha<\rho$. Then $Y \subset \bigcup \mathrm{ran}(F)=C \in V$ and $|C|^V \leqslant \sum_{\alpha<\rho}|F(\alpha)|^V=\rho$. Clearly, $Y \subset C$ implies $|C|^V=\rho$ hence in V there is a bijection $g:\rho \to C$. Since $g^{-1}[Y]$ is an unbounded subset of ρ and the algebra B is ρ-supported, there exists $A \in ([\rho]^\rho)^V$ such that $A \subset g^{-1}[Y]$. Now $g[A]\in V$ is a subset of Y of size ρ required in (ii).

Let condition (c) hold. Firstly, we suppose $\rho>\omega$. Then, in $V[G], Y$ is an uncountable set of ordinals so, by Jensen's Covering Lemma, there exists $C \in L^{V[G]}=L^V$ such that $Y \subset C$ and $|C|^{V[G]}=\rho$. Since $\rho^+ \in \mathrm{Card}^{V[G]}$ we have $|C|^V=\rho$ and consequently there is a bijection $g:\rho \to C$ belonging to V. Now, as above we obtain $A \in ([\rho]^\rho)^V$ such that $A \subset g^{-1}[Y]$ and $g[A]$ is an old subset of Y of size ρ. Secondly, let $\rho=\omega$. Then $\aleph_1^{V[G]}=\aleph_1^V$, since the collapse of $\aleph_1$ would produce new subsets of ω and then, by Fact 1(d), the algebra B would be ω-unsupported. Now, by Jensen's Covering Lemma, there is $C \in L^{V[G]}=L^V$ such that $Y \subset C$ and $|C|^{V[G]}=\aleph_1$. Since $\aleph_2$ is preserved in $V[G]$, we have $|C|^V=\aleph_1$ and, consequently, in V there exists a bijection $f:\omega_1 \to C$. Since $\aleph_1$ is preserved in $V[G]$ there is $\xi<\omega_1$ such that $f^{-1}[Y]\subset \xi$. Using the assumption B is ω-supported we easily find a countable set $A \in V$ such that $A \subset Y$. $\square$

Under the assumptions of the previous theorem we have $\mathrm{cf}^{V[G]}(\kappa)=\rho$ so the conditions $\rho<\mathfrak{h}^V(\mathsf{B})$ and $1 \Vdash \mathrm{cf}(\check{\kappa})<\mathfrak{h}^V(\mathsf{B})\check{}$ are equivalent and the conditions $\rho \geqslant \mathrm{c}^V(\mathsf{B})$ and $1 \Vdash \mathrm{cf}(\check{\kappa})\geqslant \mathrm{c}^V(\mathsf{B})\check{}$ are equivalent.

Remark 4. In Theorem 5 we proved that $\mathrm{cf}\,(\kappa)>\pi(\mathsf{B})$ implies B is κ-dependent. Now we give a short proof for a singular κ: by Theorem 6, B is λ-dependent for each regular cardinal λ satisfying $\pi(\mathsf{B})<\lambda<\kappa$ and, since $\mathrm{cf}(\kappa)>\pi(\mathsf{B})$ implies $\mathrm{cf}(\kappa)\geqslant \mathrm{c}(\mathsf{B})$, we apply Theorem 13.

Example 5 (Independence of $\aleph_\omega$-independence of $\mathrm{Col}(\aleph_1,\aleph_2)$). Using Theorems 10, 11 and 13 it is easy to check that the algebra $\mathrm{Col}(\aleph_1,\aleph_2)$ is $\aleph_{\omega_1}$-independent, $\aleph_{\omega_2}$-independent and that it is $\aleph_\omega$-dependent if and only if $\mathfrak{c}<\aleph_\omega$.

Using (c) of Theorem 13 we easily prove

Corollary 2. ($0^\sharp \notin V$) *Let* B *be a cardinal preserving c.B.a. and* $\kappa > \pi(\mathsf{B})$ *a singular cardinal. Then, if* B *is* $\mathrm{cf}(\kappa)$*-supported, it is* κ*-dependent.*

Assuming $0^\sharp \notin V, \kappa>\pi(\mathsf{B})$ and $\mathrm{cf}(\kappa)=\rho<\kappa$, we list the situations which are not covered by the previous theorems and ask some related questions.

1. B is ρ-unsupported, but ρ-dependent. Question: Is the Boolean completion of Sacks' forcing $\aleph_\omega$-dependent, if $\mathfrak{c}<\aleph_\omega$?

2. B is $\rho = \omega$-supported, $\mathfrak{h}(\mathsf{B}) = \omega$ and $\aleph_2$ is collapsed (then, clearly, $\mathfrak{h}_2(\mathsf{B}) = \aleph_1$ is preserved). Question: Is the Boolean completion of the Namba forcing, $\mathrm{Nm}(\omega_2)$, $\aleph_\omega$-dependent, if $2^{\aleph_2} < \aleph_\omega$? (We note that, according to Example 1, $2^{\aleph_1} < \aleph_\omega < 2^{\aleph_2}$ implies $\aleph_\omega$-independence of r.o.($\mathrm{Nm}(\omega_2)$).)

3. B is ρ-supported, $\rho > \omega$ and ρ^+ is collapsed in some extension. We do not know whether such a situation is consistent at all (see Problem 1).

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