Identification in $\mathbb{Z}^2$ using Euclidean balls

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The concept of identifying codes was introduced by Karpovsky, Chakrabarty and Levitin. These codes find their application, for example, in sensor networks. The network is modelled by a graph. In this paper, the goal is to find good identifying codes in a natural setting, that is, in a graph $G = (V, E)$ where $V = \mathbb{Z}^2$ is the set of vertices and each vertex (sensor) can check its neighbours within Euclidean distance $r$. We also consider a graph closely connected to a well-studied king grid, which provides optimal identifying codes for $E_{\sqrt{5}}$ and $E_{\sqrt{13}}$.

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1. Introduction

Let $G = (V, E)$ be a simple connected and undirected graph with $V$ as the set of vertices and $E$ as the set of edges. A nonempty subset of $V$ is called a code, and its elements are called codewords. Let $u$ and $v$ be vertices of $V$. Then we say that $u$ covers $v$ if the vertices $u$ and $v$ are adjacent, i.e. there exists an edge between the vertices. The ball centered at $u$ is defined as

$$B(G; u) = \{u\} \cup \{v \in V \mid u \text{ covers } v\}.$$ 

The ball $B(G; u)$ can also be written in short as $B(u)$ if the underlying graph $G$ is known from the context. For a subset $U \subseteq V$, we denote

$$B(U) = B(G; U) = \bigcup_{u \in U} B(G; u).$$

If $U = \{u_1, u_2, \ldots, u_k\}$, then we can also write $B(G; U) = B(G; u_1, u_2, \ldots, u_k) = B(u_1, u_2, \ldots, u_k)$.

Let $C \subseteq V$ be a code and $X$ be a subset of $V$. The size of the set $X$ is denoted by $|X|$. The $I$-set of $X$ with respect to the code $C$ is

$$I(X) = I(C; X) = I(G, C; X) = B(G; X \cap C).$$

Let also $Y$ be a subset of $V$. The symmetric difference of $X$ and $Y$ is defined as $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$.

Definition 1.1. Let $\ell$ be a positive integer. A code $C \subseteq V$ is said to be $\ell$-set-identifying in $G$ if for all $X, Y \subseteq V$ such that $|X| \leq \ell$, $|Y| \leq \ell$ and $X \neq Y$ we have

$$I(X) \neq I(Y).$$

If $\ell = 1$, then we simply say that $C$ is identifying.

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In other words, a code $C \subseteq V$ is $\ell$-set-identifying in $G$ if and only if for all $X, Y \subseteq V$ such that $|X| \leq \ell, |Y| \leq \ell$ and $X \neq Y$ we have

$$I(X) \cup I(Y) \neq \emptyset.$$ 

The $\ell$-set-identifying codes defined above are called $(1, \leq \ell)$-identifying codes in the terminology of, for example, [12].

Identifying codes were introduced in [15] for finding malfunctioning processors in a multiprocessor system. The topic forms an active field of research; see the numerous articles on the web-page [17] with various aspects considered; for a recent development we refer to [18,19]. Identifying codes [18,19] also finds applications in sensor networks. The network is modelled by a graph $G = (V, E)$. The sensors correspond to a code $C(\subseteq V)$ and $B(u)$ is the set of vertices which the sensor $u$ can check. The idea is that we determine the exact locations of objects (like a faulty processor) $X \subseteq V$ using only the alarm signals (that is, the set $I(C; X)$) obtained from the sensors of $C$—this can be done provided that $C$ is $\ell$-set-identifying and $|X| \leq \ell$.

Assume now that the vertex set $V$ is equal to $\mathbb{Z}^2$. Let then $t$ be a positive integer and $u = (x, y)$ be a vertex in $\mathbb{Z}^2$. The graph $\mathcal{S}_{t}$ with the ball

$$B(\mathcal{S}_{t}; u) = \{(x', y') \in \mathbb{Z}^2 \mid |x - x'| + |y - y'| \leq t\}$$

is called the square grid. The graph $\mathcal{K}_{t}$ with the ball

$$B(\mathcal{K}_{t}; u) = \{(x', y') \in \mathbb{Z}^2 \mid |x - x'| \leq t, |y - y'| \leq t\}$$

is called the king grid. The graphs $\mathcal{S}_{t}$ and $\mathcal{K}_{t}$ are illustrated in Fig. 1. The $\ell$-set-identification in $\mathcal{S}_{t}$ and $\mathcal{K}_{t}$ have been studied, for example, in [4,10,13,5,8].

Now let $r$ be a positive real number. Let again $V = \mathbb{Z}^2$. The graph $\mathcal{E}_{r} = (V, E)$ is defined by the edge set $E$ such that vertices $u$ and $v$ in $\mathbb{Z}^2$ are adjacent if the Euclidean distance of $u$ and $v$ is at most $r$. If $u = (x, y) \in \mathbb{Z}^2$, then the ball

$$B(\mathcal{E}_{r}; u) = \{(x', y') \in \mathbb{Z}^2 \mid (x - x')^2 + (y - y')^2 \leq r^2\}.$$ 

Obviously, $\mathcal{S}_{t} = \mathcal{E}_{1}, \mathcal{K}_{t} = \mathcal{E}_{\sqrt{2}}, \mathcal{S}_{2} = \mathcal{E}_{2}$ and $\mathcal{K}_{2} = \mathcal{E}_{\sqrt{2}}$. The graph $\mathcal{E}_{\sqrt{2}}$ is illustrated in Fig. 1. For larger values of $t$, the shape of the ball $B(u)$ in the graphs $\mathcal{K}_{t}$ and $\mathcal{S}_{t}$ is a square as can be seen in Fig. 1. In this paper, we consider identification in the case when $B(u)$ is a Euclidean ball, which is a natural area for a sensor in $\mathbb{Z}^2$ to check. In other words, the aim is to find good $\ell$-set-identifying codes in $\mathcal{E}_{r}$ for any real number $r \geq 1$. The motivation for considering different balls in $\mathbb{Z}^2$ also comes from [3] and [14, Section 5].

In order to measure codes in $\mathbb{Z}^2$, we define the notion of density of codes. For this, we first define

$$T_n = \{(x, y) \in \mathbb{Z}^2 \mid |x| \leq n, |y| \leq n\},$$

where $n$ is a positive integer. Now the density of a code $C \subseteq \mathbb{Z}^2$ is defined as

$$D(C) = \lim_{n \to \infty} \frac{|C \cap T_n|}{|T_n|}.$$ 

Naturally, we seek identifying codes with density as small as possible. We say that an $\ell$-set-identifying code is optimal, if there does not exist any identifying codes with a lower density.

In the sequel we will need the following result from [4, Proposition 1].

\textbf{Theorem 1.2} ([4]). Let $G = (V, E)$ be a simple connected and undirected graph. Let $u_1, u_2, u_3 \in V$ be three vertices of $G$ and $C$ be an identifying code in $G$. Then the set $H(u_1, u_2, u_3) = (B(u_1) \triangle B(u_2)) \cup (B(u_1) \triangle B(u_3)) \cup (B(u_2) \triangle B(u_3))$ contains at least two codewords.
2. On $\ell$-set-identifying codes with $\ell = 1$

In this section, we study 1-set-identifying codes in two families of graphs. We first start by considering identifying codes in $\mathcal{E}_r$. Then we examine identifying codes in a graph similar to the king grid. The identifying codes in this graph also provide optimal identifying codes for certain graphs $\mathcal{E}_r$.

2.1. Identifying codes in the graphs $\mathcal{E}_r$

In what follows, we construct a 1-set-identifying code for the graph $\mathcal{E}_r$, where $r \geq 1$ is an arbitrary real number, and also provide a lower bound on the density of such codes. For the considerations, we define the horizontal line as $L_i^{(h)} = \{(x',i) \mid x' \in \mathbb{Z}\}$ and the vertical line as $L_i^{(v)} = \{(i,y') \mid y' \in \mathbb{Z}\}$, where $i$ is an integer. We also define the diagonal with slope $-1$ as $D_i^{(d)} = \{(x',y') \in \mathbb{Z}^2 \mid x' + y' = i\}$ and the diagonal with slope $1$ as $D_i^{(1)} = \{(x',y') \in \mathbb{Z}^2 \mid x' - y' = i\}$. If $u$ is a vertex in $\mathbb{Z}^2$ and $X$ is a subset of $\mathbb{Z}^2$, then the sum of $u$ and $X$ is defined as $u + X = \{u + v \mid v \in X\}$. We first present the following technical lemma. The results (ii)-(iv) in the lemma are estimates (not always sharp), which are enough for our purposes in Section 3.

Lemma 2.1. Let $u = (x,y)$ be a vertex in $\mathbb{Z}^2$ and $r \geq 1$ be a real number.

(i) In $B(x,y) \setminus B(x,y - 1)$ there exist $2\lfloor r \rfloor + 1$ vertices, which lie on consecutive vertical lines $L_i^{(v)}$ with $i = x - \lfloor r \rfloor, \ldots, x + \lfloor r \rfloor$.

(ii) In $B(x,y) \setminus B(x - 1,y - 1)$ there exist $4\lfloor r/\sqrt{2} \rfloor + 1$ vertices, which lie on consecutive diagonals $D_i^{(d)}$ with $i = x - 2\lfloor r/\sqrt{2} \rfloor, \ldots, x + 2\lfloor r/\sqrt{2} \rfloor$.

(iii) In $B(x,y) \setminus B(x,y - 1), (x + 1,y)$ there exist $\lfloor r(1 - 1/\sqrt{2}) \rfloor$ vertices, which lie on consecutive vertical lines $L_i^{(v)}$ with $i = x - \lfloor r \rfloor, \ldots, x - \lfloor r \rfloor + \lfloor r(1 - 1/\sqrt{2}) \rfloor - 1$.

(iv) In $B(x,y) \setminus B(x - 1,y - 1), (x + 1,y - 1)$ there exist $2\lfloor r(1/\sqrt{2} - 1/2) \rfloor - 1$ vertices, which lie on consecutive diagonals $D_i^{(1)}$ with $i = x - 2\lfloor r/\sqrt{2} \rfloor, \ldots, x - 2\lfloor r/\sqrt{2} \rfloor + 2\lfloor r(1/\sqrt{2} - 1/2) \rfloor - 2$.

Proof. (i) Moving the center $u = (x,y)$ of a ball to $(x,y - 1)$ means that $u$ covers on $L_i^{(v)} (i = x - \lfloor r \rfloor, \ldots, x + \lfloor r \rfloor)$ exactly one vertex of $\mathbb{Z}^2$ which is not covered by $(x,y - 1)$, since the second coordinate decreases by one. The case (ii) is analogous.

(iii) Suppose $r \geq 4$, otherwise the claim is trivial. Denote $Q_2^{(d)} = \{(-a,b) \in \mathbb{Z}^2 \mid 0 \leq a, 0 \leq b \leq a\}$. It is easy to check that the vertices of $u + Q_2^{(d)}$ which are covered by $(x + 1,y)$ belong to $B(x,y - 1)$ also. Therefore, in $u + Q_2^{(d)}$, it is enough to consider the vertices, that $u = (x,y)$ covers but $(x,y - 1)$ does not. We obtain the claim using (i) for the consecutive vertical lines $L_i^{(v)}$ for $x - r \leq i \leq x - r/\sqrt{2}$. The case (iv) is again similar (non-trivial for $r \geq 5$).

Notice that analogous results to the previous lemma hold when the considered patterns are rotated by $\pi/2, \pi$ and $3\pi/2$. For example, when the pattern in (i) is rotated anti-clockwise by $\pi/2$, we have that the set $B(x,y) \setminus B(x + 1,y)$ contains vertices on $2\lfloor r \rfloor + 1$ consecutive horizontal lines.

For the construction of the identifying codes in $\mathcal{E}_r$, we first introduce the following sets of vertices

$$C^{(h)} = \{(j,0) \in \mathbb{Z}^2 \mid j \equiv 0 \mod 2\}$$

and

$$C^{(v)} = \{(0,j) \in \mathbb{Z}^2 \mid j \equiv 0 \mod 2\}.$$

Define then a code $C_k$ as follows:

$$C_k = \bigcup_{i \in \mathbb{Z}}((C^{(h)} + (0, i \cdot 2k)) \cup (C^{(h)} + (1, k + i \cdot 2k))) \cup \bigcup_{i \in \mathbb{Z}}((C^{(v)} + (i \cdot 2k, 0)) \cup (C^{(v)} + (k + i \cdot 2k, 1))).$$

where $k \in \mathbb{Z}$ and $k \geq 1$. The following theorem shows that the previous code $C_k$ provides a 1-set-identifying code for the graph $\mathcal{E}_r$.

Theorem 2.2. Let $r \geq 1$ be a real number.

(i) If $r^2 - \lfloor r \rfloor^2 \geq 1$, then the code $C_{2\lfloor r \rfloor + 1}$ is identifying in $\mathcal{E}_r$.

(ii) If $r^2 - \lfloor r \rfloor^2 < 1$, then the code $C_{\lfloor r \rfloor}$ is identifying in $\mathcal{E}_r$.

Proof. (i) Let $u = (x,y)$ be a vertex in $\mathbb{Z}^2$. Assume first that $r^2 - \lfloor r \rfloor^2 \geq 1$. This assumption implies that the vertices $(x - \lfloor r \rfloor, y - 1), (x - \lfloor r \rfloor, y + 1), (x + \lfloor r \rfloor, y - 1)$ and $(x + \lfloor r \rfloor, y + 1)$ belong to $B(u)$. Therefore, the set $\{(i,j) \in \mathbb{Z}^2 \mid x - \lfloor r \rfloor \leq i \leq x + \lfloor r \rfloor, y - 1 \leq j \leq y + 1\}$ is a subset of $B(u)$. By the construction of $C_{2\lfloor r \rfloor + 1}$, one of the $2\lfloor r \rfloor + 1$ consecutive vertical lines is such that every other vertex in the line is a codeword. Hence, the ball $B(u)$ contains a codeword. In other words, each vertex in $\mathbb{Z}^2$ is covered by a codeword.
Lemma 2.1

Included the proof.

Assume first that \( x' \geq 2 \) or \( y' \geq 2 \). Let \( y' \geq 2 \) (the other case is analogous). Denote then \( u' = (x, y) \) and \( v' = (x + x', y') \). Using similar arguments as in the proof of Lemma 2.1 (i), we conclude that each vertical line \( L^{(v)}_{k} \) with \( x - \lfloor r \rfloor \leq i \leq x + \lfloor x'/2 \rfloor \) contains two consecutive vertices in \( B(u) \setminus B(v) \). (Recall that \( r^2 - \lfloor r \rfloor^2 \geq 1 \).) Clearly, these same points are also included in \( B(u) \cap B(v) \). By symmetry, we can show that each vertical line \( L^{(v')}_{k} \) with \( x + \lfloor x'/2 \rfloor \leq i \leq x + x' + \lfloor r \rfloor \) contains two consecutive vertices in \( B(v) \setminus B(u) \). We have shown that each vertical line \( L^{(v)}_{k} \) with \( x - \lfloor r \rfloor \leq i \leq x + x' + \lfloor r \rfloor \) contains two consecutive vertices in \( B(u) \triangle B(v) \). Therefore, we conclude that there exists a codeword in \( B(u) \triangle B(v) \).

Assume now that \( x' \leq 1 \) and \( y' \leq 1 \). Then we have the following cases to consider:

1. Assume that \( x' = 0 \) and \( y' = 1 \). Let \( L^{(v)}_{k} \) be a vertical line with \( x - \lfloor r \rfloor \leq k \leq x + \lfloor r \rfloor \). By Lemma 2.1 (i), the set \( L^{(v)}_{k} \cap (B(v) \setminus B(u)) \) is nonempty. Let \( w = (k, y + 1 + a) \in \mathbb{Z}^2 \) be a vertex in \( B(v) \setminus B(u) \). Then, by symmetry, a vertex \( w' = (k, y - a) \in B(u) \setminus B(v) \). Since the Euclidean distance between \( w \) and \( w' \) is equal to \( 2a + 1 \), the parity of the second coordinates of the vertices \( w \) and \( w' \) are different. Therefore, since one of the vertical lines \( L^{(v)}_{k} \) with \( x - \lfloor r \rfloor \leq i \leq x + \lfloor r \rfloor \) is such that every other vertex in the line is a codeword, the symmetric difference \( B(u) \triangle B(v) \) contains a codeword.

2. If \( x' = 1 \) and \( y' = 0 \), then the proof goes exactly like in case (1); just replace the vertical lines by horizontal ones.

3. Assume now that \( x' = 1 \) and \( y' = 1 \). Let \( w = (k, y + 1 + a) \in L^{(v)}_{k} \), where \( x - \lfloor r \rfloor \leq k \leq x \), be a vertex such that \( w \in B(x, y - 1) \setminus B(x, y) \). By symmetry, the vertex \( w' = (k, y - a) \in B(x, y - 1) \setminus B(x, y) \). Since \( k \leq x \), the vertex \( w' \in B(x, y) \setminus B(x, y - 1) \). If \( w \in B(x + 1, y + 1) \setminus B(x, y) \), then the vertical line \( L^{(v)}_{k} \) contains two vertices \( (w, w') \) in \( B(u) \triangle B(v) \). Since the parity of their second coordinates are different, assume that \( w' \notin B(x, y + 1) \setminus B(y, x) \). By symmetry, the vertex \( w' = (k, y + 1 - a) \in B(x, y) \setminus B(x + 1, y + 1) \). Clearly, the parity of the second coordinates of \( w' \) and \( w' \) are different. Analogous arguments also apply, when we are considering the vertical lines \( L^{(v')}_{k} \) with \( x + 1 \leq k \leq x + 1 + \lfloor r \rfloor \). Hence, each line \( L^{(v)}_{k} \) with \( x - \lfloor r \rfloor \leq i \leq x + 1 + \lfloor r \rfloor \) contains two vertices in \( B(u) \triangle B(v) \) such that the parity of the second coordinates of the vertices are different. Thus, there exists a codeword in \( B(u) \triangle B(v) \).

In conclusion, we have shown that \( C_{2\lfloor r \rfloor - 1} \) is an identifying code in \( E_r \), when \( r^2 - \lfloor r \rfloor^2 \geq 1 \).

(ii) Let again \( u = (x, y) \) be a vertex in \( \mathbb{Z}^2 \). Assume then that \( r^2 - \lfloor r \rfloor^2 < 1 \). Define the set \( A = \{ (i, j) \in \mathbb{Z}^2 \mid x - \lfloor r \rfloor \leq i \leq x + \lfloor r \rfloor, y - 1 \leq j \leq y \} \setminus \{(x - \lfloor r \rfloor, y - 1), (x + \lfloor r \rfloor, y - 1)\} \). We then show that the set \( A \) contains a codeword of \( C_{2\lfloor r \rfloor} \).

If a vertical line \( L^{(v)}_{k} \) with \( x - \lfloor r \rfloor + 1 \leq i \leq x + \lfloor r \rfloor - 1 \) is such that every other vertex in the line is a codeword, then we are clearly done. Otherwise, we know that the vertical lines \( L^{(v)}_{x - \lfloor r \rfloor} \) and \( L^{(v)}_{x + \lfloor r \rfloor} \) are such that every other vertex in the lines is a codeword. Hence, by the construction of \( C_{2\lfloor r \rfloor} \), either the vertex \( (x - \lfloor r \rfloor, y) \) or \( (x + \lfloor r \rfloor, y) \) is a codeword. Since \( A \subseteq B(u) \), the word \( u \) is covered by a codeword.

Let \( v = (x + x', y + y') \) be a vertex in \( \mathbb{Z}^2 \) and \( v \neq u \). We need to show that the symmetric difference \( B(u) \setminus B(v) \) contains a codeword (when \( B(u) \cap B(v) \neq \emptyset \)). Without loss of generality, we can assume that \( x \geq 0 \) and \( y \geq 0 \). If \( x' = 0 \) and \( y' = 1 \), then the proof goes exactly as in the cases (1) and (2) of part (i), respectively. Assume that \( x' \neq 0 \) and \( y' \geq 2 \). If now a vertical line \( L^{(v)}_{k} \) with \( x - \lfloor r \rfloor + 1 \leq i \leq x + \lfloor r \rfloor - 1 \) is such that every other vertex in the line is a codeword, then we are done. Otherwise, either the vertex \( (x - \lfloor r \rfloor, y) \) or \( (x + \lfloor r \rfloor, y) \) in \( B(v) \setminus B(u) \) is a codeword. Therefore, \( I(u) \setminus I(v) \neq \emptyset \). Similar arguments also apply when \( x' \geq 2 \) and \( y' = 0 \). If \( x' = 1 \) and \( y' = 1 \), then the proof goes exactly as in the previous case (3), but we just consider the \( r^2 \) consecutive vertical lines \( L^{(v)}_{k} \) with \( x - \lfloor r \rfloor + 1 \leq i \leq x + \lfloor r \rfloor \). If \( x' \geq 1 \) and \( y' \geq 2 \), then the proof is similar to the third paragraph of the proof of the part (i), but we just consider the vertical lines \( L^{(v)}_{k} \) with \( x - \lfloor r \rfloor + 1 \leq i \leq x + x' + \lfloor r \rfloor - 1 \). The case with \( x' \geq 2 \) and \( y' \geq 1 \) goes the same way as the previous one. In conclusion, we have shown that \( C_{2\lfloor r \rfloor - 1} \) is an identifying code in \( E_r \), when \( r^2 - \lfloor r \rfloor^2 \geq 1 \).

It is easy to conclude that the density of the code \( C_r \) satisfies \( D(C_r) \leq 1/k \). Therefore, by the previous theorem, we have shown that for any real number \( r \geq 1 \) there exists an identifying code \( C \) such that the density

\[
D(C) \leq \frac{1}{2\lfloor r \rfloor}.
\]

For small values of \( r \), there exist identifying codes with smaller densities. Indeed, since \( E_{\sqrt{2}} = \mathcal{K}_1 \) and \( E_{2\sqrt{2}} = \mathcal{K}_2 \), we have optimal identifying codes in \( E_{\sqrt{2}} \) and \( E_{2\sqrt{2}} \) with densities 2/9 and 1/8, respectively (see [5]). Recall that \( \mathcal{E}_1 = \mathcal{D}_1 \) and \( \mathcal{E}_2 = \mathcal{D}_2 \). It has been shown in [6] that there exists an identifying code with density 7/20 in \( \mathcal{D}_1 \). Moreover, it was proved in [2] that there are no identifying codes in \( \mathcal{D}_1 \) with smaller density. There exists an identifying code in \( \mathcal{D}_2 \) with density 5/29 (see [13]). In [4], it has been shown that there does not exist an identifying code in \( \mathcal{D}_2 \) with density smaller than 3/20.

Consider then a lower bound on the density of an identifying code in \( E_r \). In order to provide a lower bound, we first need to present an auxiliary theorem. This theorem is a rephrased version of [11, Theorem 5]. For completeness, we have also included the proof.
Theorem 2.3. Assume that $C \subseteq \mathbb{Z}^2$ is a code. Let $S = \{s_1, s_2, \ldots, s_k\}$ be a subset containing $k$ different points of $\mathbb{Z}^2$. For each $i = 1, 2, \ldots, k$ we choose a real number $w_i \geq 0$, which we call the weight of $s_i$ and denote by $w(s_i)$. For all subsets $A$ of $S$ we define

$$w(A) = \sum_{a \in A} w(a).$$

If for all $v \in \mathbb{Z}^2$ we have $w((v + C) \cap S) \geq 1$, then the density of $C$ satisfies

$$D(C) \geq \frac{1}{w_1 + w_2 + \cdots + w_k}.$$

Proof. Since $S$ is finite, we can choose a constant $h$ such that $S \subseteq T_h$. Consider then the sum $\sum_{v \in T_{n-h}} w((v + C) \cap S)$, where $n > h$. Now we have

$$|T_{n-h}| \leq \sum_{v \in T_{n-h}} w((v + C) \cap S) \leq \sum_{i=1}^k w_i f_i(n),$$

where $f_i(n)$ denotes the number of pairs $(c, v)$ such that $c \in C$, $v \in T_{n-h}$ and $s_i = v + c$. Since $v \in T_{n-h}$ and $s_i \in T_h$, we know that $c = s_i - v \in T_n$. Hence, there are at most $|C \cap T_n|$ choices for $c$. Furthermore, for every $c$ there is at most one possible choice for $v \in T_{n-h}$ such that $s_i = v + c$. Therefore, $f_i(n) \leq |C \cap T_n|$.

Combining this result with the Eq. (1), we have

$$|T_{n-h}| \leq (w_1 + w_2 + \cdots + w_k)|C \cap T_n|.$$

Thus,

$$\frac{|C \cap T_n|}{|T_n|} \geq \frac{|T_{n-h}|}{|T_n|} \cdot \frac{1}{w_1 + w_2 + \cdots + w_k}.$$

The claim immediately follows from this equation, since $|T_{n-h}|/|T_n| \to 1$ when $n \to \infty$. \qed

In what follows, we prove a lower bound on the density of an identifying code in $\mathcal{E}_r$. The lower bound is actually attained for some graphs $\mathcal{E}_r$ (see Theorem 2.7).

Theorem 2.4. If $C \subseteq \mathbb{Z}^2$ is an identifying code in $\mathcal{E}_r$, then the density satisfies

$$D(C) \geq \frac{3}{4[r] + 4|b| + 4[\sqrt{r^2 - (|b| + 1)^2}] + 8},$$

where $b = -1/2 + 1/2 \cdot \sqrt{2r^2 - 1}$.

Proof. Let $C \subseteq \mathbb{Z}^2$ be an identifying code in $\mathcal{E}_r$. Denote $u_1 = (0, 0)$, $u_2 = (-1, 0)$, $u_3 = (0, -1)$ and $u_4 = (-1, -1)$. Then define the set

$$H = (B(u_1) \Delta B(u_2)) \cup (B(u_1) \Delta B(u_3)) \cup (B(u_2) \Delta B(u_3)) \cup (B(u_2) \Delta B(u_4)) \cup (B(u_3) \Delta B(u_4)) \cup (B(u_3) \Delta B(u_2))$$

and $H'$ as the set of vertices that belong to $H$ and are covered by exactly two of the vertices $u_1$, $u_2$, $u_3$ and $u_4$. Notice that if $v \in H \setminus H'$, then $v$ is covered by exactly one or three of the vertices $u_1$, $u_2$, $u_3$ and $u_4$. If a codeword $c \in C$ belongs to $H \setminus H'$, then, by Theorem 1.2, there exist at least three codewords in $H$. On the other hand, if there does not exist any codeword in $H \setminus H'$, then there clearly exist at least two codewords in $H'$.

Using the notations of Theorem 2.3, we choose $S = H$. The weight of a vertex $s \in H$ is now defined as follows: if $s \in H'$, then $w(s) = 1/2$, else $w(s) = 1/3$. By the considerations in the previous paragraph, we conclude that for every $v \in \mathbb{Z}^2$ we have $w((v + C) \cap H) \geq 1$. By Theorem 2.3, we have

$$D(C) \geq \frac{1}{1/2 \cdot |H'| + 1/3 \cdot (|H| - |H'|)} = \frac{3}{|H| + 1/2 \cdot |H'|}.$$

For the lower bound, it is now enough to calculate the number of vertices in $H$ and $H'$.

For the calculations, define the set $Q = \{(x, y) \in \mathbb{Z}^2 \mid x \geq 0, y \geq 0\}$. It is clear that a vertex $u \in Q \cap H$ if and only if $u \in B(0, 0) \setminus B(-1, -1)$ and $u \in Q$. Now, by straightforward computations, we have that the number of vertices in $Q \cap H$ is equal to

$$\sum_{i=0}^{[r]-1} ([\sqrt{r^2 - r^2}] - [\sqrt{r^2 - (i + 1)^2}] - 1) + [\sqrt{r^2 - [r]^2}] + 1 = 2[r] + 1.$$
Consider then the number of vertices in \( H' \). It is easy to see that the circles of radius \( r \) centered at the points \((-1,0)\) and \((0,-1)\) intersect each other in the point \((b,b)\), where \( b = -1/2 + 1/2 \cdot \sqrt{2r^2 - 1} \). Then define the set \( Q_b = \{(x,y) \in \mathbb{Z}^2 \mid 0 \leq x \leq b, y \geq 0\} \). It is clear that a vertex \( u \in Q_b \cap H' \) if and only if \( u \in (B(0, 0) \cup B(0, 1)) \setminus (B(-1, 0) \cup B(-1, -1)) \) and \( u \in Q_b \). Hence, by straightforward computations, we have that the number of vertices in \( Q_b \cap H' \) is equal to

\[
\sum_{i=0}^{[b]} (|\sqrt{r^2 - (i + 1)^2}| - |\sqrt{r^2 - i^2} - 1|) = |\sqrt{r^2 - ([b] + 1)^2}| + [b] - [r] + 1.
\]

Therefore, by symmetry, the number of vertices in \( H' \) is equal to

\[
8(|\sqrt{r^2 - ([b] + 1)^2}| + [b] - [r] + 1).
\]

Thus, we obtain the lower bound on the density

\[
D(C) \geq \frac{3}{4[r] + 4[b] + 4|\sqrt{r^2 - ([b] + 1)^2}| + 8}. \quad \Box
\]

Let us then consider more closely the lower bound given by the previous theorem. As in the theorem, let \( C \subseteq \mathbb{Z}^2 \) be an identifying code in \( \mathcal{E}_t \) and denote \( b = -1/2 + 1/2 \cdot \sqrt{2r^2 - 1} \). Denote further \([b] = k \in \mathbb{Z}\). Since now \( b < k + 1 \), we have that \( r < \sqrt{1/2 \cdot (2k + 3)^2 + 1/2} \). Therefore, we have

\[
|\sqrt{r^2 - ([b] + 1)^2}| \leq \sqrt{(\sqrt{1/2 \cdot (2k + 3)^2 + 1/2})^2 - ([b] + 1)^2} = k + 2.
\]

Hence, we further obtain that \(|\sqrt{r^2 - ([b] + 1)^2}| \leq [b] + 1 \). Thus, the denominator of the lower bound can be estimated as follows:

\[
4[r] + 4[b] + 4|\sqrt{r^2 - ([b] + 1)^2}| + 8 \leq 4[r] + 8[b] + 12 \leq 4(\sqrt{2} + 1)r + 12.
\]

Therefore, we have the following approximation for the lower bound on the density of an identifying code \( C \) in \( \mathcal{E}_t \):

\[
D(C) \geq \frac{3}{4(\sqrt{2} + 1)r + 12} \geq \frac{1}{3, 22r + 4}.
\]

2.2. Identifying codes in the king grids without corners

In this section, we consider 1-set-identification in a graph closely related to the king grid. These considerations provide two optimal identifying codes in \( \mathcal{E}_t \), as is shown in Theorem 2.7. The vertex set \( V \) is again equal to \( \mathbb{Z}^2 \). Let then \( t \) be a positive integer and \( u = (x, y) \) be a vertex in \( \mathbb{Z}^2 \). The edge set \( E \) of the considered graph \( \mathcal{K}_t \) is such that

\[
B(\mathcal{K}_t; u) = B(\mathcal{K}_t; u) \setminus \{(x + t, y + t), (x + t, y - t), (x - t, y + t), (x - t, y - t)\}.
\]

The graph \( \mathcal{K}_t \) is called the king grid without corners. Notice that \( \mathcal{K}_t = \mathcal{S}_t \). As was mentioned in Section 2.1, there exists an optimal identifying code in \( \mathcal{S}_t \) with density \( 7/20 \).

Define a code

\[
C_t = \bigcup_{\alpha \in \mathbb{Z}}\{(2t \cdot i + \alpha, \alpha) \mid \alpha \in \mathbb{Z} \text{ and } \alpha \text{ is even}\}.
\]

The code \( C_t \) is illustrated in Fig. 1 when \( t = 2 \). Clearly, the density \( D(C_t) \) is equal to \( 1/(4t) \). It has been shown in [5] that \( C_t \) is an optimal identifying code in \( \mathcal{S}_t \). The following theorem shows that \( C_t \) is also an identifying code in \( \mathcal{K}_t \) — notice that now the ball in \( \mathcal{K}_t \) is smaller than the one in \( \mathcal{S}_t \)!

In Theorem 2.6, we prove that identifying codes in \( \mathcal{K}_t \) with a lower density do not exist.

**Theorem 2.5.** Let \( t \geq 2 \) be an integer. Then the code \( C_t \) is identifying in \( \mathcal{K}_t \).

**Proof.** Let \( w = (x, y) \) be a vertex in \( \mathbb{Z}^2 \). Then define sets

\[
A_h(w) = \{(i, j) \in \mathbb{Z}^2 \mid x \leq i \leq x + 2t - 1, y \leq j \leq y + 1\}
\]

and

\[
A_v(w) = \{(i, j) \in \mathbb{Z}^2 \mid x \leq i \leq x + 1, y \leq j \leq y + 2t - 1\}.
\]

Let \( i \) be an integer. If \( i \) is even, then the horizontal line \( L_i^{(h)} \) is such that one of the \( 2t \) consecutive vertices in the line is a codeword of \( C_t \). The same also holds for the vertical lines. Thus, the sets \( A_h(w) \) and \( A_v(w) \) both contain at least one codeword.
Let $u = (x_1, x_2)$ and $v = (x_2, x_3)$ be vertices in $\mathbb{Z}^2$. The $I$-set $l(u)$ is nonempty, since the ball $B(u)$ contains the set $A_0(w)$ with a suitable choice of $w$, when $t \geq 2$. In order to prove the claim, we have to show that the symmetric difference $B(u) \triangle B(v)$ always contains a codeword. Assume first that $|x_1 - x_2| \geq 3$ or $|y_1 - y_2| \geq 3$. Then the symmetric difference $B(u) \triangle B(v)$ contains the set $A_0(w)$ or $A_0(w)$. Thus, $l(u) \triangle l(v) \neq \emptyset$.

Assume now that $|x_1 - x_2| \leq 2$ and $|y_1 - y_2| \leq 2$. Then we have the following cases to consider (other cases are analogous):

1. Assume that $v = (x_1 + 1, y_1)$ or $v = (x_1 + 2, y_1)$. Denote $X_1 = \{ (x_1 - t, y_1 - t + 1), (x_1 - t, y_1 - t + 2), \ldots, (x_1 - t, y_1 + t - 1) \}$ and $X_2 = \{ (x_1 + t + 1, y_1 - t + 1), (x_1 + t + 1, y_1 - t + 2), \ldots, (x_1 + t + 1, y_1 + t) \}$. It is easy to see that $X_1, X_2 \subseteq B(u) \triangle B(v)$ and $(x_1 + t + 1, y_1 - 1), (x_1 + t, y_1 + t) \in B(u) \triangle B(v)$. Assume first that $x_1 - t$ is even. Then, by the previous considerations, either $X_1$ contains a codeword or the vertex $(x_1 - t, y_1 - t)$ is a codeword. If $X_1$ contains a codeword, we are done. Otherwise, the vertex $(x_1 - t, y_1 - t)$ is a codeword. Therefore, the construction of $C_r$, the vertex $(x_1 - t + 2t, y_1 - t + 2t) = (x_1 + t, y_1 + t)$ is a codeword. Assume then that $x_1 - t$ is odd. Hence, $x_1 + t + 1$ is clearly even. The proof is now similar to the first case.

2. Assume that $v = (x_1 + 1, y_1 + 1)$. Denote $Y_1 = Y_2 = (0, 1) + X_2$. It is easy to see that $Y_1, Y_2 \subseteq B(u) \triangle B(v)$ and $(x_1 + t + 1, y_1 - t + 1), (x_1 + t, y_1 + t) \in B(u) \triangle B(v)$. Assume first that $x_1 - t$ is even. If $Y_1$ contains a codeword, we are done. Otherwise, the vertex $(x_1 - t, y_1 - t)$ is a codeword. Therefore, the vertex $(x_1 - t + 2t, y_1 - t + 2t) = (x_1 + t, y_1 + t)$ is a codeword. If $x_1 - t$ is odd, then $x_1 + t + 1$ is even and the proof is similar to the first case.

3. Assume that $v = (x_1 + 2, y_1 + 1)$. The proof is now analogous to the case (2).

In conclusion, we have shown that the symmetric difference $l(u) \triangle l(v)$ is always nonempty. Hence, the claim follows. □

The following theorem provides a lower bound on the density of an identifying code in $K'_{r}$.

**Theorem 2.6.** If $C$ is an identifying code in $K'_{r}$, then the density

$$D(C) \geq \frac{1}{4t}.$$ 

**Proof.** Let $C$ be an identifying code in $K'_{r}$. Define the vertices $u_1, u_2, u_3, u_4 \in \mathbb{Z}^2$ and the sets $H, H' \subseteq \mathbb{Z}^2$ as in the proof of Theorem 2.4. Using similar arguments as in the proof of Theorem 2.4, we have

$$D(C) \geq \frac{3}{|H| + 1/2 \cdot |H'|}.$$ 

It is easy to calculate that $|H| = 8t + 4$ and $|H'| = 4(2t - 2)$. Therefore,

$$D(C) \geq \frac{3}{8t + 4 + 1/2 \cdot 4(2t - 2)} = \frac{1}{4t}.$$ □

In conclusion, we have shown that $C_r$ is an optimal identifying code in $K'_{r}$. Hence, we have the following theorem concerning identifying codes in $E_r$, where $r = \sqrt{5}$ or $r = \sqrt{13}$.

**Theorem 2.7.** The codes $C_2$ and $C_3$ are optimal identifying codes in $E_{\sqrt{5}}$ and $E_{\sqrt{13}}$, respectively.

**Proof.** The claim immediately follows from the fact that $E_{\sqrt{5}} = K'_{2}$ and $E_{\sqrt{13}} = K'_{3}$. □

3. On $\ell$-set-identifying codes with $\ell > 1$

Let $r \geq 1$ be a real number and let $\mathbb{Z}_r$ denote the set of positive integers. In what follows, we show that there exists a 2-set-identifying code $C_r$ in $E_r$ such that the density satisfies $D(C_r) = \Theta(1/r)$. We also prove that the density of a 2-set-identifying code in $E_r$ is always at least $1/(2 |r| + 1)$. In Theorem 2.2, we consider for which $r$ a 3-set-identifying code can exist in $E_r$. Theorem 3.3 shows that there does not exist a 4-set-identifying code in $E_r$ for any $r$.

The following theorem considers 2-set-identifying codes in $E_r$.

**Theorem 3.1.** Let $C_r$ be a 2-set-identifying code in $E_r$, $r \geq 1$. Then $C_r$ satisfies $D(C_r) \geq \frac{1}{2 |r| + 1}$. Moreover, there exists a sequence of 2-set-identifying codes $C_r$ such that $D(C_r) = \Theta(\frac{1}{r})$.

**Proof.** Let $C_r$ be any 2-set-identifying code in $E_r$. The lower bound $D(C_r) \geq 1/(2 |r| + 1)$ comes from comparing the sets $B(x)$ and $B(x, x + (1, 0))$, where $x \in \mathbb{Z}^2$. By Lemma 2.1(i) $|B(x) \triangle B(x, x + (1, 0))| = 2 |r| + 1$ and there must be at least one
Theorem 2.3

Lemma 2.1

We claim that the code

\[ C_r = C_{1,r} \cup C_{2,r} \]

is 2-set-identifying in \( E_r \). Clearly, \(|C| \leq 4/P_1\).

We need to show that for \( C \), we have \( I(X) \neq I(Y) \) for any two sets \( X, Y \subset \mathbb{Z}^2 \), where \( X \neq Y \) and \(|X| \leq 2 \) and \(|Y| \leq 2 \).

Suppose to the contrary that there exist distinct subsets \( X \) and \( Y \) of \( \mathbb{Z}^2 \) such that

\[ I(X) = I(Y) \]

where \(|X|, |Y| \leq 2\).

Clearly, if \( X \) or \( Y \) is the emptyset, we get \( I(X) \neq I(Y) \). Therefore, assume that \(|X| \geq 1 \) and \(|Y| \geq 1 \).

Let \( L_1 \) (resp. \( L_2 \)) be a horizontal line \( L_i^{(h)} \) where \( i \) is such that \( L_i^{(h)} \) contains at least one element of \( X \cup Y \) but for any \( j > i \) (resp. \( j < i \)) the line \( L_j^{(h)} \) contains no elements of \( X \cup Y \). Similarly, let \( L_3 \) (resp. \( L_4 \)) be a vertical line \( L_i^{(v)} \) where \( i \) is such that \( L_j^{(v)} \) contains at least one element of \( X \cup Y \) and for any \( j < i \) (resp. \( j > i \)) the line \( L_j^{(v)} \) contains no elements of \( X \cup Y \). Denote by \( R \) the set of the vertices that belong to a rectangle or a line segment bordered by the the four lines \( L_1, L_2, L_3, L_4 \). Clearly, all the vertices of \( X \cup Y \) belong to \( R \).

Of course, \( R \) is a line segment if (and only if) \( L_1 = L_2 \) or \( L_3 = L_4 \). Suppose first that this is the case: without loss of generality, let \( L_1 = L_2 \). We can also assume that on (at least) one end of the line segment \( R \) there is \( x \in \Delta \). Without loss of generality, let \( x \in X \) be on the left end of \( R \). Now, by Lemma 2.1(i) (rotated anti-clockwise by \( \pi/2 \)), we know that \( B(x) \) contains \( 2|r| + 1 \) vertices on consecutive horizontal lines, which \( x + (1, 0) \) does not cover. Since none of the vertices \( x + (a, 0), a \in \mathbb{Z}_+ \), covers them either, the elements of \( Y \) cannot cover them. By the definition of \( C_1 \), the set \( I(C_1; x) \triangle I(C_1; Y) \neq \emptyset \). Hence we get a contradiction with (2).

Consequently, we can assume that \( R \) is a rectangle.

(1) Suppose first that (at least) one of the four corners of \( R \) contains \( x \in \Delta \). Without loss of generality, we may assume that \( x \in X \) is in the north–west corner of \( R \).

By Lemma 2.1(iii) there are at least \( P_1 \) vertices on consecutive vertical lines in \( B(x) \setminus B(u, w) \) where \( u = x + (0, -1) \) and \( w = x + (1, 0) \). It is easy to verify that none of these \( P_1 \) vertices is covered by any vertex in \( S = \{ x + (a, b) \in \mathbb{Z}^2 \mid a \geq 0, b \geq 0, (a, b) \neq (0, 0) \} \). Since \( Y \subset S \), these \( P_1 \) vertices belong to \( B(x) \setminus B(y) \). Now the code \( C_{2,r} \) guarantees that there is at least one codeword among these \( P_1 \) vertices, a contradiction with (2).

(2) Suppose then that there are no vertices of \( X \cup Y \) in any of the corners of \( R \). Consequently, there must be an element of \( X \cup Y \) on each line \( L_i, i = 1, 2, 3, 4 \). Therefore, \(|X| = |Y| = 2\); denote \( X = \{ x, y \} \) and \( Y = \{ u, w \} \).

(2.1) Assume first that the elements of \( X \) are on two non-intersecting lines, without loss of generality, let \( x \in L_1 \) and \( y \in L_2 \). Assume further \( u \in L_3 \) and \( w \in L_4 \).

By Lemma 2.1(iv), there are \( P_2 \) vertices of \( B(x) \cap \{ x + (a, b) \mid a, b \in \mathbb{Z}_+, a \leq b \} \) on consecutive diagonals, which are either in \( B(x + (1, 1)) \) or in \( B(x + (1, -1)) \). Again, none of the vertices \( u \in U = \{ x + (a, -b) \mid a, b \in \mathbb{Z}_+ \} \) can cover these \( P_2 \) points. It is also easy to verify that none of the vertices \( u \in T = \{ x + (a, -b) \mid a, b \in \mathbb{Z}_+, a \leq b \} \) can cover these \( P_2 \) points either. Consequently, if \( u \in T \), the code \( C_{2,r} \) gives the codeword to the set \( I(X) \triangle I(Y) \), which contradicts (2). Assume then that \( u \not\in T \), that is, \( u = x + (a, -b) \) where \( a, b \in \mathbb{Z}_+ \) and \( a > b \). In this case, we examine the vertices of \( B(u) \cap \{ u + (c, d) \mid c, d \in \mathbb{Z}_+, c \geq d \} \) which are not covered by \( u + (1, 1) \) and \( u + (1, -1) \) there are again \( P_2 \) of them by a result symmetrical to Lemma 2.1(iv). We observe that neither the vertex \( y = u + (c, -d) \) for any \( c, d \in \mathbb{Z}_+ \) nor the vertex \( x = u + (a, b) \) cannot cover these \( P_2 \) vertices in \( B(u) \) (because the assumption \( a > b \) now gives symmetric situation to the above case \( u \in T \)). Therefore, there must be a codeword of \( C_{2,r} \) in \( I(X) \triangle I(Y) \) to give the contradiction.

(2.2) Assume then that the elements of \( X \) are on two intersecting lines, and without loss of generality, let \( x \in L_1 \), \( y \in L_2 \), \( u \in L_3 \), and \( w \in L_4 \). Let \( I_x = \{ x + (0, -a) \mid a \in \mathbb{Z}_+ \} \). Again \( w \in U \).

If \( u \in T \cup U \cup I_x \), then the previous arguments give us the contradiction with (2). Indeed, if \( u \in U \cup I_x \), the argument of (1) applies although \( x \) is not in a corner of \( R \). If \( u \in T \), the case (2.1) yields the needed contradiction. Therefore, it suffices to assume that \( w \in U \) and \( u = x + (a, -b) \) where \( a, b \in \mathbb{Z}_+, a > b \). Now consider \( y \) in the role of \( x \). In this case \( u \in U \) and \( w \) belong to the area \( T' \cup U' \cup I_y \) where \( T' = \{ y + (c, d) \mid c, d \in \mathbb{Z}_+, c \geq d \}, U' = \{ y + (c, -d) \mid c, d \in \mathbb{Z}_+ \} \) and \( I_y = \{ y + (c, 0) \mid c \in \mathbb{Z}_+ \} \). Now the area \( T' \cup U' \cup I_y \) for \( y \) is analogous to \( T \cup U \cup I_x \) for \( x \). This contradicts (2).

(3) Finally, it suffices to check the case where there is a vertex \( y \in X \cap Y \) in one of the corners of \( R \). By (1) we can assume that there is no other vertex of \( X \cup Y \) in any corner. Consequently, we may assume that \( y \) is in the south–east corner and \( x \in L_1 \) and \( u \in L_3 \). This situation goes exactly like in (2.1). \( \square \)
The graphs \( \mathcal{K}_r \) and \( \mathcal{H}_r \) have balls of equal size, for example, when \( r = 347 \) and \( t = 307 \). In the king grid \( \mathcal{K}_{307} \) the optimal density of a 2-set-identifying code equals 0.25 (see [8]) and by our previous construction we have a 2-set-identifying code in \( \mathcal{K}_{347} \) with density at most 0.0396. Similarly, the square grid \( \mathcal{S}_r \) and \( \mathcal{D}_r \) have the same cardinality of vertices in a ball when, for instance, \( r = 385 \) and \( t = 482 \). The smallest possible density of a 2-set-identifying code in \( \mathcal{S}_{482} \) is at least 0.125 (see [10]) and our construction gives a 2-set-identifying code of density at most 0.0357.

In general, an optimal 2-set-identifying code \( C_r \) in the king grid \( \mathcal{K}_r, r \geq 3 \), satisfies \( D(C_r) = 1/4 \) (see [8]). Similarly, in the square grid \( \mathcal{S}_r \) we know (by Honkala [10]) that \( D(C_r) \geq 1/8 \) for any code \( C_r \) which is 2-set-identifying. In \( \mathcal{S}_r \), however, the density of such codes can be arbitrarily small by the previous theorem. For the 2-set-identifying codes in \( \mathcal{S}_r = \mathcal{S}_1 \), \( \mathcal{S}_2 = \mathcal{S}_2 \), \( \mathcal{S}_3 = \mathcal{K}_1 \) and \( \mathcal{S}_4 = \mathcal{K}_2 \), we refer to [10].

Consider next the \( \ell \)-set-identifying codes in \( \mathcal{S}_r \) when \( \ell = 3 \). Since the sets \( I((1, 0), (−1, 0)) \) and \( I((1, 0), (−1, 0), (0, 0)) \) must differ and also the sets \( I((−1, −1), (1, 1)) \) and \( I((−1, −1), (1, 1), (0, 0)) \) must differ, we obtain the following statement for 3-set-identifying codes.

**Theorem 3.2.** Let \( r \in \mathbb{R}, r \geq 1 \). If there exists a 3-set-identifying code in \( \mathcal{S}_r \), then we must have

\[
|\{r\}| > \sqrt{r^2 - 1}
\]

and, if \( r \geq \sqrt{2} \), we also must have

\[
|\{r/\sqrt{2}\}| > \sqrt{2^2/2 - 1}.
\]

By the previous theorem we obtain that \( 1, 3, 17, 99, 577, 3363, 19601, \ldots \) are the first values of an integer \( r \) such that the graphs \( \mathcal{S}_r \) can have 3-set-identifying codes. By Honkala [9, Theorem 2], we know that there exists a 3-set-identifying code in \( \mathcal{S}_1 = \mathcal{S}_1 \). However, it remains open whether there exists a 3-set-identifying code in \( \mathcal{S}_r \) for all possible values of \( r \) (listed above).

Moreover, if there exists a 3-set-identifying code \( C \) in \( \mathcal{S}_r = (V, E) \) for \( r = 3 \) or \( r = 17 \), then necessarily \( C = V \) and thus the density equals one due to the fact that

\[
B((-1, -1), (1, 2)) \triangle B((-1, -1), (1, 2), (0, 0)) = \{(2, -2)\}
\]

(only one vertex!) for \( r = 3 \) and \( B((-2, -1), (3, 1)) \triangle B((-2, -1), (3, 1), (0, 0)) = \{(-8, 15)\} \) for \( r = 17 \).

**Theorem 3.3.** Let \( r \geq 1 \). There does not exist an \( \ell \)-set-identifying code in \( \mathcal{S}_r \) for any \( \ell \geq 4 \).

**Proof.** The claim follows since

\[
B((-1, 0), (1, 1), (1, -1)) = B((-1, 0), (1, 1), (1, -1), (0, 0))
\]

and thus these sets of three and four vertices cannot be distinguished. \( \square \)

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