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Equivalent embeddings of the dynamics on an invariant manifold

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Abstract

A dynamical system admitting an invariant manifold can be interpreted as a single element of an infinite class of dynamical systems that all exhibit the same behaviour on the invariant manifold. This observation is used in the context of autonomous ordinary differential equations to generalize a global stability result of Li and Muldowney. The new result is demonstrated on an epidemiological model.

Keywords: Invariant manifold; Global stability; Compound matrix

1. Introduction

Invariant manifolds appear in many areas of research. For example, in physics one frequently studies systems in which quantities such as mass and energy are conserved; in population studies one may analyze systems for which the total population size is fixed; in ecological studies one may assume that the total biomass remains constant. In each case, the conservation law corresponds to an invariant manifold.

Often when studying a dynamical system that exhibits an invariant manifold, one is only interested in the behaviour of solutions on the invariant manifold. Suppose that Γ is an invariant manifold with respect to the flow φ. Then φ is just one of an infinite number of flows that leave Γ invariant. It is the purpose of this paper to show that a careful choice of flow $\tilde{\varphi}$ such that $\tilde{\varphi} = \varphi$ on Γ, may simplify an analysis of the dynamics on Γ.

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In this paper, we work with autonomous ordinary differential equations and consider global stability within an invariant manifold in $\mathbb{R}^n$. With that in mind, some global stability results of Li and Muldowney are given in Section 2; a detailed exposition of these techniques may be found in [2–5,7]. The main result of this paper appears in Section 3 where the theory of Li and Muldowney for invariant manifolds is extended. In Section 4, this innovation is applied to an example from mathematical epidemiology.

2. Mathematical preliminaries

Consider the differential equation

$$x' = f(x), \tag{2.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Let the solution to Eq. (2.1) that passes through x_0 at time 0 be denoted by $\varphi(t; x_0)$.

The time-dependent linear differential equation

$$y' = \left[\frac{\partial f}{\partial x}\big(\varphi(t; x_0)\big)\right] y \tag{2.2}$$

describes the evolution of line segments near $\varphi(t; x_0)$. It is shown in [7] that the evolution of k-dimensional volumes near $\varphi(t; x_0)$ is described by the time-dependent linear differential equation

$$z' = \left[\frac{\partial f}{\partial x}\big(\varphi(t; x_0)\big)\right]^{[k]} z, \tag{2.3}$$

where for an $n \times n$ matrix M, the $\binom{n}{k} \times \binom{n}{k}$ matrix $M^{[k]}$ is the kth additive compound [7] of M. If $y_1(t), \ldots, y_k(t)$ are solutions to (2.2), then $z(t) = y_1 \wedge \cdots \wedge y_k$ is a solution to (2.3). Note that z is treated as an element of $\mathbb{R}^{\binom{n}{k}}$, which is diffeomorphic to the space of k-forms on $\mathbb{R}^n$. It has been shown by Li and Muldowney [3] that if $\varphi(t; x_0)$ is bounded and (2.3) is asymptotically stable for $k = 2$, then the omega limit set of $\varphi(t; x_0)$ either contains an equilibrium or is an orbitally asymptotically stable periodic orbit. Some global stability results involving compound matrices are given here; see [2–5,7] for a detailed exposition.

In order to study certain measures of k-dimensional volume, let $w = Q(x)z$ where Q is a C^1 non-singular $\binom{n}{k} \times \binom{n}{k}$ matrix-valued function such that the norm of Q^{-1} is bounded. Then

$$w' = \left[Q_f Q^{-1} + Q \frac{\partial f}{\partial x}^{[k]} Q^{-1}\right]_{\varphi(t; x_0)} w, \tag{2.4}$$

where Q_f is the directional derivative of Q in the direction of the vector field f. In other words, Q_f can be constructed by replacing each entry of Q with its time derivative. Note that if w goes to zero, then so does z. The following theorem [2, Theorem 3.5] relates bounds on the rate of growth of solutions to (2.4) to the global behaviour of (2.1).

Theorem 2.1. *Suppose Δ is a compact absorbing set for* (2.1) *containing a unique equilibrium $\bar{x}$. If for $k = 2$, there exist $\epsilon > 0$ and a norm $\|\cdot\|$ on $\mathbb{R}^{\binom{n}{2}}$ such that $D_+\|w\| \leqslant -\epsilon\|w\|$ for all solutions w of Eq.* (2.4) *and all $x_0 \in \Delta$, then $\bar{x}$ is globally asymptotically stable under the flow* (2.1).

Note that the condition that there is a norm such that $D_+\|w\| \leqslant -\epsilon\|w\|$, is equivalent to there being a Lozinskii measure μ such that $\mu(Q_f Q^{-1} + Q \frac{\partial f}{\partial x}^{[2]} Q^{-1}) \leqslant -\epsilon$ on the compact set Δ;

see [1, p. 41]. Here, $-\epsilon$ is a bound on the exponential behaviour of solutions to Eq. (2.4), with $\|w(t)\| \leqslant \|w(0)\| \, e^{-\epsilon t}$.

When studying the dynamics on an invariant manifold, an extension of this method can be used. In this situation, it is not necessary to obtain bounds on the rate of growth of all two-dimensional areas, but just the two-dimensional areas that are contained in the invariant manifold. In order to do this, it is necessary to account for the component of the dynamics which is normal to the manifold.

Definition. Let $g:\mathbb{R}^n \to \mathbb{R}^m$ be C^2 and let $\Gamma = \{x \in \mathbb{R}^n\colon\ g(x) = 0\}$. Then Γ is a manifold of dimension $n-m$ if $\operatorname{rank}(\partial g/\partial x) = m$ for $g(x) = 0$, and Γ is invariant with respect to (2.1) if $x_0 \in \Gamma$ implies $\varphi(t; x_0) \in \Gamma$ for all t.

The following result is Proposition 3.1 in [4].

Proposition 2.2. *Let $g:\mathbb{R}^n \to \mathbb{R}^m$ be C^2 and satisfy* $\operatorname{rank}(\partial g/\partial x) = m$ *for* $g(x) = 0$. *Then the manifold $\Gamma = \{x \in \mathbb{R}^n\colon\ g(x) = 0\}$ is invariant with respect to* (2.1) *if and only if there is a continuous $m \times m$ matrix valued function $N(x)$ defined in a neighbourhood of Γ such that*

$$g_f = Ng.$$

Let Γ be a simply connected manifold, which is invariant with respect to (2.1), given by $g(x) = 0$ where $g:\mathbb{R}^n \to \mathbb{R}^m$. We define the scalar function ν by

$$\nu = \operatorname{trace} N. \tag{2.5}$$

While N is not necessarily uniquely defined in a neighbourhood of Γ, it is shown in [4] that N, and hence ν, is unique on Γ. The function ν is related to the rate of growth of m-dimensional volumes that are normal to Γ.

Consider the equation

$$w' = \left[Q_f Q^{-1} + Q \frac{\partial f}{\partial x}^{[m+2]} Q^{-1} - \nu J \right]_{\varphi(t; x_0)} w, \tag{2.6}$$

where J is the identity matrix and Q is a C^1 non-singular $\binom{n}{m+2} \times \binom{n}{m+2}$ matrix-valued function such that the norm of Q^{-1} is bounded. The following theorem [4, Theorem 6.1] relates bounds on the rate of growth of solutions to (2.6) to global stability within Γ.

Theorem 2.3. *Suppose that for Eq.* (2.1), *$\Delta \subseteq \Gamma$ is simply connected, contains a unique equilibrium $\bar{x}$, and is a compact absorbing set relative to Γ. If there exist $\epsilon > 0$ and a norm $\|\cdot\|$ on $\mathbb{R}^{\binom{n}{m+2}}$ such that $D_+\|w\| \leqslant -\epsilon\|w\|$ for all solutions w of* (2.6) *and all $x_0 \in \Delta$, then $\bar{x}$ is globally asymptotically stable in Γ under the flow* (2.1).

3. Main result

Consider Eq. (2.1) and suppose Γ is an invariant manifold of codimension m given by $g(x) = 0$. Further, suppose that we are only interested in the dynamics of (2.1) on Γ. Any vector field $\tilde{f}$ that is equal to f on Γ, exhibits the same dynamics on Γ. Thus, we can replace Eq. (2.1) with

$$x' = \tilde{f}(x),$$

where

$$\tilde{f}(x) = f(x) + E(x)g(x)$$

and $E:\mathbb{R}^n \to \mathbb{M}_{n\times m}$ is defined continuously in a neighbourhood of Γ, without changing the dynamics on Γ. While $\tilde{f}|_\Gamma = f|_\Gamma$, the associated Jacobians may differ. Noting that g is zero on Γ we see that, on Γ

$$\frac{\partial \tilde{f}}{\partial x} = \frac{\partial}{\partial x}(f + Eg) = \frac{\partial f}{\partial x} + E\frac{\partial g}{\partial x} + \frac{\partial E}{\partial x}g = \frac{\partial f}{\partial x} + E\frac{\partial g}{\partial x}.$$

Since $\frac{\partial g}{\partial x}$ is non-zero on Γ, it is clear that E can be chosen so that $\frac{\partial \tilde{f}}{\partial x}$ and $\frac{\partial f}{\partial x}$ are not equal. When performing calculations involving the Jacobian matrix in order to study the dynamics on Γ, E may be chosen to facilitate the calculations.

We now calculate $\tilde{\nu}$ in terms of ν, E and g. Since Γ is invariant under the flow given by f, it is also invariant under the flow given by $\tilde{f}$. Thus, there exists an $m\times m$ matrix $\widetilde{N}$ such that

$$\widetilde{N}g = g_{\tilde{f}} = \frac{\partial g}{\partial x}\cdot \tilde{f} = \frac{\partial g}{\partial x}\cdot (f + Eg) = Ng + \frac{\partial g}{\partial x}Eg = \left(N + \frac{\partial g}{\partial x}E\right)g.$$

While $\widetilde{N}$ is not necessarily unique in a neighbourhood of Γ, $\tilde{\nu}$ is uniquely defined on Γ. Thus, we may choose $\widetilde{N} = N + \frac{\partial g}{\partial x}E$, and take the trace of each side, giving

$$\tilde{\nu} = \nu + \operatorname{trace}\left(\frac{\partial g}{\partial x}E\right). \tag{3.1}$$

Noting that $Q_{\tilde{f}} = Q_f$ on Γ, it is clear that

$$\begin{aligned} &Q_{\tilde{f}}Q^{-1} + Q\frac{\partial \tilde{f}}{\partial x}^{[m+2]}Q^{-1} - \tilde{\nu}J \\ &\quad = Q_f Q^{-1} + Q\frac{\partial f}{\partial x}^{[m+2]}Q^{-1} - \nu J + Q\left(E\frac{\partial g}{\partial x}\right)^{[m+2]}Q^{-1} - \operatorname{trace}\left(\frac{\partial g}{\partial x}E\right)J. \end{aligned}$$

Thus, Theorem 2.3 is generalized to the following.

Theorem 3.1. *Suppose that for Eq.* (2.1), $\Delta \subseteq \Gamma$ *is simply connected, contains a unique equilibrium* $\bar{x}$, *and is a compact absorbing set relative to* Γ. *Let* $\|\cdot\|$ *be a norm on* $\mathbb{R}^{\binom{n}{m+2}}$ *and let* E *be a* $\mathbb{M}_{n\times m}$*-valued function that is continuously defined in a neighbourhood of* Γ. *If there exists* $\epsilon > 0$ *such that* $D_+\|w\| \leqslant -\epsilon\|w\|$ *for all solutions* w *of*

$$\begin{aligned} w' = \Bigg[&Q_f Q^{-1} + Q\frac{\partial f}{\partial x}^{[m+2]}Q^{-1} - \nu J \\ &+ Q\left(E\frac{\partial g}{\partial x}\right)^{[m+2]}Q^{-1} - \operatorname{trace}\left(\frac{\partial g}{\partial x}E\right)J \Bigg]_{\varphi(t;x_0)} w \end{aligned} \tag{3.2}$$

and all $x_0 \in \Delta$, *then* $\bar{x}$ *is globally asymptotically stable in* Γ *under the flow* (2.1).

Proof. All that remains to be shown for this theorem to be proven, is that it is sufficient for E to be continuous. Implicit in the above discussion is the fact that E is differentiable. Suppose that E is continuous, but not differentiable, and that $D_+\|w\| \leqslant -\epsilon\|w\|$. Note that the expressions in Eq. (3.2) depend on E and not on the derivatives of E, so the condition on $D_+\|w\|$ is robust under small C^0 perturbations to E. Thus, if there exists a continuous vector field E such that the derivative condition holds, then there also exists a differentiable vector field close to E satisfying the derivative condition, concluding the argument. □

4. An example from mathematical epidemiology

We now consider a model of an infectious disease of long duration in a population of fixed size, perhaps HIV/AIDS in a jail that is filled to capacity. We assume that the duration of infection is long enough that the dynamics are better modelled by having individuals pass through three successive infective stages, rather than just one. This allows for different parameters to be used to describe the characteristics of individuals who are at different stages of infection. A similar model with varying total population size is studied in [6].

A population of total size T is divided into a susceptible group of size S and three infective groups with sizes I_1, I_2, I_3. Thus, $T = S + I_1 + I_2 + I_3$. The transfer diagram is as follows:

$$\begin{tikzcd}
{} \arrow[r, "B"] & S \arrow[r, "\sum_{j=1}^{3} c_j\beta_j I_j S"] \arrow[d, "dS"] & I_1 \arrow[r, "k_1 I_1"'] \arrow[d, "dI_1"] & I_2 \arrow[l, bend right, "r_2 I_2"'] \arrow[r, "k_2 I_2"'] \arrow[d, "dI_2"] & I_3 \arrow[l, bend right, "r_3 I_3"'] \arrow[r, "k_3 I_3"'] \arrow[d, "dI_3"] & {} \\
 & {} & {} & {} & {} &
\end{tikzcd}$$

For $j = 1, 2$, the parameters k_j and r_{j+1} are the rate constants for movement from I_j to I_{j+1} and from I_{j+1} to I_j, respectively. The flow rate $k_j I_j$ represents individuals progressing to a more advanced stage of infection, including removal from the population for disease related reasons for $j = 3$. The flow rate $r_j I_j$ represents individuals undergoing a limited recovery or amelioration, from a more advanced stage of infection to a less advanced stage of infection. For many diseases, these amelioration rates are zero.

The rate constant for removal from the population, not directly related to the disease, is d. Thus, in the absence of disease, the average time spent in the population is $1/d$. In the context of a jail setting, this would be the mean duration of incarceration.

The mean number of contacts that a susceptible individual has with individuals in infective class I_j per unit time is $c_j I_j$. The probability that a contact between a susceptible and an infective in class I_j results in transmission of the disease is β_j. Thus, the rate at which new infections occur is $\sum_{j=1}^{3} c_j \beta_j I_j S$.

The recruitment rate B of new individuals into the population is chosen so that the total population size remains constant. Thus, $B = d(S + I_1 + I_2 + I_3) + k_3 I_3$. We assume that k_1, k_2, k_3, d, and at least one of the products $c_j \beta_j$ is positive and that all other parameters are non-negative.

The differential equation for the sizes of the population subgroups is

$$\begin{aligned}
S' &= d(I_1 + I_2 + I_3) - \sum_{j=1}^{3} c_j \beta_j I_j S + k_3 I_3, \\
I_1' &= \sum_{j=1}^{3} c_j \beta_j I_j S - (k_1 + d) I_1 + r_2 I_2, \\
I_2' &= k_1 I_1 - (k_2 + r_2 + d) I_2 + r_3 I_3, \\
I_3' &= k_2 I_2 - (k_3 + r_3 + d) I_3.
\end{aligned} \tag{4.1}$$

We denote the right-hand side of (4.1) by $f(x)$ where $x = (S, I_1, I_2, I_3)^T$. The biologically relevant region is the non-negative orthant $\{(S, I_1, I_2, I_3)\colon\ S, I_1, I_2, I_3 \geqslant 0\}$. We restrict our analysis to this region.

Every point on the disease-free axis is an equilibrium. We label these by $P_0(T) = (T, 0, 0, 0)$. For every T, the point $P_*(T) = (S_*, I_{1*}, I_{2*}, I_{3*})$ is an equilibrium, where

$$S_* = \frac{(k_1+d)[(k_2+r_2+d)(k_3+r_3+d) - k_2r_3] - k_1r_2(k_3+r_3+d)}{c_1\beta_1[(k_2+r_2+d)(k_3+r_3+d) - k_2r_3] + c_2\beta_2k_1(k_3+r_3+d) + c_3\beta_3k_1k_2},$$

$$I_{1*} = \frac{(k_2+r_2+d)(k_3+r_3+d) - k_2r_3}{(k_2+r_2+d)(k_3+r_3+d) - k_2r_3 + k_1(k_3+r_3+d) + k_1k_2}(T - S_*),$$

$$I_{2*} = \frac{k_1(k_3+r_3+d)}{(k_2+r_2+d)(k_3+r_3+d) - k_2r_3 + k_1(k_3+r_3+d) + k_1k_2}(T - S_*),$$

$$I_{3*} = \frac{k_1k_2}{(k_2+r_2+d)(k_3+r_3+d) - k_2r_3 + k_1(k_3+r_3+d) + k_1k_2}(T - S_*).$$

Note that S_* is independent of T, and so $P_*(T)$ is in the interior of the non-negative orthant if and only if $T > S_*$. In this case, P_* is called an endemic equilibrium. If $T = S_*$, then P_* coincides with P_0. If $T < S_*$, then P_* lies outside the non-negative orthant and is not biologically relevant. Thus, there is only an endemic equilibrium if the population size is large enough.

Since $T' = 0$, the total population size T is constant, meaning T is a first integral. Of interest here, is the fact that for any particular T, the three-dimensional manifold $\Gamma = \{x\colon\ g(x) = 0,\ S, I_1, I_2, I_3 \geqslant 0\}$ is positively invariant under the flow described by (4.1), where

$$g = S + I_1 + I_2 + I_3 - T.$$

We will study the stability of the equilibria relative to the invariant manifold in which they lie.

Let $h = -[c_1\beta_1, c_2\beta_2, c_3\beta_3]^T$ and

$$L = \begin{bmatrix} -(k_1+d) & k_1 & 0 \\ r_2 & -(k_2+r_2+d) & k_2 \\ 0 & r_3 & -(k_3+r_3+d) \end{bmatrix}.$$

If $a = [a_1, a_2, a_3]^T$ is defined by the equation $La = h$, then it can be shown that each a_j is positive and that $W = a_1I_1 + a_2I_2 + a_3I_3$ is a Lyapunov function satisfying

$$W' = \left(\frac{S}{S_*} - 1\right)\sum_{j=1}^{3} c_j\beta_jI_j.$$

Thus, if $T \leqslant S_*$ then W is decreasing in $\Gamma \setminus P_0$ and so P_0 is globally stable in Γ. On the other hand, if $T > S_*$ then W is increasing near P_0 and so P_0 is repelling. Note that for $T > S_*$, the boundary of Γ is repelling towards the interior. Thus, there is a compact absorbing set Δ contained in the interior of Γ.

We make the simplifying assumption that $r_2 = r_3 = r$ and $k_j = k$, $c_j\beta_j = c\beta$ for $j = 1, 2, 3$. We will use the ideas developed in Section 3 to show that for $T > S_*$, the equilibrium P_* is globally asymptotically stable in $\Gamma \setminus P_0$.

Consider the new system

$$x' = \tilde{f}(x) := f(x) + E(x)g, \tag{4.2}$$

where

$$E = [c\beta S - d, -c\beta T, 0, 0]^T. \tag{4.3}$$

In order to illustrate the constructive manner in which this method is expected to be used, the calculations will be performed with a general vector E which will, at the appropriate step, be replaced with the particular vector given by (4.3).

On Γ, Eqs. (4.1) and (4.2) describe the same dynamics. The associated Jacobians, however, are different. On Γ, the Jacobian associated with (4.2) is given by

$$\frac{\partial \tilde{f}}{\partial x} = \frac{\partial f}{\partial x} + E\frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} + \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix} [1, 1, 1, 1].$$

Calculating $\frac{\partial f}{\partial x}$, and then using the relationship $T = S + I_1 + I_2 + I_3$ yields

$$\frac{\partial \tilde{f}}{\partial x} = \begin{bmatrix} c\beta(S-T) & d - c\beta S & d - c\beta S & k + d - c\beta S \\ c\beta(T-S) & c\beta S - (k+d) & r + c\beta S & c\beta S \\ 0 & k & -(k+r+d) & r \\ 0 & 0 & k & -(k+r+d) \end{bmatrix} + \begin{bmatrix} E_1 & E_1 & E_1 & E_1 \\ E_2 & E_2 & E_2 & E_2 \\ E_3 & E_3 & E_3 & E_3 \\ E_4 & E_4 & E_4 & E_4 \end{bmatrix}.$$

Note that the codimension of Γ is $m = 1$, and so $m + 2 = 3$. The third additive compound [4, Appendix] of $\frac{\partial \tilde{f}}{\partial x}$ is

$$\frac{\partial \tilde{f}}{\partial x}^{[3]} = \begin{bmatrix} \begin{pmatrix} c\beta(2S-T) \\ -(2k+r+2d) \end{pmatrix} & r & -c\beta S & k + d - c\beta S \\ k & \begin{pmatrix} c\beta(2S-T) \\ -(2k+r+2d) \end{pmatrix} & r + c\beta S & c\beta S - d \\ 0 & k & \begin{pmatrix} c\beta(S-T) \\ -2(k+r+d) \end{pmatrix} & d - c\beta S \\ 0 & 0 & c\beta(T-S) & \begin{pmatrix} c\beta S \\ -(3k+2r+3d) \end{pmatrix} \end{bmatrix} + \begin{bmatrix} E_1+E_2+E_3 & E_3 & -E_2 & E_1 \\ E_4 & E_1+E_2+E_4 & E_2 & -E_1 \\ -E_4 & E_3 & E_1+E_3+E_4 & E_1 \\ E_4 & -E_3 & E_2 & E_2+E_3+E_4 \end{bmatrix}.$$

Since $m = 1$, the matrix N is a scalar and so $\nu = N$. Furthermore, by differentiating g it is clear that $g_f = 0$ and so Eq. (2.5) implies $\nu = 0$. Thus, Eq. (3.1) gives $\tilde{\nu} = E_1 + E_2 + E_3 + E_4$. Letting $\bar{M} = \frac{\partial \tilde{f}}{\partial x}^{[3]} - \tilde{\nu} J$ yields

$$\bar{M}=\begin{bmatrix} \begin{pmatrix} c\beta(2S-T) \\ -(2k+r+2d) \end{pmatrix} & r & -c\beta S & k+d-c\beta S \\ k & \begin{pmatrix} c\beta(2S-T) \\ -(2k+r+2d) \end{pmatrix} & r+c\beta S & c\beta S-d \\ 0 & k & \begin{pmatrix} c\beta(S-T) \\ -2(k+r+d) \end{pmatrix} & d-c\beta S \\ 0 & 0 & c\beta(T-S) & \begin{pmatrix} c\beta S \\ -(3k+2r+3d) \end{pmatrix} \end{bmatrix}$$

$$+\begin{bmatrix} -E_4 & E_3 & -E_2 & E_1 \\ E_4 & -E_3 & E_2 & -E_1 \\ -E_4 & E_3 & -E_2 & E_1 \\ E_4 & -E_3 & E_2 & -E_1 \end{bmatrix}.$$

Let $Q=\frac{1}{I_1}J$. Then $Q_{\tilde{f}}Q^{-1}=-\frac{I_1'}{I_1}J$ and Q commutes with $\frac{\partial \tilde{f}}{\partial x}^{[3]}$, so Eq. (3.2) takes the form

$$w'=\left(-\frac{I_1'}{I_1}J+\bar{M}\right)w. \tag{4.4}$$

Note that since Δ is a compact absorbing set in the interior of Γ, it is bounded and so the norm of Q^{-1} is bounded on Δ. Noting that $\frac{I_1'}{I_1}=c\beta S-(k+d)+\phi$ where $\phi=\frac{c\beta S(I_2+I_3)+rI_2}{I_1}>0$, the matrix $M=-\frac{I_1'}{I_1}J+\bar{M}$ is given by

$$M=\begin{bmatrix} \begin{pmatrix} c\beta(S-T)-\phi \\ -(k+r+d) \end{pmatrix} & r & -c\beta S & k+d-c\beta S \\ k & \begin{pmatrix} c\beta(S-T)-\phi \\ -(k+r+d) \end{pmatrix} & r+c\beta S & c\beta S-d \\ 0 & k & -\begin{pmatrix} c\beta T+\phi \\ +k+2r+d \end{pmatrix} & d-c\beta S \\ 0 & 0 & c\beta(T-S) & -(\phi+2k+2r+2d) \end{bmatrix}$$

$$+\begin{bmatrix} -E_4 & E_3 & -E_2 & E_1 \\ E_4 & -E_3 & E_2 & -E_1 \\ -E_4 & E_3 & -E_2 & E_1 \\ E_4 & -E_3 & E_2 & -E_1 \end{bmatrix}.$$

The stability of (4.4) is shown by using the l_∞ norm as a Lyapunov function. In doing so, we find [1] that $D_+\|w\|_\infty\leqslant\mu\|w\|_\infty$, where

$$\mu=\max_{i=1,2,3,4}\left\{m_{ii}+\sum_{j\neq i}|m_{ij}|\right\} \tag{4.5}$$

is the l_∞ Lozinskii measure of $M=[m_{ij}]$ (i.e., a sum is calculated for each row of M, and μ is the maximum of these sums). Thus, if $\mu\leqslant-\epsilon$ for some constant $\epsilon>0$, then the conditions of Theorem 3.1 are satisfied.

Hence, we want to choose E in such a way that the off-diagonal entries of M are close to zero, while the diagonal entries are negative. By choosing E_3 and E_4 to be zero, we ensure that the first two columns of M contain some zeroes. By choosing $E_1=c\beta S-d$, we make two of the off-diagonal entries in the fourth column of M equal to zero. The choice of $E_2 = -c\beta T$ is less obvious, but is made in order to make μ negative. Thus, the choice of E given in (4.3) yields

$$M = \begin{bmatrix} \begin{pmatrix} c\beta(S-T)-\phi \\ -(k+r+d) \end{pmatrix} & r & c\beta(T-S) & k \\ k & \begin{pmatrix} c\beta(S-T)-\phi \\ -(k+r+d) \end{pmatrix} & r+c\beta(S-T) & 0 \\ 0 & k & -(\phi+k+2r+d) & 0 \\ 0 & 0 & -c\beta S & -\begin{pmatrix} c\beta S+\phi+ \\ 2k+2r+d \end{pmatrix} \end{bmatrix}.$$

Evaluating μ according to (4.5) gives

$$\mu = \max\Bigl\{-(\phi+d), c\beta(S-T)-(\phi+r+d)+\bigl|r+c\beta(S-T)\bigr|, -(\phi+2r+d), \\ -(\phi+2k+2r+d)\Bigr\}.$$

Since $T \geqslant S$, it follows that $|r + c\beta(S-T)| \leqslant r + c\beta(T-S)$, and so it is clear that $\mu = -(\phi+d) < -d$. Thus, we may choose $\epsilon = d$, obtaining $D_+\|w\|_\infty \leqslant -\epsilon\|w\|_\infty$.

Therefore, by Theorem 3.1, P_* is globally stable in $\Gamma \setminus P_0$ for $T > S_*$.

Thus, we have shown that if $T \leqslant S_*$, then the disease-free equilibrium is globally attracting and the disease dies out; if $T > S_*$ then the disease persists in the population and there is a unique endemic equilibrium. Furthermore, for $T > S_*$, if $r_2 = r_3$, $k_1 = k_2 = k_3$, and $c_1\beta_1 = c_2\beta_2 = c_3\beta_3$, and the disease is present, then the disease will eventually go to the endemic equilibrium levels.

By performing the same calculations while allowing the parameters for the different infective groups to differ, a global stability result for system (4.1) with limited heterogeneity can be obtained.

References

[1] W.A. Coppel, Stability and Asymptotic Behaviour of Differential Equations, Heath, Boston, 1965.
[2] M.Y. Li, J.S. Muldowney, A geometric approach to global-stability problems, SIAM J. Math. Anal. 27 (1996) 1070–1083.
[3] M.Y. Li, J.S. Muldowney, Phase asymptotic semiflows, Poincaré's condition and the existence of stable limit cycles, J. Differential Equations 124 (1996) 425–448.
[4] M.Y. Li, J.S. Muldowney, Dynamics of differential equations on invariant manifolds, J. Differential Equations 168 (2000) 295–320.
[5] Y. Li, J.S. Muldowney, On Bendixson's Criterion, J. Differential Equations 106 (1993) 27–39.
[6] C.C. McCluskey, A model of HIV/AIDS with staged progression and amelioration, Math. Biosci. 181 (2003) 1–16.
[7] J.S. Muldowney, Compound matrices and ordinary differential equations, Rocky Mountain J. Math. 20 (1990) 857–872.