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Koszul duality for monoids and the operad of enriched rooted trees

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ABSTRACT

We introduce here the notion of Koszul duality for monoids in the monoidal category of species with respect to the ordinary product. To each Koszul monoid we associate a class of Koszul algebras in the sense of Priddy, by taking the corresponding analytic functor. The operad $\mathscr{A}_M$ of rooted trees enriched with a monoid M was introduced by the author. One special case of that is the operad of ordinary rooted trees, called in the recent literature the permutative non-associative operad. We prove here that $\mathscr{A}_M$ is Koszul if and only if the corresponding monoid M is Koszul. In this way we obtain a wide family of Koszul operads, extending a recent result of Chapoton and Livernet, and providing an interesting link between Koszul duality for associative algebras and Koszul duality for operads.

1. Introduction

The present paper is an overdue followup of the program initiated in [25,26], some years before the introduction in [17] of the notion of Koszul duality for operads, and whose initial step in the above mentioned references we proceed to describe. Unformally, a (set) Joyal species is a family of labelled combinatorial structures that is invariant under relabelling. Being more formal, it can be defined as a functor from the category $\mathbb{B}$ of finite sets and bijections, to the category $\mathcal{F}$ of finite sets and arbitrary functions.

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Each set species has three associated formal power series with non-negative coefficients (see [4, Section 1.1]), related to the enumeration of the family of combinatorial structures involved. The program intended to overcome a gap in the theory of species as originally introduced by Joyal in [21]. That was the lack of a combinatorial interpretation for the formal power series with possible negative coefficients. Even though this task could be performed by means of the formal differences between ordinary species (virtual species) [20], the problem of finding *the inverses* by the process of sieving known as Möbius inversion on a partially ordered set [33] was open by then. The term *inverse* means here inverse with respect to each one of the usual binary operations between species: sum, product, and substitution. The motivation for this approach was that many combinatorial identities come in pairs. The two identities belonging to a pair are usually said to be *inverses* to each other, because one comes from the other by Möbius inversion on a partially ordered set (e.g., identities involving many families of polynomials of binomial type and their umbral inverses [27,19,32]).

A Möbius species is defined as a functor from the category of finite sets and bijections to the category of finite disjoint union of finite partially ordered sets with unique maxima and minima. All the main features of Joyal's set species are extended to this context; the generating function of a Möbius species is computed by replacing the cardinality of a finite set by the evaluation of the Möbius function on partially ordered sets. In this way, generating functions having negative coefficients are allowed. To define the Möbius inverses for ordinary Joyal set species, we had to introduce the concept of left-cancellative monoid (c-monoid) in the context of certain kinds of monoidal categories. We studied three instances of such monoidal categories on species, one for each of the above mentioned operations. In order to construct the Möbius inverse of a given species with respect to an operation one has to presuppose a c-monoidal structure on it. For example, if a given set species M is a c-monoid in the monoidal category related to the operation of product, we can define a Möbius species M^{-1} whose Möbius generating function is the inverse with respect to the product of formal power series of the generating function of M. A similar procedure can be applied to the operations of sum and substitution in order to define the Möbius inverses with respect to each of these operations. A c-monoid in the monoidal category of set species with respect to the operation of substitution is a cancellative set operad (*c-operad* from now on). In this way we introduced there many c-operads and their associated families of partially ordered sets, some of them rediscovered in the more recent literature. For example: the c-operad of pointed sets $E^{\bullet}$ that induces the poset of pointed partitions [26, Example 3.13], studied in [9] (the permutative operad). We also introduced the c-operad of M-enriched rooted trees $\mathscr{A}_M$, M being a c-monoid with respect to the product (see [25, Section 5] and [26, Section 3]). When M is the species of sets E, $\mathscr{A}_E = \mathscr{A}$ is the permutative non-associative operad [29]. We proved by bijective methods that the Möbius species $\mathscr{A}_M^{\langle -1\rangle}$ (substitutional inverse of $\mathscr{A}_M$) and XM^{-1} have the same Möbius valuation. In this way we related the substitutional Möbius inverse of $\mathscr{A}_M$ with the multiplicative Möbius inverse of M, prefiguring the main result of this paper (see Theorem 63). This result has an intimate connection with the Lagrange inversion formula and the combinatorial proofs given by G. Labelle [39] and W. Chen [11]. Its corollary (Corollary 64) extends a recent result of Chapoton and Livernet [10].

The following natural step in this program is to extend this procedure of inversion from set to tensor species [20], Koszul duality for operads can be seen as a way of doing that for the operation of substitution. The Möbius function is the Euler characteristic of the complex associated to the chains in a partially ordered set [33,6]. As a matter of fact B. Vallette rediscovered recently the construction of posets from c-operads [40] and proved that a quadratic c-operad is Koszul if and only if all the posets in the associated inverse Möbius species are Cohen–Macaulay. In this vein, we introduce here the notion of Koszul duality for quadratic monoids with respect to the product of species, and prove an analogous result for quadratic c-monoids (see Theorem 41).

Given a Koszul monoid, we obtain a family of Koszul algebras in the sense of Priddy [31], by taking the corresponding analytic functor. By the Schur correspondence we can go back in this construction. In this way we can translate many classical results about Koszul duality for associative algebras to this context. However, the present approach has the advantage of embodying Koszul duality for associative algebras into the realm of representation theory of the symmetric groups and symmetric functions.

In a forthcoming paper we shall explore the connections with Hopf algebras induced by Hopf monoids as in [1] and with the antipode of incidence Hopf algebras induced by c-operads as in [25].

As a conclusion we can say that from the point of view of a combinatorialist, Koszul duality, for operads or for product monoids, is a sophisticated method of defining the inverses of some 'good' generating functions. An inversion procedure that goes deeper from the generating function, into the structure of the algebraic and combinatorial objects that it enumerates. For example, André's generating function for alternating permutations of even length ($\sec(x)$) [2] can be obtained as the generating function of dimensions of the Koszul dual of a c-monoid introduced in [25]. The generating function for alternating permutations of odd length ($\tan(x)$) is associated in the same way to the Koszul dual of a module on this c-monoid. Moreover this construction gives, by means of the orthogonal relations of Koszul duality, the ribbon representations of the symmetric groups studied in [15], and in [14, Theorem 6.1] (see also [36]). These generating functions as well as the character generating functions given in [7], are instances of the inversion formulas associated to Koszul duality (Proposition 20). The same procedure can be applied to the combinatorial interpretation of the Bessel function $J_0(2x)$ given in [8] (see also [28]), and its modules, whose graded generating functions corresponds to the Bessel functions on two parameters $J_{r,s}(2x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+r+s}}{(k+r)!(k+s)!}$. These and many other classical and new enumerative formulas will be presented in a forthcoming paper.

2. Set, tensor, g-tensor, and dg-tensor species

Let $\mathbb{K}$ be a field of characteristic zero. Let $\mathrm{Vec}_{\mathbb{K}}$ be the category whose objects are vector spaces over $\mathbb{K}$ and linear transformations as morphisms. Denote by $\mathrm{gVec}_{\mathbb{K}}$ the category whose objects are graded vector spaces $V^{\bullet} = \bigoplus_{i\in\mathbb{Z}} V^i$ and morphisms are linear maps preserving the grading. $\mathrm{gVec}_{\mathbb{K}}$ is a symmetric monoidal category with respect to the tensor product of graded vector spaces

$$(V^{\bullet} \otimes W^{\bullet})^n = \bigoplus_{i+j=n} V^i \otimes W^j. \tag{1}$$

The symmetry isomorphism is given by

$$v \otimes w \mapsto (-1)^{\deg(v)\deg(w)} w \otimes v. \tag{2}$$

Denote by $\mathrm{dgVec}_{\mathbb{K}}$ the category of differential graded vector spaces, or complex over $\mathbb{K}$. The objects of $\mathrm{dgVec}_{\mathbb{K}}$ are pairs $(V^{\bullet}, d)$ where $V^{\bullet}$ is a graded vector space, and $d : V^{\bullet} \to V^{\bullet}$ is a linear map of degree 1 satisfying $d^2 = 0$. The tensor product $(V^{\bullet}, d_1) \otimes (W^{\bullet}, d_2)$ is defined to be $(V^{\bullet} \otimes W^{\bullet}, d)$, where $V^{\bullet} \otimes W^{\bullet}$ is as in Eq. (1) and d is given by

$$d(v \otimes w) = d_1(v) \otimes w + (-1)^{\deg(v)} v \otimes d_2(w). \tag{3}$$

The direct sum of objects in the categories $\mathrm{gVec}_{\mathbb{K}}$ and $\mathrm{dgVec}_{\mathbb{K}}$ is defined in a trivial way. The cohomologies of a dg-vector space $(V^{\bullet}, d)$, $H^i(V^{\bullet}, d) = \operatorname{Ker} d^i / \operatorname{Im} d^{i-1}$, $i \in \mathbb{Z}$, defines a functor

$$H : \mathrm{dgVec}_{\mathbb{K}} \to \mathrm{gVec}_{\mathbb{K}}, \qquad H(V^{\bullet}, d) := \bigoplus_{i\in\mathbb{Z}} H^i(V^{\bullet}, d). \tag{4}$$

It preserves tensor products:

$$H\big((V^{\bullet}, d_1) \otimes (W^{\bullet}, d_2)\big) = H(V^{\bullet}, d_1) \otimes H(W^{\bullet}, d_2). \tag{5}$$

Denote by $\mathcal{F}$ the category whose objects are finite sets and whose morphisms arbitrary functions. Let $\mathbb{B}$ be the groupoid subjacent in $\mathcal{F}$, the objects of $\mathbb{B}$ are finite sets and the morphisms are bijective functions.

Definition 1. When clear from the context, the common term 'species' will designate a covariant functor from $\mathbb{B}$ to $\mathcal{V}$, where $\mathcal{V}$ is either of the categories $\mathcal{F}$, $\mathrm{Vec}_{\mathbb{K}}$, $\mathrm{gVec}_{\mathbb{K}}$, or $\mathrm{dgVec}_{\mathbb{K}}$. Specifically, we shall call $M : \mathbb{B} \to \mathcal{V}$ a set, tensor, **g**-tensor, or a **dg**-tensor species if its codomain category $\mathcal{V}$ is respectively $\mathcal{F}$, $\mathrm{Vec}_{\mathbb{K}}$, $\mathrm{gVec}_{\mathbb{K}}$, or $\mathrm{dgVec}_{\mathbb{K}}$. For a species F, we denote by $F[U]$ the image under F of the finite set U in the corresponding category. In this article we only deal with finite-dimensional (tensor, **g**-tensor, and **dg**-tensor) species, i.e., species where $F[U]$ is a finite-dimensional vector space for every U in $\mathbb{B}$. Similarly, for a bijection between finite sets $\sigma : U \to V$, $F[\sigma]$ will denote the corresponding isomorphism $F[\sigma] : F[U] \to F[V]$.

A morphism $\alpha : F \to G$ between two species of the same kind, is a natural transformation between F and G as functors. Two isomorphic species F and G will be considered as equal and we write $F = G$. Unless otherwise stated, for $k \geqslant 0$, F_k will denote the subspecies of F concentrated on sets of cardinality k. The (left) truncated species $\sum_{j \geqslant k} F_j$ will be denoted by F_{k^+}. We also denote the species F truncated in 1, F_{1^+}, by F_+. By a standard construction, a tensor species T is equivalent to a sequence $\{T[n]\}_{n=0}^{\infty}$ of representations of the symmetric groups (see [20]). Let μ and λ be two partitions such that $\mu \subseteq \lambda$. We denote by $\mathcal{S}_{\lambda/\mu}$ the Specht representation corresponding to the skew shape λ/μ, as well as its corresponding tensor species.

Given a set species M, we can construct a tensor species by composing with the functor l (linear span), that goes from $\mathcal{F}$ to $\mathrm{Vec}_{\mathbb{K}}$, and sends each finite set S to $\mathbb{K} \cdot S$, the free $\mathbb{K}$-vector space generated by S. The category $\mathrm{Vec}_{\mathbb{K}}$ can be thought of as subcategory of $\mathrm{gVec}_{\mathbb{K}}$ or of $\mathrm{dgVec}_{\mathbb{K}}$, by considering a vector space as a graded vector space concentrated in degree zero in the first case, or as a trivial complex concentrated in degree zero in the second case. Similarly, the category $\mathrm{gVec}_{\mathbb{K}}$ is naturally imbedded into $\mathrm{dgVec}_{\mathbb{K}}$ by providing a graded vector space with the zero differential. Then, a tensor species can be thought of either as a **g**-tensor species or as a **dg**-tensor species, and a **g**-tensor species as a **dg**-tensor species. In other words, we have the following category imbeddings

$$\mathcal{F}^{\mathbb{B}} \subset \mathrm{Vec}_{\mathbb{K}}^{\mathbb{B}} \subset \mathrm{gVec}_{\mathbb{K}}^{\mathbb{B}} \subset \mathrm{dgVec}_{\mathbb{K}}^{\mathbb{B}}. \tag{6}$$

In this article we frequently denote the classical set species and their corresponding tensor, **g**-tensor, and **dg**-tensor species with the same symbol.

2.1. Operations on species

Definition 2. Let F and G be two species of the same kind. We define the operations of sum, product, Hadamard product, substitution and derivative,

$$(F + G)[U] := F[U] \oplus G[U], \tag{7}$$

$$(F \cdot G)[U] := \bigoplus_{U_1 \uplus U_2 = U} F[U_1] \otimes G[U_2], \tag{8}$$

$$(F \odot G)[U] := F[U] \otimes G[U], \tag{9}$$

$$F(G)[U] = (F \circ G)[U] := \bigoplus_{\pi \in \Pi[U]} \left(\bigotimes_{B \in \pi} G[B] \right) \otimes F[\pi], \tag{10}$$

$$DF[U] = F'[U] := F\big[U \uplus \{\star\}\big], \tag{11}$$

$$D^k F[U] := F\big[U \uplus \{\star_1, \star_2, \dots, \star_k\}\big], \tag{12}$$

$$F^{\bullet}[U] := \mathbb{K} \cdot U \otimes F[U] \tag{13}$$

with $\{\star\}$ standing for any one-element set, and $\{\star_1, \star_2, \dots, \star_k\}$ for any k-element set.

A family of species F_i, $i \in I$, is said to be summable if for every finite set U, $F_i[U]=0$ for almost every $i \in I$. We can then define the sum $\sum_{i\in I} F_i$ by

$$\left(\sum_{i\in I} F_i\right)[U] = \bigoplus_{i:F_i[U]\neq 0} F_i[U]. \tag{14}$$

These definitions are valid for all the kinds of species studied here, the tensor product has to be interpreted in the corresponding category. In the case of set species, tensor product has to be interpreted as cartesian product, and direct sum as disjoint union. In the substitution (Eq. (10)), $\Pi[U]$ is the set of partitions of the set U, and we require that $G[\emptyset]=0$. The tensor product over the blocks of a partition in the right-hand side of it has to be interpreted as an unordered tensor product in the corresponding monoidal category. Because all the monoidal categories in consideration are symmetric, unordered tensor products have a precise meaning in each context as coinvariants under the action of the symmetric group:

$$\bigotimes_{i\in I} V_i := \left(\bigoplus_{i_1,i_2,\ldots,i_k} V_{i_1}\otimes V_{i_2}\otimes\cdots\otimes V_{i_k}\right)_{S_k}, \tag{15}$$

where the direct sum is taken over all the total orderings of the set I.

A **g**-tensor species could be thought of as a summable family of tensor species $\{F^{\underline{k}} \mid k\in\mathbb{Z}\}$. We usually denote it with the symbol $F^{\underline{g}}$, $F^{\underline{g}} = \sum_{k\in\mathbb{Z}} F^{\underline{k}}$. In the same vein, a **dg**-tensor species could be thought of as a summable family as above, plus a family of natural transformations $d=\{d_k\}_{k\in\mathbb{Z}}$, $d_k : F^{\underline{k}} \to F^{\underline{k+1}}$, $k\in\mathbb{Z}$, such that $d_{k+1}\circ d_k = 0$, for every $k\in\mathbb{Z}$.

Definition 3 *(Dual species).* Let F be a tensor species. The dual F^* of F is defined by

$$F^*[U] = \big(F[U]\big)^*, \quad \text{for every finite set } U, \tag{16}$$

$$F^*[f]h = h\circ F\big[f^{-1}\big], \quad \text{for every bijection } f: U\to V \text{ and every } h\in F^*[U]. \tag{17}$$

For a **g**-tensor species $F^{\underline{g}}$, the dual **g**-species $(F^{\underline{g}})^*$ is obtained by dualizing each component and reversing the grading: $(F^{\underline{g}})^{*\underline{k}} := (F^{\underline{-k}})^*$, $k\in\mathbb{Z}$. For a **dg**-tensor species $(F^{\underline{g}}, d)$, its dual is defined to be $((F^{\underline{g}})^*, d^*)$, where d_k^* is the adjoint of d_{-k-1}, $k\in\mathbb{Z}$.

Example 4. Observe that the species 1 (empty set indicator) and X (see [4] for definitions), are respectively the identities for the operations of product and substitution. We have that $1\cdot F = F\cdot 1 = F$ for every species F, and $F(X)=X(F)=F$ for every species such that $F[\emptyset]=0$.

For a tensor species F, the following two generating functions are defined: the generating function of the dimensions and the Frobenius character (see [20]),

$$F(x) = \sum_{n=0}^{\infty} \dim\big(F[n]\big)\frac{x^n}{n!}, \tag{18}$$

$$\mathrm{Ch}\,F(\mathbf{x}) = \sum_{n=0}^{\infty}\sum_{\alpha\vdash n} \mathrm{tr}\,F[\alpha]\frac{p_\alpha(\mathbf{x})}{z_\alpha}. \tag{19}$$

In the right-hand side of Eq. (19) we use Macdonald's notation for the power sum symmetric function (see [23]), as in the rest of this article when dealing with symmetric functions. When restricted to set species, we get the exponential generating function of the cardinals $|F[n]|$ in the first case, and the cycle index series $Z_F(p_1, p_2, \ldots)$ in the second case. These definitions are easily extended to **g** and **dg**-tensor species, by taking respectively the Euler–Poincaré characteristic and the alternating sum of traces

$$\chi\left(F^{\underline{g}}[n]\right) = \sum_{i\in\mathbb{Z}} (-1)^i \dim F^i[n], \qquad \operatorname{tr} F^{\underline{g}}[\alpha] := \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{tr} F^i[\alpha] \tag{20}$$

instead of $\dim F[n]$, and of $\operatorname{tr} F[\alpha]$. More explicitly, we have

$$F^{\underline{g}}(x) = \sum_{k\in\mathbb{Z}} (-1)^k F^{\underline{k}}(x), \tag{21}$$

$$\operatorname{Ch} F^{\underline{g}}(\mathbf{x}) = \sum_{k\in\mathbb{Z}} (-1)^k \operatorname{Ch} F^{\underline{k}}(\mathbf{x}). \tag{22}$$

The operations of above are preserved by taking generating functions;

$$(F+G)(x) = F(x) + G(x), \tag{23}$$

$$\operatorname{Ch}(F+G)(\mathbf{x}) = \operatorname{Ch} F(\mathbf{x}) + \operatorname{Ch} G(\mathbf{x}), \tag{24}$$

$$(F\cdot G)(x) = F(x)G(x), \tag{25}$$

$$\operatorname{Ch}(F\cdot G)(\mathbf{x}) = \operatorname{Ch} F(\mathbf{x}) \cdot \operatorname{Ch} G(\mathbf{x}), \tag{26}$$

$$(F \odot G)(x) = F(x) \odot G(x), \tag{27}$$

$$\operatorname{Ch}(F \odot G)(\mathbf{x}) = \operatorname{Ch} F(\mathbf{x}) \odot \operatorname{Ch} G(\mathbf{x}), \tag{28}$$

$$F(G)(x) = F\big(G(x)\big), \tag{29}$$

$$\operatorname{Ch} F(G)(\mathbf{x}) = \operatorname{Ch} F(\mathbf{x}) * \operatorname{Ch} G(\mathbf{x}), \tag{30}$$

$$DF(x) = F'(x), \tag{31}$$

$$\operatorname{Ch} DF(\mathbf{x}) = \frac{\partial \operatorname{Ch} F(\mathbf{x})}{\partial p_1}, \tag{32}$$

$$F^{\bullet}(x) = xF'(x), \tag{33}$$

$$\operatorname{Ch} F^{\bullet}(\mathbf{x}) = p_1(\mathbf{x}) \frac{\partial \operatorname{Ch} F(\mathbf{x})}{\partial p_1}. \tag{34}$$

In the right-hand side of Eqs. (27) and (28), $\odot$ means respectively Hadamard product of exponential formal power series and internal product of symmetric functions. In the right-hand side of Eq. (30), $*$ means plethysm of symmetric functions.

Example 5. The set species 1 and X have as generating functions $1(x) = 1$, $\operatorname{Ch} 1(\mathbf{x}) = 1$, $X(x) = x$, and $\operatorname{Ch} X(\mathbf{x}) = p_1(\mathbf{x})$. The set species E (exponential, or uniform species), and the species $\mathbb{L} = \sum_{k\geqslant 0} X^k$ of totally ordered sets (see [4]) have as generating functions

$$E(x) = e^x, \qquad \operatorname{Ch} E(\mathbf{x}) = \sum_{n=0}^{\infty} h_n(\mathbf{x}) = \prod_{n=1}^{\infty} \frac{1}{1-x_n}, \tag{35}$$

$$\mathbb{L}(x) = \frac{1}{1-x}, \qquad \operatorname{Ch}\mathbb{L}(\mathbf{x}) = \frac{1}{1-p_1(\mathbf{x})} = \frac{1}{1-\sum_{n=1}^{\infty} x_n}. \tag{36}$$

Example 6. The tensor species Λ, defined by

$$\Lambda[U] = \bigwedge^{|U|}(\mathbb{K} \cdot U), \quad U \in \mathbb{B}, \tag{37}$$

has as generating functions

$$\Lambda(x) = e^x \quad \text{and} \quad \operatorname{Ch}\Lambda(\mathbf{x}) = \sum_{k=0}^{\infty} e_k(\mathbf{x}) = \prod_{n=1}^{\infty}(1 + x_n). \tag{38}$$

Definition 7 *(The functor $\mathcal{H}$).* Let F be a **dg**-tensor species. Let $\mathcal{H}F = H \circ F$ be the **g**-tensor species defined as the functorial composition of the cohomology functor H with F. $\mathcal{H}$ is a functor from the category $\mathrm{dgVec}_{\mathbb{K}}{}^{\mathbb{B}}$ of **dg**-tensor species, to the category $\mathrm{gVec}_{\mathbb{K}}{}^{\mathbb{B}}$ of **g**-tensor species.

It is not difficult to check, since H preserves tensor products (Eq. (5)), that $\mathcal{H}$ preserves all the operations in Definition 2:

$$\mathcal{H}(F+G) = \mathcal{H}F + \mathcal{H}G, \qquad \mathcal{H}(F \cdot G) = \mathcal{H}F \cdot \mathcal{H}G,$$
$$\mathcal{H}(F \odot G) = \mathcal{H}F \odot \mathcal{H}G, \qquad \mathcal{H}\big(F(G)\big) = \mathcal{H}F(\mathcal{H}G). \tag{39}$$

Because of the identities

$$\chi\left(F^{\underline{g}}[n]\right) = \chi\left(H\left(F^{\underline{g}}[n]\right)\right) \quad \text{and} \quad \operatorname{tr} F^{\underline{g}}[\alpha] = \operatorname{tr} HF^{\underline{g}}[\alpha], \tag{40}$$

the generating functions (18), and (19) are also preserved by $\mathcal{H}$,

$$\mathcal{H}M(x) = M(x), \qquad \operatorname{Ch}\mathcal{H}M(\mathbf{x}) = \operatorname{Ch} M(\mathbf{x}). \tag{41}$$

3. Additive, multiplicative and substitutional inverses

In this section, by working in the general context of **dg**-tensor species, we introduce procedures to define the "inverse" of a species, for each of the fundamental operations of sum, product, and substitution. In each case, these procedures of inversion behave well with respect to generating functions. The goal of this is the construction of a theoretical framework to place the notion of Koszul duality in the appropriate context of the problem of "inverting a species".

Denote by s^m, $m \in \mathbb{Z}$, the shift endofunctor of the category $\mathrm{dgVec}_{\mathbb{K}}$; $s^m : \mathrm{dgVec}_{\mathbb{K}} \to \mathrm{dgVec}_{\mathbb{K}}$ defined by $(s^m(V^{\bullet}))^i = V^{i+m}$, $s^m(d_{V^{\bullet}}) = (-1)^m d_{V^{\bullet}}$. The shift can also be defined (by restriction) as an endofunctor of $\mathrm{gVec}_{\mathbb{K}}$.

Definition 8 *(Additive inverses).* Let F be a species of any kind. F can be always thought of as a **dg**-tensor species or as a **g**-tensor species. There are two kinds of 'additive inverses' of F

$$\leftharpoondown F := s^{-1} \circ F, \tag{42}$$

$$\rightharpoondown F := s^{1} \circ F. \tag{43}$$

Clearly, $(\leftharpoondown F)(x) = -F(x) = (\rightharpoondown F)(x)$ and $\operatorname{Ch}(\leftharpoondown F)(\mathbf{x}) = \operatorname{Ch}(\rightharpoondown F)(\mathbf{x}) = -\operatorname{Ch} F(\mathbf{x})$. There are two other interesting ways of shifting F. Define $F^{\leftarrow}$ and $F^{\rightarrow}$ as follows,

$$F^{\leftarrow} := \sum_{k=0}^{\infty} s^{-k} \circ F_k, \tag{44}$$

$$F^{\rightarrow} := \sum_{k=0}^{\infty} s^{k} \circ F_k. \tag{45}$$

Proposition 9. *We have the identities,*

$$F(\leftharpoonup X) = \Lambda^{\leftarrow} \odot F, \tag{46}$$

$$F(\rightharpoonup X) = \Lambda^{\rightarrow} \odot F. \tag{47}$$

Proof. Let us prove Eq. (46), the proof of (47) is similar. By the definition of substitution we have that

$$\begin{aligned} F(\leftharpoonup X)[U] &= \left(\bigotimes_{b\in U} (\leftharpoonup X)\big[\{b\}\big] \right) \otimes F[U] = \left(\bigotimes_{b\in U} s^{-1}\mathbb{K}\cdot\{b\} \right) \otimes F[U] \\ &= \left(s^{-|U|}\Lambda[U]\right) \otimes F[U]. \end{aligned} \tag{48}$$

The last step because we have the isomorphism (see [17, Lemma 3.2.9])

$$\bigotimes_{j\in J} s^{-1} V_j^{\bullet} = s^{-|J|} \left(\bigotimes_{j\in J} V_j^{\bullet} \right) \otimes \Lambda[J]. \qquad \square \tag{49}$$

Their generating functions are:

$$F(\leftharpoonup X)(x) = F(\rightharpoonup X)(x) = \sum_{k=0}^{\infty} (-1)^k F_k(x) = F(-x), \tag{50}$$

$$\operatorname{Ch} F(\leftharpoonup X)(\mathbf{x}) = \operatorname{Ch} F(\rightharpoonup X)(\mathbf{x}) = \sum_{k=0}^{\infty} (-1)^k \operatorname{Ch} F_k(\mathbf{x}) \odot e_k(\mathbf{x}) = \operatorname{Ch} F(-\mathbf{x}) \odot \sum_{k=0}^{\infty} e_k(\mathbf{x}). \tag{51}$$

Example 10 *(Koszul complexes).* We have the identities

$$E(\leftharpoonup X) = \Lambda^{\leftarrow} \odot E = \Lambda^{\leftarrow}, \qquad E(\rightharpoonup X) = \Lambda^{\rightarrow} \odot E = \Lambda^{\rightarrow}. \tag{52}$$

Let $P = \leftharpoonup X + X$ and $Q = \rightharpoonup X + X$ be the **dg**-tensor species with differential $d_{-1} : X = P^{\underline{-1}} \to P^{\underline{0}} = X$ (resp. $d'_0 : X = Q^{\underline{0}} \to Q^{\underline{1}} = X$) in each case the trivial isomorphism and zero for $k \neq -1$ (resp. $k \neq 0$). As **g**-tensor species $E(P)$ and $E(Q)$ are respectively equal to

$$E(\leftharpoonup X + X) = E(\leftharpoonup X) \cdot E = \Lambda^{\leftarrow} \cdot E, \tag{53}$$

$$E(\rightharpoonup X + X) = E \cdot E(\rightharpoonup X) = E \cdot \Lambda^{\rightarrow}. \tag{54}$$

As **dg**-tensor species, the differentials that come from P and Q by using the definition of substitution of species are easily seen to be:

$$d_k : \big((\Lambda^{\leftarrow})^{\underline{k}} \cdot E\big)[U] \to \big((\Lambda^{\leftarrow})^{\underline{k+1}} \cdot E\big)[U], \quad k = -|U|, \ldots, 0, \tag{55}$$

$$d_k(a_1 \wedge \cdots \wedge a_k \otimes U_2) = \sum_{i=1}^{k} (-1)^{i-1} a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_k \otimes U_2 \cup \{a_i\}, \tag{56}$$

$$d'_k : \big(E \cdot (\Lambda^{\rightarrow})^{\underline{k}}\big)[U] \to \big(E \cdot (\Lambda^{\rightarrow})^{\underline{k+1}}\big)[U], \quad k = 0, \ldots, |U|, \tag{57}$$

$$d'_k(U_1 \otimes a_1 \wedge a_2 \wedge \cdots \wedge a_k) = \sum_{a \in U_1} \big(U_1 - \{a\}\big) \otimes a \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_k. \tag{58}$$

By Eq. (39), $\mathcal{H}E(\leftharpoonup X + X) = \mathcal{H}E(\rightharpoonup X + X) = E(0) = 1$.

Let F be of the form $F = 1 + F_+$. The inverse of its generating function is

$$F(x)^{-1} = \big(1 + F_+(x)\big)^{-1} = 1 - F_+(x) + F_+(x)^2 - \cdots = \mathbb{L}\big(-F_+(x)\big). \tag{59}$$

This motivates the following definition.

Definition 11 *(Multiplicative inverses).* Let F be as above. We define two 'multiplicative inverses' of F,

$$F^{\leftharpoonup 1} := \mathbb{L}(\leftharpoonup F_+), \tag{60}$$

$$F^{\rightharpoonup 1} := \mathbb{L}(\rightharpoonup F_+). \tag{61}$$

Obviously,

$$F^{\leftharpoonup 1}(x) = F^{\rightharpoonup 1}(x) = F(x)^{-1}, \tag{62}$$

$$\mathrm{Ch}\, F^{\leftharpoonup 1}(\mathbf{x}) = \mathrm{Ch}\, F^{\rightharpoonup 1}(\mathbf{x}) = \sum_{k=0}^{\infty} (-1)^k \big(\mathrm{Ch}\, F_+(\mathbf{x})\big)^k = \big(\mathrm{Ch}\, F(\mathbf{x})\big)^{-1}. \tag{63}$$

Denote by $\mathscr{F}$ the set species of commutative Schröder trees, or generalized commutative parenthesizations. It satisfies the implicit equation

$$\mathscr{F} = X + E_{2^+}(\mathscr{F}). \tag{64}$$

The structures of $\mathscr{F}[U]$ are rooted trees whose leaves are labelled with the elements of U, and whose internal vertices, each one with at least two sons, are unlabelled. More generally, let G be a species of the form $G = X + G_{2^+}$. The species of G-enriched Schröder trees (see for example [13] or [21]) is the solution (fixed point) of the implicit equation

$$\mathscr{F}_G = X + G_{2^+}(\mathscr{F}_G). \tag{65}$$

$\mathscr{F}_G$ is the free operad generated by G. This kind of structures should be called Hipparchus trees, who computed in the second century B.C. the number of composed propositions out of 10 simple propositions (see [37, p. 213], and [38]). In modern combinatorial language, the number of generalized non-commutative parenthesizations (or bracketings) using 10 undistinguishable symbols. In our notation: $|\mathscr{F}_{\mathbb{L}_+}[10]|/10!$. The general problem of enumerating non-commutative bracketings was solved by Schröder in [34].

Let us introduce some notation in order to give an explicit description of the object $\mathscr{F}_G[U]$, for a finite set U. For a tree $t \in \mathscr{F}[U]$, we denote by $\mathrm{Iv}(t)$ the set of internal vertices of t. For a vertex

$v \in \mathrm{Iv}(t)$ denote by t_v the subtree of t having v as a root and as vertices all the descendants of v in t. Let U_v be the set of leaves of t_v. We shall identify the internal (unlabelled) vertex v with U_v. The set U_v will be used as a label for v. In the same way we identify $\mathrm{Iv}(t)$ with the set $\{U_v \mid v \in \mathrm{Iv}(t)\}$. Let $\{v_1, v_2, \dots, v_k\}$ be the set of sons of v in t. We denote by π_v the partition $\{U_{v_i} \mid i = 1, 2, \dots, k\}$ of the set U_v of leaves of t_v, generated by the leaves of the set of trees $\{t_{v_i} \mid i = 1, 2, \dots, k\}$ attached to v. Consistently, we identify the partition π_v with the set of sons of v. The object $\mathscr{F}_G[U]$ is explicitly given by the formula

$$\mathscr{F}_G[U] = \bigoplus_{t \in \mathscr{F}[U]} \bigotimes_{v \in \mathrm{Iv}(t)} G_{2^+}[\pi_v]. \tag{66}$$

$\mathscr{F}_G$ has a natural grading $\mathscr{F}_G^g = \sum_{k=0}^{\infty} \mathscr{F}_G^k$, where $\mathscr{F}_G^0 = X$ and for $k \geqslant 1$, $\mathscr{F}_G^k$ is the species of the G-enriched Shröder trees with exactly k internal vertices.

From Eq. (65), the generating function $\mathscr{F}_G(x)$ is the solution of the implicit equation

$$\mathscr{F}_G(x) = x + G_{2^+}\big(\mathscr{F}_G(x)\big). \tag{67}$$

Then, we have that

$$\mathscr{F}_G(x) - G_{2^+}\big(\mathscr{F}_G(x)\big) = x, \tag{68}$$

and equivalently, that the substitutional inverse $(x - G_{\geqslant 2}(x))^{\langle -1 \rangle}$ of $(x - G_{2^+}(x))$, is $\mathscr{F}_G(x)$. Similarly, the Frobenius character, $\mathrm{Ch}\,\mathscr{F}_G(\mathbf{x})$ is the plethystic inverse of $(p_1(\mathbf{x}) - \mathrm{Ch}\,G_{2^+}(\mathbf{x}))$;

$$\mathrm{Ch}\,\mathscr{F}_G(\mathbf{x}) = \big(p_1(\mathbf{x}) - \mathrm{Ch}\,G_{2^+}(\mathbf{x})\big)^{\langle -1 \rangle}. \tag{69}$$

This motivates the following definition.

Definition 12 *(Substitutional inverse).* Let G be a species as above. We define two 'substitutional inverses' of G,

$$G^{\langle \leftharpoonup 1 \rangle} := \mathscr{F}_{\leftharpoonup G}, \tag{70}$$

$$G^{\langle \rightharpoonup 1 \rangle} := \mathscr{F}_{\rightharpoonup G}. \tag{71}$$

It is easy to prove that

$$G^{\langle \leftharpoonup 1 \rangle}(x) = G^{\langle \rightharpoonup 1 \rangle}(x) = G^{\langle -1 \rangle}(x), \tag{72}$$

and that

$$\mathrm{Ch}\,G^{\langle \leftharpoonup 1 \rangle}(\mathbf{x}) = \mathrm{Ch}\,G^{\langle \rightharpoonup 1 \rangle}(\mathbf{x}) = \big(\mathrm{Ch}\,G(\mathbf{x})\big)^{\langle -1 \rangle}. \tag{73}$$

See Fig. 1 for a graphical representation of the right shifted substitutional inverse.

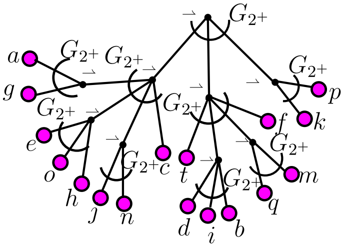

Fig. 1. Graphical representation of a typical structure of the (right shifted) substitutional inverse of $G = X + G_{2^+}$.

4. Monoidal categories on species

The category $\mathcal{V}^{\mathbb{B}}$ is monoidal with respect to the product of species, having 1 as identity object. A monoid in $(\mathcal{V}^{\mathbb{B}}, \cdot, 1)$ is a species M together with morphisms $\nu : M \cdot M \to M$ and $e : 1 \to M$, satisfying the associative and unity properties (see [24] for the general definition of monoidal categories and of a monoid in a monoidal category). In this article we only consider monoids of the form $M = 1 + M_+$. Observe that in that case the morphism $e : 1 \to M$ is unique. Then, its monoidal structure is completely determined by giving the morphism $\nu : M \cdot M \to M$ satisfying the associative and identity properties:

For every finite set U and $m_{U_1} \otimes m_{U_2} \otimes m_{U_3} \in (M.M.M)[U]$

$$\nu\big(\nu(m_{U_1} \otimes m_{U_2}) \otimes m_{U_3}\big) = \nu\big(m_{U_1} \otimes \nu(m_{U_2} \otimes m_{U_3})\big). \tag{74}$$

For $e \otimes m \in (M_0 \cdot M)[U] = (1 \cdot M)[U]$, and $m \otimes e \in (M \cdot M_0)[U] = (M \cdot 1)[U]$ we have

$$\nu(e \otimes m) = \nu(m \otimes e) = m. \tag{75}$$

Given a monoid (M, ν), a species N is called a (right) M-module if there exists a right action $\tau : N.M \to N$, of M on N that satisfies the associative and identity properties. For every finite set U and $n_{U_1} \otimes m_{U_2} \otimes m_{U_3} \in (N.M.M)[U]$ we have

$$\tau\big(\tau(n_{U_1} \otimes m_{U_2}) \otimes m_{U_3}\big) = \tau\big(n_{U_1} \otimes \nu(m_{U_2} \otimes m_{U_3})\big). \tag{76}$$

For $n \otimes e \in (N \cdot M_0)[U] = (N \cdot 1)[U]$

$$\tau(n \otimes e) = n. \tag{77}$$

In a standard way we define the dual notions of comonoid and comodule with respect to a comonoid. See [1] for an extensive treatment of monoids, comonoids, and Hopf monoids in the monoidal category of species with respect to the product.

Let us denote by $\mathcal{V}_+^{\mathbb{B}}$ the category of species F satisfying $F[\emptyset] = 0$. $\mathcal{V}_+^{\mathbb{B}}$ is a monoidal category with respect to the substitution of species having X as identity.

An *operad* $(\mathscr{O}, \eta, e)$ is defined to be a monoid in the monoidal category $(\mathcal{V}_+^{\mathbb{B}}, X, \circ)$. Equivalently, $\eta : \mathscr{O}(\mathscr{O}) \to \mathscr{O}$ and $e : X \to \mathscr{O}$ are morphisms in $\mathcal{V}_+^{\mathbb{B}}$ (natural transformations) that satisfy the associativity and identity properties. $\mathscr{O}$ will be called a set-operad when $\mathcal{V} = \mathcal{F}$. Similarly, $\mathscr{O}$ will be called an operad, a **g**-operad or a **dg**-operad if the category $\mathcal{V}$ is respectively $\text{Vec}_{\mathbb{K}}$, $\text{gVec}_{\mathbb{K}}$ or $\text{dgVec}_{\mathbb{K}}$.

We only consider here operads of the form $\mathscr{O} = X + \mathscr{O}_{2^+}$, whose structure is completely determined by giving the morphism $\eta : \mathscr{O}(\mathscr{O}) \to \mathscr{O}$.

4.1. Koszul duality for quadratic monoids and modules

Definition 13 *(Quadratic monoids).* Let F be a tensor species such that $F[\emptyset]=\emptyset$. The species

$$\mathbb{L}(F)=1+F+F^2+F^3+\cdots$$

is the free monoid generated by F. Let R be a subspecies of F^2 and let $\mathcal{R}_M=\langle R\rangle$ be the monoid ideal generated by R in $\mathbb{L}(F)$. Explicitly

$$\mathcal{R}_M=\sum_{k=0}^{\infty}\mathcal{R}_M^k, \tag{78}$$

where

$$\mathcal{R}_M^k=\sum_{i=0}^{k-2}F^iRF^{k-2-i}\subseteq F^k. \tag{79}$$

The monoid $M=\mathbb{L}(F)/\mathcal{R}_M=1+F+F^2/R+F^3/\mathcal{R}_M^3+\cdots$, will be called the quadratic monoid with generators in F, quadratic relations in R_M, and denoted by $M=\mathcal{M}(F,R)$. There is a natural grading on the monoid M, the corresponding graded monoid will be denoted by $M^{\underline{g}}$, $M^{\underline{k}}=F^k/\mathcal{R}_M^k$.

Definition 14 *(Quadratic M-modules).* Let M be a quadratic monoid as above and G an arbitrary tensor species. The species

$$G\cdot\mathbb{L}(F)=G+G\cdot F+G\cdot F^2+\cdots$$

is the free (right) $\mathbb{L}(F)$-module generated by G. Let $R_N\subseteq G.F$ be a subspecies of $G.F$, and $\mathcal{R}_{M,N}$ the graded species

$$\mathcal{R}_{M,N}=\sum_{k=0}^{\infty}\mathcal{R}_{M,N}^k, \tag{80}$$

where

$$\mathcal{R}_{M,N}^0=0\subseteq G, \tag{81}$$

$$\mathcal{R}_{M,N}^1=R_N\subseteq G\cdot F, \tag{82}$$

$$\mathcal{R}_{M,N}^k=R_N\cdot F^{k-1}+G\cdot\sum_{i=0}^{k-2}F^iR_MF^{k-2-i}\subseteq G\cdot F^k,\quad k\geqslant 2. \tag{83}$$

The M-module

$$N=\big(G\cdot\mathbb{L}(F)\big)/\mathcal{R}_{M,N}=G+(G\cdot F)/R_N+\big(G\cdot F^2\big)/\mathcal{R}_{M,N}^2+\cdots \tag{84}$$

will be called the quadratic M-module generated by G with relations in R_N, and denoted by

$$N=\mathcal{M}_M(G,R_N). \tag{85}$$

N has a natural grading, the corresponding $M^{\underline{g}}$-graded module will be denoted by $N^{\underline{g}}$,

$$N^{\underline{k}} := \left(G \cdot F^{k}\right)/\mathcal{R}^{\underline{k}}_{M,N} \quad (k = 0, 1, 2, \ldots). \tag{86}$$

Definition 15 *(Quadratic duality).* Let $M = \mathcal{M}(F, R_M)$ be a quadratic monoid. Define the quadratic dual of M by

$$M^{!\cdot} := \mathcal{M}\left(F^{*}, R_M^{\perp}\right), \tag{87}$$

where $R_M^{\perp}$ is the annihilator of R_M in $(F^2)^* = (F^*)^2$. Let $N = \mathcal{M}_M(G, R_N)$ be a quadratic M-module. The quadratic dual of N is the $M^{!\cdot}$-module defined by

$$N^{!\cdot} = \mathcal{M}_{M^{!\cdot}}\left(G^{*}, R_N^{\perp}\right), \tag{88}$$

where $R_N^{\perp}$ is the annihilator of R_N in $(F \cdot G)^* = F^* \cdot G^*$. The dual $M^{¡\cdot} := (M^{!\cdot})^*$ is a comonoid. Similarly, for a quadratic M-module N, $N^{¡\cdot} := (N^{!\cdot})^*$ is an $M^{¡}$-comodule.

Observe that the species of relations R_M and R_N are respectively the kernels of

$$\nu^{\underline{2}} : F \cdot F \to M^{\underline{2}} \quad \text{and} \quad \tau^{\underline{1}} : G \cdot F \to N^{\underline{1}}.$$

Then, the orthogonal relations are respectively the images of $(\nu^{\underline{2}})^*$ and $(\tau^{\underline{1}})^*$.

Example 16. $\mathbb{L}$ is a quadratic monoid, $\nu : \mathbb{L} \cdot \mathbb{L} \to \mathbb{L}$ being the concatenation of linear orders. $\mathbb{L} = \mathcal{M}(X, 0)$, then $\mathbb{L}^{!\cdot} = \mathcal{M}(X^*, (X^*)^2) = \mathbb{L}(X^*)/\langle (X^*)^2\rangle = 1 + X^*$.

Example 17. The species E is a monoid. There is a unique natural transformation $\nu : E \cdot E \to E$ sending each product $U_1 \otimes U_2 \in (E \cdot E)[U]$, to $U_1 \uplus U_2 = U$. (E, ν) is quadratic, $E = \mathcal{M}(X, R_E)$, where

$$R_E\left[\{a, b\}\right] = \mathbb{K}\{a \otimes b - b \otimes a\} \subset X^2\left[\{a, b\}\right]. \tag{89}$$

$R_E^{\perp} = \mathbb{K}\{a^* \otimes b^* + b^* \otimes a^*\}$, and clearly its quadratic dual is Λ^*. Its monoidal structure given by $\nu' : \Lambda^* \cdot \Lambda^* \to \Lambda^*$, ν' being the concatenation of wedge monomials.

Consider the species E_{j^+}, for a fixed integer j, $j \geqslant 1$. E_{j^+} is an E-module, $\tau : E_{j^+} \cdot E \to E_{j^+}$ being the restriction of ν. Similarly, the species Λ_{j^+} is a Λ-module, with $\tau' : \Lambda_{j^+} \cdot \Lambda \to \Lambda_{j^+}$ being the restriction of ν'.

We have the following proposition.

Proposition 18. *For every $j \geqslant 1$ we have that E_{j^+} is a quadratic E-module whose dual (a Λ^*-module) is isomorphic to the following sum of Specht representations with hook shapes,*

$$E^{!\cdot}_{j^+} = \sum_{k \geqslant 0} \mathcal{S}_{(j,1^k)}. \tag{90}$$

Moreover, $(E^{!\cdot}_{j^+})^{\underline{k}} = \mathcal{S}_{(j,1^k)}$. In a similar way, Λ_{j^+} is a quadratic Λ-module. Its dual (an E^*-module) is isomorphic to a sum of Specht representations with hook shapes, as follows

$$\Lambda^{!\cdot}_{j^+} = \sum_{k \geqslant 0} \mathcal{S}_{(k,1^j)}, \tag{91}$$

$(\Lambda^{!\cdot}_{j^+})^{\underline{k}}$ being isomorphic to $\mathcal{S}_{(k,1^j)}$.

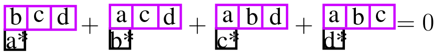

Fig. 2. Garnir relation on $\mathcal{S}_{(3,1)}[\{a,b,c,d\}]$.

Proof. The generator of E_{j+} is obviously the species E_j. Let U be a set of cardinal $j+1$. By the definition of product of tensor species, $(E_j\cdot X)[U]$ is the vector space with basis

$$\big\{\big(U-\{a\}\big)\otimes a \;\big|\; a\in U\big\}.$$

Since $\tau^{\underline{1}}((U-\{a\})\otimes a)=U-\{a\}\cup\{a\}=U$, for every $a\in U$, $R^{\perp}_{E_{j+}}[U]$ is the one-dimensional vector space generated by

$$\sum_{a\in U}\big(U-\{a\}\big)^*\otimes a^*. \tag{92}$$

The consequence of relation (92) is the Garnir element of a (dual) Specht module of shape $(j,1)$. Fig. 2 is a tableau form of the consequences of Eq. (92) for $j=3$ and $U=\{a,b,c,d\}$. Then, $(E_jX)^*/R^{\perp}_{E_{j+}}=\mathcal{S}_{(j,1)}$. For a set U, $|U|=k+j$, $(E_j\cdot X^k)[U]$ is the vector space generated by vectors of the form $U_2\otimes l$, where $l=(u_1,u_2,\dots,u_k)$ is a linear order of length k over a set U_2, $|U_2|=k$, U_1 is a set of cardinal j and $U_1\uplus U_2=U$. Recall that $\mathcal{R}^{\perp}_E=\mathcal{R}_{\Lambda^*}$ is the species of anticommuting relations. The space of relations $\mathcal{R}^{k}_{\Lambda^*,E^{!}_{j+}}[U]$ is generated by

$$U_2^*\otimes(u_1,\dots,u_i,u_{i+1},\dots,u_k)^*+U_2^*\otimes(u_1,\dots,u_{i+1},u_i,\dots,u_k)^*,\quad i=1,2,\dots,k-1, \tag{93}$$

$$U_2^*\otimes(u_1,\dots,u_k)^*+\sum_{u\in U_2}\big(U_2\cup\{u_1\}-\{u\}\big)^*\otimes(u,u_2,\dots,u_k)^*. \tag{94}$$

It is not difficult to see that $(E_j\cdot X^k)^*[U]/(\mathcal{R}^{k}_{\Lambda^*,E^{!}_{j+}}[U])$ is isomorphic to the Specht module of hook shape $(j,1^k)$. Equivalently, $(E^{!}_{j+})^{\underline{k}}$ is isomorphic to $\mathcal{S}_{(j,1^k)}$.

Eq. (91) is proved in a similar way. □

4.2. Bar and cobar constructions

Let $M=\mathcal{M}(F,R)$ be a quadratic monoid. Let $M^{\underline{g}}$ be the associated graded monoid. Consider the free **g**-comonoid generated by $\leftharpoonup(M^{\underline{g}})$

$$\mathcal{B}\mathrm{ar}(M):=\mathbb{L}\big(\leftharpoonup M^{\underline{g}}_{+}\big)=\big(M^{\underline{g}}\big)^{\leftharpoonup 1}. \tag{95}$$

For a quadratic M-module $N=\mathcal{M}_M(G,R_N)$ and $N^{\underline{g}}$ its associated graded $M^{\underline{g}}$-module, define the **g**-species $\mathcal{B}\mathrm{ar}(M,N)$ by

$$\mathcal{B}\mathrm{ar}(M,N):=N^{\underline{g}}\cdot\mathbb{L}\big(\leftharpoonup M^{\underline{g}}_{+}\big)=N^{\underline{g}}\big(M^{\underline{g}}\big)^{\leftharpoonup 1}. \tag{96}$$

From Eqs. (95) and (96) we get

$$\mathcal{B}\mathrm{ar}(M)^{\underline{k}}=\sum_{r=1}^{\infty}\Bigg(\sum_{j_1+j_2+\cdots+j_r=k+r}M^{\underline{j_1}}_{+}M^{\underline{j_2}}_{+}\dots M^{\underline{j_r}}_{+}\Bigg),\quad k\geqslant 0, \tag{97}$$

and

$$\mathcal{B}\mathrm{ar}(M,N)^{\underline{k}} = \sum_{r=1}^{\infty}\left(\sum_{j_1+j_2+\cdots+j_r+j_{r+1}=k+r} N^{\underline{j_1}} M_+^{\underline{j_2}} \dots M_+^{\underline{j_r}} M_+^{\underline{j_{r+1}}}\right), \quad k \geqslant 0. \tag{98}$$

Hence $\mathcal{B}\mathrm{ar}(M)^{\underline{0}} = \mathbb{L}(F)$ and $\mathcal{B}\mathrm{ar}(M,N)^{\underline{0}} = G \cdot \mathbb{L}(F)$.

We now transform $\mathcal{B}\mathrm{ar}(M)$ and $\mathcal{B}\mathrm{ar}(M,N)$ into **dg**-tensor species, by providing them with differentials. For a fixed finite set U let us define $d_U : \mathcal{B}\mathrm{ar}(M)[U] \to \mathcal{B}\mathrm{ar}(M)[U]$ and $d_U^N : \mathcal{B}\mathrm{ar}(M,N) \to \mathcal{B}\mathrm{ar}(M,N)$ by

$$d_U(m_1 \otimes m_2 \otimes \cdots \otimes m_r) = \sum_{i=1}^{r-1}(-1)^{i-1} m_1 \otimes m_2 \otimes \cdots \otimes \nu(m_i \otimes m_{i+1}) \otimes \cdots \otimes m_r, \tag{99}$$

$$d_U^N(n_1 \otimes m_2 \otimes \cdots \otimes m_{r+1}) = \tau(n_1 \otimes m_2) \otimes \cdots \otimes m_r \otimes m_{r+1} - n_1 \otimes d_U(m_2 \otimes \cdots \otimes m_{r+1}), \tag{100}$$

where $m_1 \otimes m_2 \otimes \cdots \otimes m_r$, is a generic decomposable element of $\mathcal{B}\mathrm{ar}(M)^{\underline{k}}[U]$ for some $k \geqslant 0$, and similarly with respect to $n_1 \otimes m_2 \otimes \cdots \otimes m_{r+1}$.

The cobar **dg**-tensor species $\mathrm{Cob}(M)$ and $\mathrm{Cob}(M,N)$ are defined to be respectively the dual of $\mathcal{B}\mathrm{ar}(M)$ and $\mathcal{B}\mathrm{ar}(M,N)$;

$$\mathrm{Cob}(M) = \mathcal{B}\mathrm{ar}(M)^* = \left(\mathbb{L}\left(\rightharpoonup\left(M^{\underline{g}}\right)^*\right), d^*\right) = \left(\left(\left(M^{\underline{g}}\right)^*\right)^{\rightharpoonup 1}, d^*\right), \tag{101}$$

$$\begin{aligned}\mathrm{Cob}(M,N) = \mathcal{B}\mathrm{ar}(M,N)^* &= \left(\left(N^{\underline{g}}\right)^* \cdot \mathbb{L}\left(\rightharpoonup\left(M^{\underline{g}}\right)^*\right), \left(d^N\right)^*\right) \\ &= \left(\left(N^{\underline{g}}\right)^* \cdot \left(\left(M^{\underline{g}}\right)^*\right)^{\rightharpoonup 1}, \left(d^N\right)^*\right).\end{aligned} \tag{102}$$

The generating functions of $\mathcal{B}\mathrm{ar}(M)$, $\mathcal{B}\mathrm{ar}(M,N)$, $\mathrm{Cob}(M)$ and $\mathrm{Cob}(M,N)$ are

$$\mathcal{B}\mathrm{ar}(M)(x) = \mathrm{Cob}(M)(x) = \left(M^{\underline{g}}(x)\right)^{-1}, \tag{103}$$

$$\mathrm{Ch}\,\mathrm{Cob}(M)(\mathbf{x}) = \mathrm{Ch}\,\mathcal{B}\mathrm{ar}(M)(\mathbf{x}) = \left(\mathrm{Ch}\,M^{\underline{g}}(\mathbf{x})\right)^{-1}, \tag{104}$$

$$\mathcal{B}\mathrm{ar}(M,N)(x) = \mathrm{Cob}(M,N)(x) = \frac{N^{\underline{g}}(x)}{M^{\underline{g}}(x)}, \tag{105}$$

$$\mathrm{Ch}\,\mathcal{B}\mathrm{ar}(M,N)(\mathbf{x}) = \mathrm{Ch}\,\mathrm{Cob}(M,N)(\mathbf{x}) = \frac{\mathrm{Ch}\,N^{\underline{g}}(\mathbf{x})}{\mathrm{Ch}\,M^{\underline{g}}(\mathbf{x})}. \tag{106}$$

From Eq. (97), $\mathrm{Cob}(M)^{\underline{-1}} = \sum_{n\geqslant 2}\sum_{i=0}^{n-2}(F^*)^i (M^{\underline{2}})^*(F^*)^{n-i-2}$, and $\mathrm{Cob}(M)^{\underline{0}} = \mathbb{L}(F^*)$. From that, the image of $(d)^*_{-1}$ is equal to

$$\begin{aligned}\mathrm{Im}(d)^*_{-1} &= \sum_{n\geqslant 2}\sum_{i=0}^{n-2}\left(F^*\right)^i \nu^*\left(M^{\underline{2}}\right)^*\left(F^*\right)^{n-i-2} = \sum_{n\geqslant 2}\sum_{i=0}^{n-2}\left(F^*\right)^i R^{\perp}\left(F^*\right)^{n-i-2} \\ &= \left\langle R^{\perp}\right\rangle \subseteq \mathbb{L}\left(F^*\right)\end{aligned} \tag{107}$$

and we have $H^0(\mathrm{Cob}(M)) = \mathbb{L}(F^*)/\langle R^{\perp}\rangle = M^{!\cdot}$. In a similar way, for N as above, we get $H^0(\mathrm{Cob}(M,N)) = N^{!\cdot}$. By duality we have $H^0(\mathcal{B}\mathrm{ar}(M)) = (M^{!\cdot})^* = M^{¡\cdot}$, $H^0(\mathcal{B}\mathrm{ar}(M,N)) = (N^{!\cdot})^* = N^{¡\cdot}$.

Observe that the bar and cobar complexes are the sum of subcomplexes

$$\mathcal{B}\text{ar}(M)=\sum_k \mathcal{B}\text{ar}^{(k)}(M),\qquad \mathcal{B}\text{ar}(M,N)=\sum_k \mathcal{B}\text{ar}^{(k)}(M,N),$$

$$\text{Cob}(M)=\sum_k \text{Cob}^{(k)}(M),\qquad \text{Cob}(M,N)=\sum_k \text{Cob}^{(k)}(M,N).$$

For example, the vectors in $\mathcal{B}\text{ar}^{(k)}(M)$ (resp. $\mathcal{B}\text{ar}^{(k)}(M,N)$) are those of degree k before the shifting:

$$0\to F^k \xrightarrow{d} \sum_{i=0}^{k-2} F^i M^{\underline{2}} F^{k-i-2} \xrightarrow{d} \cdots \xrightarrow{d} M^{\underline{k}} \to 0, \tag{108}$$

$$0\to G\cdot F^k \xrightarrow{d} N^{\underline{1}}F^{k-1}+\sum_{i=0}^{k-2} G\cdot F^i M^{\underline{2}} F^{k-i-2} \xrightarrow{d} \cdots \xrightarrow{d} N^{\underline{k}} \to 0. \tag{109}$$

Clearly $H^0\,\mathcal{B}\text{ar}^{(k)}(M)=(M^{!\cdot})^{\underline{k}}$ and $H^0\,\mathcal{B}\text{ar}^{(k)}(M,N)=(N^{!\cdot})^{\underline{k}}$.

Definition 19 *(Koszul monoids and modules).* A quadratic monoid M is said to be *Koszul* if $H^i(\mathcal{B}\text{ar}(M))=0$, $i>0$ (equivalently, if $H^i(\text{Cob}(M))=0$, $i<0$). In other words, if the **g**-tensor species $\mathcal{H}\mathcal{B}\text{ar}(M)$ and $\mathcal{H}\,\text{Cob}(M)$ are both concentrated in degree zero. For a Koszul monoid M, a quadratic M-module N is said to be Koszul if $\mathcal{H}\mathcal{B}\text{ar}(M,N)$ is concentrated in degree zero (equivalently, if $\mathcal{H}\,\text{Cob}(M,N)$ is concentrated in degree zero).

Proposition 20. *Let M be a Koszul monoid and N a Koszul M-module. Then*

$$M^{!\cdot}(x)=\left(M^{\underline{g}}(x)\right)^{-1}, \tag{110}$$

$$\text{Ch}\,M^{!\cdot}(\mathbf{x})=\left(\text{Ch}\,M^{\underline{g}}(\mathbf{x})\right)^{-1}, \tag{111}$$

$$N^{!\cdot}(x)=\frac{N^{\underline{g}}(x)}{M^{\underline{g}}(x)}, \tag{112}$$

$$\text{Ch}\,N^{!\cdot}(\mathbf{x})=\frac{\text{Ch}\,N^{\underline{g}}(\mathbf{x})}{\text{Ch}\,M^{\underline{g}}(\mathbf{x})}. \tag{113}$$

Proof. The result follows from Eqs. (103)–(106), and (41). □

The previous proposition gives the following necessary conditions for a quadratic monoid to be Koszul.

Proposition 21. *Let M be a quadratic monoid. If M is Koszul then the coefficients of the exponential generating function $(M^{\underline{g}}(x))^{-1}$ are non-negative integers. More generally, the coefficient of the symmetric function $(\text{Ch}\,M^{\underline{g}}(\mathbf{x}))^{-1}$, when expanded in terms of Schur functions, are non-negative integers.*

4.3. Koszul analytic algebras

We study here the connection of Koszul monoids in species with classical Koszul algebras by means of the Schur correspondence between tensor species and analytic functors. This correspondence is an equivalence of categories when the characteristic of the field $\mathbb{K}$ is zero.

Let $\widetilde{F} : \mathrm{Vec}_{\mathbb{K}} \to \mathcal{V}$ be the analytic functor associated to a species $F : \mathbb{B} \to \mathcal{V}$ ($\mathcal{V} = \mathrm{Vec}_{\mathbb{K}}$, $\mathrm{gVec}_{\mathbb{K}}$, or $\mathrm{dgVec}_{\mathbb{K}}$);

$$\widetilde{F}(V) = \bigoplus_{n=0}^{\infty} \big(F[n] \otimes V^{\otimes n}\big)_{S_n}. \tag{114}$$

It is well known that the $\widetilde{F.G}(V) = \widetilde{F}(V) \otimes \widetilde{G}(V)$. Then, a monoid (M, ν) defines an analytic functor $(\widetilde{M}, \tilde{\nu})$, with a structure of graded algebra when evaluated on a finite-dimensional vector space V,

$$\tilde{\nu}_V : \widetilde{M}(V) \otimes \widetilde{M}(V) \to \widetilde{M}(V). \tag{115}$$

Assuming that $\widetilde{M^{\underline{k}}}(V)$ is finite-dimensional for every $k \geqslant 0$, its Hilbert series is defined by

$$\widetilde{M}(V, t) = \sum_{k=0}^{\infty} \dim \widetilde{M^{\underline{k}}}(V) t^k. \tag{116}$$

Since $\dim \widetilde{M^{\underline{k}}}(V) = \mathrm{Ch}\, M^{\underline{k}}(\overbrace{1, 1, \ldots, 1}^{n}, 0, 0, \ldots)$, $n = \dim V$ (see [23]), we have

$$\widetilde{M}(V, t) = \sum_{k=0}^{\infty} \mathrm{Ch}\, M^{\underline{k}}\big(\overbrace{1, 1, \ldots, 1}^{n}, 0, 0, \ldots\big) t^k. \tag{117}$$

A species F is called *polynomial* if the number of integers k, such that $F[k] \neq 0$, is finite. In this section we assume that all the quadratic monoids and modules are generated by polynomial species. The reader may verify that in this case, for every finite-dimensional space V, the corresponding algebra $(\widetilde{M}(V), \tilde{\nu}_V)$ is quadratic with generators in $W = \widetilde{F}(V)$ (a finite-dimensional vector space), and relations in $\widetilde{R}_M(V) \subseteq W \otimes W = \widetilde{F}(V) \otimes \widetilde{F}(V)$. In a similar way, for every quadratic M-module $N = \mathcal{M}_M(G, R_M)$, the corresponding $\widetilde{M}(V)$-module $\widetilde{N}(V)$ is quadratic with generators in $Q = \widetilde{G}(V)$, and relations in $R_N(V) \subseteq \widetilde{F}(V) \otimes \widetilde{G}(V) = W \otimes Q$.

The bar and cobar constructions in Section 4.2 become the classical bar and cobar constructions for the algebra $(\widetilde{M}(V), \tilde{\nu}_V)$ and module $(\widetilde{N}(V), \tilde{\tau}_V)$. Similarly with respect to the definitions of Koszul algebra, Koszul module, and quadratic duals (see [31,30]).

We invoke Schur classical equivalence between representations of the symmetric groups and homogeneous polynomial representations of the general linear group (see [23]). It is well known that this equivalence can be restated by saying that the category of tensor species and the category of analytic functors are equivalent (see [20]). So, we can go backwards in this construction and obtain the following proposition.

Proposition 22. *The monoid (M, ν) is quadratic (resp. Koszul) if and only if $(\widetilde{M}(V), \tilde{\nu}_V)$ is a quadratic (resp. Koszul) algebra, for every finite-dimensional vector space V. In a similar way, an M-module (N, τ) is quadratic (resp. Koszul) if and only if $(\widetilde{N}(V), \tilde{\tau}_V)$ is a quadratic (resp. Koszul) $\widetilde{M}(V)$-module for every finite-dimensional vector space V.*

Quadratic algebras A and $A^!$ are Koszul simultaneously (see [30, Corollary 3.3]). By the previous proposition we obtain the following.

Proposition 23. *Let M be a quadratic monoid. Then, M is Koszul if and only if its quadratic dual $M^!$ is so.*

Definition 24. For a monoid (M, ν) we call the pair $(\widetilde{M}, \tilde{\nu})$ an *analytic algebra*. It will be called quadratic (resp. Koszul) if the corresponding monoid (M, ν) is quadratic (resp. Koszul).

Example 25. Consider the monoids $\mathbb{L}$, E and Λ. The corresponding analytic algebras are the tensor ($\mathbb{T}$), symmetric ($\mathbb{S}$) and exterior ($\bigwedge$) algebras. The three of them are Koszul. Then $\mathbb{L}$, E and Λ are Koszul monoids. The Koszul dual of E is Λ^*, and the Koszul dual of $\mathbb{L}$ is $1^* + X^* = 1 + X$.

Definition 26. Let $M^{\underline{g}}$ be a positively graded monoid and k a positive integer. We define the Veronese power $M_{(k)}$ as the tensor species $M_{(k)} = \sum_{j=0}^{\infty} M^{\underline{kj}}$. $M_{(k)}$ inherits the monoidal structure from $M^{\underline{g}}$ and is grading is given by $M_{(k)}^{\underline{r}} = M^{\underline{kr}}$. Let $N^{\underline{g}}$ be a graded $M^{\underline{g}}$-module. For every $k \geqslant 0$ we define the truncated M-module $N^{[k]}$ as $(N^{[k]})^{\underline{j}} = N^{\underline{k+j}}$.

Proposition 27. *Let M and N be as above. If the monoid M is quadratic (Koszul), then $M_{(k)}$ is quadratic (resp. Koszul). Similarly with respect to N, if the M-module N is quadratic (Koszul), then $N^{[k]}$ is quadratic (Koszul) for each $k \geqslant 0$.*

Proof. The Veronese power of a quadratic (Koszul) algebra is quadratic (Koszul) (see [3]). Let A be an algebra and B and A-module. Then if A and B are quadratic or Koszul, then the same is true for $B^{[k]}$ (see [30, Proposition 1.1]). The result then follows by Proposition 22. □

By this proposition, the modules in Proposition 18 are Koszul.

Definition 28 *(Segre and Manin products of monoids).* Let $M^{\underline{g}}$ and $N^{\underline{g}}$ be two graded monoids. The Segre product $M \underline{\circ} N$ is the graded monoid defined by

$$(M \underline{\circ} N)^{\underline{k}} = M^{\underline{k}} \cdot N^{\underline{k}}, \quad k \geqslant 0 \tag{118}$$

and obvious product. For a pair of graded modules $R^{\underline{g}}$ and $S^{\underline{g}}$ over M and N respectively, the Segre product $R \underline{\circ} S$ is the graded $M \underline{\circ} N$-module $(R \underline{\circ} S)^{\underline{k}} = R^{\underline{k}} \cdot S^{\underline{k}}$, $k \geqslant 0$. If M and N are quadratic, the black circle or Manin product is defined as follows

$$M \underline{\bullet} N = \left(M^{!_\cdot} \underline{\circ} N^{!_\cdot}\right)^{!_\cdot}. \tag{119}$$

The black circle product is similarly defined for R and S quadratic modules over M and N respectively.

The black circle product for quadratic algebras was introduced by Manin in [22].

Proposition 29. *If M and N are quadratic (resp. Koszul), then $M \underline{\circ} N$ and $M \underline{\bullet} N$ are quadratic (resp. Koszul). If both M-module R and N-module S are quadratic (resp. Koszul), then the $M \underline{\circ} N$-module $R \underline{\circ} S$ is quadratic (resp. Koszul).*

Proof. Similar to the proof of Proposition 27. See [3] for the analogous result about Koszulness of the Segre and Manin products of Koszul algebras and modules. □

Note that if M and N are quadratic, $M \underline{\circ} N$ is generated by $M^{\underline{1}} \cdot N^{\underline{1}}$, and

$$R_{M \underline{\circ} N} = R_M \cdot \left(N^{\underline{1}}\right)^2 + \left(M^{\underline{1}}\right)^2 \cdot R_N.$$

As a consequence,

$$R_{M \underline{\bullet} N} = R_M \cdot R_N.$$

5. Möbius species, c-monoids, and Cohen–Macaulyness

A *partially ordered set* or *poset* $(P, \leqslant)$ is a set together with a partial order. We usually refer to a poset by its underlying set P. If P and Q are two posets we define the direct product $P \times Q$ as the poset with underlying set the cartesian product of the respective underlying sets and order relation $(p, q) \leqslant_{P\times Q} (p', q')$ if $p \leqslant_P p'$ and $q \leqslant_Q q'$. The direct sum or coproduct $P \sqcup Q$ is the poset with underlying set the disjoint union of the respective underlying sets and such that $x \leqslant y$ in $P \sqcup Q$ either if $x, y \in P$ and $x \leqslant_P y$ or $x, y \in Q$ and $x \leqslant_Q y$.

Denote respectively by $\mathrm{Max}(P)$ and $\mathrm{Min}(P)$ the set of maximal and minimal elements of P. When $\mathrm{Min}(P)$ has one element, $\hat{0}$ denotes that unique element. Similarly, if $\mathrm{Max}(P)$ has only one element it is denoted by $\hat{1}$. A set with $\hat{0}$ and $\hat{1}$ is called *bounded*.

An interval $[x, y]$ in P, $x, y \in P$ is the subposet $[x, y] = \{z \mid x \leqslant z \leqslant y\}$. We say that y covers z if $|[x, y]| = 2$ and denote it by $x \prec y$. A *chain* is a totally ordered subset $x_0 < x_1 < \cdots < x_l$ of P. Its length is defined to be l. A *maximal chain* of an interval $[x, y]$ is one of the form

$$x = x_0 \prec x_1 \prec \cdots \prec x_l = y.$$

A poset is *pure* if for every interval $[x, y]$, all the maximal chains have the same length. A bounded poset that is pure is called a *graded poset*. The length of a maximal chain is called the rank of P ($\mathrm{rk}(P)$).

We follow the conventions in [40] for the definition of order complexes, homology of posets and Cohen–Macaulay posets. This definition of Cohen–Macaulay poset agrees with the usual one. We refer the reader to [6] and the more recent notes [41] for tools and techniques on poset topology.

Denote by $\Delta(P)$ the set of chains $x_0 < x_1 < \cdots < x_l$ of P such that $x_0 \in \mathrm{Min}(P)$ and $x_l \in \mathrm{Max}(P)$. $\Delta(P) = \biguplus_l \Delta_l(P)$, $\Delta_l(P)$ being the set of chains of length l. Define $\partial_l : \mathbb{K}\Delta_l(P) \to \mathbb{K}\Delta_{l-1}(P)$,

$$\partial_i(x_0 < x_1 < \cdots < x_l) = \sum_{i=1}^{l-1} (-1)^{i-1} (x_0 < x_1 < \cdots < x_{i-1} < x_{i+1} < \cdots < x_l).$$

The chain complex $(\mathbb{K}\Delta(P), \partial)$ is called the *order complex* of P. The homology of P with coefficients in $\mathbb{K}$ is the homology of the complex $\mathbb{K}\Delta(P)$. It is denoted by $H_*(P, \mathbb{K})$.

Definition 30. Let P be a graded poset. It is said to be Cohen–Macaulay over $\mathbb{K}$ if for every interval $[x, y]$ of P the homology is concentrated in top rank:

$$H_r\big([x, y], \mathbb{K}\big) = 0, \quad \text{for } r \neq \mathrm{rk}\big([x, y]\big). \tag{120}$$

Definition 31. Let **Int** be the category of finite families of finite posets with $\hat{0}$ and $\hat{1}$, where for two objects A and B of **Int**, a morphism $f : A \to B$ is an isomorphism of posets $f : \coprod_{I\in A} I \to \coprod_{I'\in B} I'$. A *Möbius* species is a functor $P : \mathbb{B} \to \mathbf{Int}$. For an object $A \in \mathbf{Int}$, define the Möbius cardinality $|A|_\mu = \sum_{I\in A} \mu_I(\hat{0}, \hat{1})$, where μ_I is the Möbius function of the poset I. If $A = \emptyset$, we set $|A|_\mu = 0$. For a permutation $\sigma : U \to U$, let $\mathrm{Fix}_P[\sigma] = \{I_\sigma \mid I \in M[U],\ P[\sigma](I) = I\}$, where I_σ is the subposet of elements of I fixed by σ; $I_\sigma = \{x \in I \mid P[\sigma]x = x\}$. The Möbius cardinal $|\mathrm{Fix}\, P[\sigma]|_\mu$ depends only on the cycle type of σ. This leads to analogous of the generating functions for ordinary species.

$$\text{Möb}\, P(x) = \sum_{n=0}^{\infty} \big|P[n]\big|_\mu \frac{x^n}{n!}, \tag{121}$$

$$\mathrm{Ch}\, P(\mathbf{x}) = \sum_\alpha \big|\mathrm{Fix}\, P[\alpha]\big|_\mu \frac{p_\alpha(\mathbf{x})}{z_\alpha}. \tag{122}$$

P is said to be Cohen–Macaulay if for every finite set U, all the posets in $P[U]$ are Cohen–Macaulay.

Definition 32. Let (M,ν) be a monoid in the monoidal category of set-species $(\mathcal{F}^{\mathbb{B}},\cdot,1)$. Recall that the elements of the set $(M\cdot M)[U]$ are ordered pairs of the form (m_{U_1},m_{U_2}), $U_1\uplus U_2=U$, where $m_{U_i}\in M[U_i]$, $i=1,2$. (M,ν) is called a c-monoid if it satisfies the left cancellation law

$$\nu(m_{U_1},m_{U_2})=\nu\bigl(m_{U_1},m'_{U_2}\bigr)\quad\Rightarrow\quad m_{U_2}=m'_{U_2}. \tag{123}$$

By the category imbeddings of Eq. (6), a c-monoid (as a set species) can be thought of as a tensor species. We will say that a c-monoid M is quadratic, if it is so as a tensor species, $M=\mathcal{M}(F,R)$, where the species F of generators is also a set species.

5.1. The posets associated to a c-monoid

From a c-monoid (M,ν) we can define a partially ordered set

$$\Biggl(\biguplus_{U_1\subseteq U}M[U_1],\leqslant_\nu\Biggr)=\bigl((E\cdot M)[U],\leqslant_\nu\bigr)$$

for each finite set U. The relation $\leqslant_\nu$ is defined by

$$\begin{aligned} m_{U_1}\leqslant_\nu m_{U_2}\quad&\text{if there exists } m'_{U'_2},\\ U_1\uplus U'_2=U_2,\quad&\text{such that } \nu\bigl(m_{U_1},m'_{U'_2}\bigr)=m_{U_2}. \end{aligned} \tag{124}$$

This posets have a zero the unique element of $M[\emptyset]$. All the elements of $M[U]$ are maximal in $(\biguplus_{U_1\subseteq U}M[U_1],\leqslant_\nu)$. In general an element $m_1\in M[U_1]$, $U_1\neq U$, could also be maximal. To avoid this situation we define $\mathcal{P}_M[U]$ to be the subposet of $(\biguplus_{U_1\subseteq U}M[U_1],\leqslant_\nu)$ whose set of maximal elements is equal to $M[U]$,

$$\mathcal{P}_M[U]:=\bigl\{m_1\bigm|\text{such that there exists } m\in M[U] \text{ with } m_1\leqslant_\nu m\bigr\}.$$

In other words, $\mathcal{P}_M[U]=\bigcup_{m\in M[U]}[\hat{0},m]$.

The partial order of $\mathcal{P}_M[U]$ is functorial, for every bijection $f:U\to V$ between finite sets we define $\mathcal{P}_M[f]:\mathcal{P}_M[U]\to\mathcal{P}_M[V]$ by

$$\mathcal{P}_M[f]m_1=M[f_{U_1}]m_1=(EM)[f]m_1, \tag{125}$$

where $m_1\in M[U_1]$ and f_{U_1} is the restriction of f to U_1. From the cancellation property in c-monoids we obtain the following theorem and corollaries (see [26, Theorem 3.2]).

Theorem 33. *The family of posets* $\{\mathcal{P}_M[U]\mid U$ *a finite set*$\}$ *satisfies the following properties*:

(1) $\mathcal{P}_M[f]:\mathcal{P}_M[U]\to\mathcal{P}_M[V]$ *is an order isomorphism.*
(2) $\mathcal{P}_M[U]$ *has a* $\hat{0}$ *the unique element of* $M[\emptyset]$.
(3) *For a finite set* $U_1\subseteq U$, *and* $m_1\in M[U_1]$ *an element of* $\mathcal{P}_M[U]$, *the order coideal*

$$\mathcal{C}_{m_1}=\bigl\{m_2\in\mathcal{P}_M[U]\bigm|m_2\geqslant m_1\bigr\},$$

is isomorphic to $\mathcal{P}_M[U-U_1]$.

Corollary 34. *If $f : U \to V$ is a bijection, and $[\hat{0}, m]$, $m \in M[U]$, an interval of $\mathcal{P}_M[U]$, then, the restriction of $\mathcal{P}_M[f]$ to $[\hat{0}, m]$, $\mathcal{P}_M[f]|_{[\hat{0},m]} : [\hat{0}, m] \to [\hat{0}, M[f]m]$ is an isomorphism of posets.*

Corollary 35. *Every interval $[m_1, m_2]$ of $\mathcal{P}_M[U]$, $m_i \in M[U_i]$, $i = 1, 2$, $U_1 \subseteq U_2 \subseteq U$, is isomorphic to the interval $[\hat{0}, m_2']$ of $\mathcal{P}_M[U_2 - U_1]$, m_2' being the unique element of $M[U_2 - U_1]$ such that $\nu(m_1, m_2') = m_2$.*

Now we can define the inverse Möbius species M^{-1}.

Definition 36. Let M be a c-monoid as above. Define the Möbius species

$$M^{-1}[U] = \bigl\{[\hat{0}, m] \bigm| m \in M[U]\bigr\},$$

for a bijection $f : U \to U'$, the isomorphism $M^{-1}[f] : \coprod_{m \in M[U]} [\hat{0}, m] \to \coprod_{m' \in M[U']} [\hat{0}, m']$ is given by

$$M^{-1}[f]m_1 = M[f_{U_1}]m_1, \tag{126}$$

where U_1 is the subset of U such that $m_1 \in M[U_1]$. It is clear that if $m_1 \in [\hat{0}, m]$, then $M^{-1}[f]m_1 \in [\hat{0}, M[f]m]$.

By Möbius inversion in the poset $\mathcal{P}_M[U]$ we obtain (see [26, Theorem 3.3]),

$$\text{Möb}\, M^{-1}(x) = \bigl(M(x)\bigr)^{-1}. \tag{127}$$

Let $\mathcal{P}_M[U, \sigma]$ be the subposet of $\mathcal{P}_M[U]$ of the elements fixed by the permutation $\sigma : U \to U$, under the action given by Eq. (125). In a similar way, by Möbius inversion on $\mathcal{P}_M[U, \sigma]$, we obtain

$$\text{Ch}\, M^{-1}(\mathbf{x}) = \bigl(\text{Ch}\, M(\mathbf{x})\bigr)^{-1}. \tag{128}$$

Proposition 37. *Assume that $M = \mathcal{M}(F, R)$ is a quadratic c-monoid. Then, every interval $[\hat{0}, m]$ with $m \in M^{\underline{k}}[U]$ is graded with rank k.*

Proof. It is easy to see that for m_1 and m_2 elements of M, $m_1 \prec m_2$ if and only if there exists f, element of F, such that $m_2 = \nu(m_1, f)$. Then for $m \in M^{\underline{k}}[U]$, every maximal chain in $[\hat{0}, m]$ has to be of the form

$$\hat{0} = m_0 \prec m_1 \prec m_2 \prec \cdots \prec m_l = m,$$

where $m_i = \nu(m_{i-1}, f_i)$ for $i = 1, 2, \ldots, l$, each f_i being an element of F. Hence

$$m = \nu\bigl(\cdots\bigl(\nu(f_1 \otimes f_2) \otimes \cdots\bigr) \otimes f_l\bigr) = \overline{f_1 \otimes f_2 \otimes \cdots \otimes f_l} \in F^l / \langle R\rangle[U] = M^{\underline{k}}[U],$$

and $l = k$. □

From the previous proposition we obtain the corollary.

Corollary 38. *The poset $\mathcal{P}_M^k[U] := \bigcup_{m \in M^{\underline{k}}[U]} [\hat{0}, m]$, is pure.*

Theorem 39. *Let $M = \mathcal{M}(F, R)$ be a quadratic c-monoid. Then, the tensor species $U \mapsto H_k(P^{k}_{M}[U], \mathbb{K})$ is isomorphic to $(M^{¡\cdot})^{\underline{k}}$. Moreover, M is Koszul if and only if for every finite set U and every k, the homology $H_*(\mathcal{P}^{k}_{M}[U], \mathbb{K})$ is concentrated in top rank; $H_r(\mathcal{P}^{k}_{M}[U], \mathbb{K}) = 0$, for $r \neq k$.*

Proof. We reverse the enumeration in the complex $\mathbb{K}\Delta_*(\mathcal{P}^{k}_{M}[U])$ to define an appropriate **dg**-tensor species. Let $C^{(k),\underline{g}}$ be defined by

$$C^{(k),\underline{r}}[U] := \mathbb{K}\Delta_{k-r}\big(\mathcal{P}^{k}_{M}[U]\big), \tag{129}$$

where $r = 0, 1, 2, \ldots, k$. The differential $\hat{d}_r : C^{(k),\underline{r}} \to C^{(k),\underline{r+1}}$ is equal to ∂_{k-r}. Let

$$m_{U_1} \otimes m_{U_2} \otimes \cdots \otimes m_{U_r} \in \mathcal{B}\mathrm{ar}^{(k)}(M)^{\underline{j}}[U], \quad m_{U_i} \in M[U_i],\ i = 1, 2, \ldots, r. \tag{130}$$

Let $V_i = \biguplus_{s=1}^{i} U_s$, and define recursively,

$$\overline{m}_{V_1} = m_{U_1}, \qquad \overline{m}_{V_i} = \nu(\overline{m}_{V_{i-1}}, m_{U_i}), \quad i = 2, \ldots, r.$$

It is clear that $\hat{0} <_\nu \overline{m}_{V_1} <_\nu \overline{m}_{V_2} <_\nu \cdots <_\nu \overline{m}_{V_r}$ is a chain in $C^{(k),\underline{k-r}}[U]$. By the left cancellation law, the correspondence

$$\alpha : m_{U1} \otimes m_{U_2} \otimes \cdots \otimes m_{U_r} \mapsto \hat{0} <_\nu \overline{m}_{V_1} <_\nu \overline{m}_{V_2} <_\nu \cdots <_\nu \overline{m}_{V_r} \in M[U] \tag{131}$$

is bijective between basic elements of $\mathcal{B}\mathrm{ar}(M)^{(k),\underline{k-r}}[U]$ and $C^{(k),\underline{k-r}}[U]$ respectively. Since

$$\begin{aligned}&\alpha\big(m_{U1} \otimes \cdots \otimes \nu(m_{U_i}, m_{U_{i+1}}) \otimes \cdots \otimes m_{U_r}\big)\\ &\quad = \hat{0} <_\nu \overline{m}_{V_1} <_\nu \overline{m}_{V_2} <_\nu \cdots <_\nu \overline{m}_{V_{i-1}} <_\nu \overline{m}_{V_{i+1}} <_\nu \overline{m}_{V_r},\end{aligned}$$

α extends by linearity to an isomorphism of **dg**-tensor species $\alpha : \mathcal{B}\mathrm{ar}^{(k)}(M) \to C^{(k)}$. Because of this we have $H_k(\mathcal{P}^{k}_{M}[U], \mathbb{K}) = H^0(C^{(k),\underline{g}}[U]) = H^0\,\mathcal{B}\mathrm{ar}^{(k)}(M) = (M^{¡\cdot})^{\underline{k}}$, and that $\mathcal{H}\mathcal{B}\mathrm{ar}^{(k)}(M)$ is concentrated in degree zero if and only if $\mathcal{H}C^{(k),\underline{g}}$ is so for every $k = 0, 1, 2, \ldots.$ □

Lemma 40. *Let $M = \mathcal{M}(F, R)$ be a quadratic c-monoid as above. The inverse species M^{-1} is Cohen–Macaulay if and only if for every finite set U and every poset $[\hat{0}, m] \in M^{-1}[U]$, the homology $H_*([\hat{0}, m])$ is concentrated in top rank $k = \mathrm{rk}([\hat{0}, m])$.*

Proof. The result follows directly from Corollary 35. □

Theorem 41. *Let M be a c-monoid as above. Then, M is Koszul if and only if the Möbius species M^{-1} is Cohen–Macaulay.*

Proof. We have the identity

$$H_r\big(\mathcal{P}^{k}_{M}[U], \mathbb{K}\big) = \bigoplus_{m \in M^{\underline{k}}[U]} H_r\big([\hat{0}, m], \mathbb{K}\big), \quad r = 0, 1, 2, \ldots, k. \tag{132}$$

Then, $H_r(\mathcal{P}^{k}_{M}[U], \mathbb{K}) = 0$ if and only if $H_r([\hat{0}, m], \mathbb{K}) = 0$ for every $m \in M^{\underline{k}}[U]$. The result follows from the previous lemma and Theorem 39. □

If M is a Koszul quadratic c-monoid, since $((M^{\text{!}\cdot})^{\underline{g}}(x))^{-1} = M(x)$ and $((\mathrm{Ch}\, M^{\text{!}\cdot})^{\underline{g}})^{-1}(\mathbf{x}) = \mathrm{Ch}\, M(\mathbf{x})$, we have

$$\text{Möb}\, M^{-1}(x) = \left(M^{\text{!}\cdot}\right)^{\underline{g}}(x), \tag{133}$$

$$\mathrm{Ch}\, M^{-1}(\mathbf{x}) = \mathrm{Ch}\left(M^{\text{!}\cdot}\right)^{\underline{g}}(\mathbf{x}). \tag{134}$$

Moreover, these identities can be refined a little more. Using standard arguments, we can prove that the Möbius cardinalities have the following interpretations:

$$\sum_{m\in M^{\underline{k}}[n]} \mu(\hat{0}, m) = (-1)^k \dim\left(M^{\text{!}\cdot}\right)^{\underline{k}}[n], \tag{135}$$

$$\sum_{m\in M^{\underline{k}}[n],\, M[\sigma]m=m} \mu\left([\hat{0}, m]_\sigma\right) = (-1)^k \operatorname{tr}\left(M^{\text{!}\cdot}\right)^{\underline{k}}[\sigma]. \tag{136}$$

Example 42. For the c-monoid E, $\mathcal{P}_E[U] = [\emptyset, U] = \mathscr{B}(U)$ is the Boolean algebra of the parts of U. It is well known that $\mathscr{B}(U)$ is Cohen–Macaulay. This is another way of obtaining that E is a Koszul monoid (over $\mathbb{Z}$).

5.2. The posets associated to a c-module

Definition 43. Let M be a c-monoid and N a right (set) M-module such that $N[\emptyset] = \emptyset$. N is said to be a c-(M)-module if it satisfies the left cancellation law: $\tau(n, m) = \tau(n, m') \Rightarrow m = m'$. We will say that N is quadratic if it is so as a tensor species, $N = \mathcal{M}_M(G, R_N)$, where the species G is also a set species.

Define a partial order on $E.N[U] = \biguplus_{U_1\subseteq U} N[U_1]$ by: $n_1 \leqslant_\tau n_2$ if there exists $m \in M[U_2 - U_1]$ such that $\tau(n_1, m) = n_2$. $E.N[U]$ does not have a $\hat{0}$. In that case define $\widehat{E.N}[U] = \hat{0} + E.N[U]$, then trim the maximal elements not in $N[U]$, to obtain

$$\mathcal{P}_{M,N}[U] = \bigcup_{n\in N[U]} [\hat{0}, n] \tag{137}$$

$[\hat{0}, n]$ being an interval of $\widehat{E.N}[U]$. As in the case of posets induced by c-monoids, the order is functorial, and we have the following theorem and corollaries.

Theorem 44. *The family of posets $\{\mathcal{P}_{M,N}[U] \mid U$ a finite set$\}$ satisfies the following properties*:

(1) *For a bijection $f : U \to V$, $\mathcal{P}_{M,N}[f] : \mathcal{P}_{M,N}[U] \to \mathcal{P}_{M,N}[V]$ is an order isomorphism.*
(2) *For a finite set $U_1 \subseteq U$, and $n_1 \in N[U_1]$ an element of $\mathcal{P}_{M,N}[U]$, the order coideal*

$$\mathcal{C}_{n_1} = \left\{n_2 \in \mathcal{P}_{M,N}[U] \,\middle|\, n_2 \geqslant n_1\right\},$$

is isomorphic to $\mathcal{P}_{M,N}[U - U_1]$.

Corollary 45. *If $f : U \to V$ is a bijection, and $[\hat{0}, n]$, $n \in N[U]$, an interval of $\mathcal{P}_{M,N}[U]$, then, the restriction of $\mathcal{P}_{M,N}[f]$ to $[\hat{0}, n]$, $\mathcal{P}_{M,N}[f]|_{[\hat{0},n]} : [\hat{0}, n] \to [\hat{0}, N[f]n]$ is an isomorphism of posets.*

Lemma 46. *Let $[n_1, n_2]$ be an interval of $\mathcal{P}_{M,N}[U]$, $n_i \in N[U_i]$, $U_1 \subseteq U_2 \subseteq U$. Then, $[n_1, n_2]$ is isomorphic to $[\hat{0}, m]$, $m \in M[U_2 - U_1]$ being the unique element such that $\tau(n_1, m) = n_2$.*

Theorem 47. *Let $Q_{N,M}$ be the Möbius species defined as*

$$Q_{N,M}[U] = \{[\hat{0}, n] \mid n \in N[U]\}. \tag{138}$$

Then

$$\text{Möb}\, Q_{N,M}(x) = -N(x)\,\text{Möb}\, M^{-1}(x) = -\frac{N(x)}{M(x)}, \tag{139}$$

$$\text{Ch}\, Q_{N,M}(\mathbf{x}) = -\frac{\text{Ch}\, N(\mathbf{x})}{\text{Ch}\, M(\mathbf{x})}. \tag{140}$$

Proof.

$$\sum_{n\in N[U]} \sum_{\hat{0}\leqslant n_1\leqslant n} \mu(n_1, n) = \sum_{n\in N[U]} \mu(\hat{0}, n) + \sum_{n\in N[U]} \sum_{\hat{0}<n_1\leqslant n} \mu(n_1, n) = 0, \tag{141}$$

$$= \big|Q_{N,M}[U]\big|_\mu + \sum_{n\in N[U]} \sum_{\emptyset\neq U_1\subseteq U} \sum_{n_1\in N[U_1]} \mu(n_1, n) = 0, \tag{142}$$

$$= \big|Q_{N,M}[U]\big|_\mu + \sum_{\emptyset\neq U_1\subseteq U} \big|N[U_1]\big| \sum_{m\in M[U-U_1]} \mu(\hat{0}, m) = 0, \tag{143}$$

$$= \big|Q_{N,M}[U]\big|_\mu + \sum_{\emptyset\neq U_1\subseteq U} \big|N[U_1]\big|\big|M^{-1}[U - U_1]\big|_\mu = 0. \tag{144}$$

From that, we get

$$\text{Möb}\, Q_{N,M}(x) + N(x)\,\text{Möb}\, M^{-1}(x) = 0. \tag{145}$$

The proof of the character formula is similar, by Möbius inversion on the poset

$$\mathcal{P}_{M,N}[U, \sigma] = \{n \in \mathcal{P}_{M,N}[U] \mid \mathcal{P}_{M,N}[\sigma]n = n\}. \qquad \square$$

Analogously to the c-monoid case we have the following results.

Proposition 48. *The poset $\mathcal{P}^k_{M,N}[U] := \bigcup_{n\in N^{\underline{k-1}}[U]}[\hat{0}, n]$ is pure with $\text{rk}([\hat{0}, n]) = k$ for every n maximal in $\mathcal{P}^k_{M,N}[U]$.*

Theorem 49. *Let M be a Koszul c-monoid, and N a quadratic c-module on M. Then, the tensor species $U \mapsto H_k(P^k_{M,N}[U], \mathbb{K})$ is isomorphic to $(N^{!\cdot})^{\underline{k-1}}$. Moreover, N is Koszul if and only if for every finite set U and every k, the homology $H_*(\mathcal{P}^k_{M,N}[U], \mathbb{K})$ is concentrated in top rank; $H_r(\mathcal{P}^k_{M,N}[U], \mathbb{K}) = 0$, for $r \neq k$.*

Corollary 50. *The M-module N is Koszul if and only if the Möbius species $Q_{N,M}$ is Cohen–Macaulay. Moreover we have*

$$\text{Möb}\, Q_{N,M}(x) = -\big(N^{!\cdot}\big)^{\underline{g}}(x), \tag{146}$$

$$\text{Ch}\, Q_{M,N}(\mathbf{x}) = -\,\text{Ch}\big(N^{!\cdot}\big)^{\underline{g}}(\mathbf{x}), \tag{147}$$

$$\sum_{n\in N^{\underline{k}}[n]} \mu(\hat{0}, m) = (-1)^{k+1} \dim\bigl(N^{\text{!`}}\bigr)^{\underline{k}}[n], \tag{148}$$

$$\sum_{n\in N^{\underline{k}}[n],\, N[\sigma]n=n} \mu\bigl([\hat{0}, n]_\sigma\bigr) = (-1)^{k+1} \operatorname{tr}\bigl(N^{\text{!`}}\bigr)^{\underline{k}}[\sigma]. \tag{149}$$

Example 51 *(Truncated posets).* Let $M = \mathcal{M}(F, R)$ be a Koszul c-monoid. For a fixed positive integer l, $M^{[l]} = \sum_{j\geqslant l} M^{\underline{j}}$ is a cancellative Koszul M-module. $Q_{M,M^{[l]}}[U]$ is the poset obtained from $\mathcal{P}_M[U]$ by removing the elements of rank j, $0 < j < l$. The orthogonal relation $R^{\perp}_{M^{[l]}}[U]$ is the linear span of elements of the form

$$\sum_{\nu(m_1, f)=m} m_1^* \otimes f^* \in \bigl(M^{\underline{l}}\bigr)^* \cdot F^*,$$

for m in $M^{\underline{l+1}}[U]$. If $M = E$, $\mathcal{P}_{E,E^{[l]}}[U]$ is the truncated Boolean algebra $\mathscr{B}^{[l]}[U]$. Its top homology is given in the first part of Proposition 18 (see [35] for a general result on the top homology of rank selected Boolean algebras).

6. c-Operads

Let R and S be two set-species such that $S[\emptyset] = \emptyset$. Recall that the elements of the substitution $R(S)[U]$ are pairs of the form (a, r), a being an assembly of S-structures,

$$a = \{s_B\}_{B\in\pi}, \quad s_B \in S[B],\ B \in \pi,\ \pi \text{ a partition of } U,$$

and $r \in R[\pi]$.

Definition 52 *(c-Operads).* Let $(\mathscr{C}, \eta)$ be a set-operad. We say that $\mathscr{C}$ is a c-operad if it satisfies the left cancellation law: For every pair of elements $(a, c), (a, c') \in \mathscr{C}(\mathscr{C})[U]$, if $\eta(a, c) = \eta(a, c')$, then $c = c'$.

Example 53. Let $\mathscr{G}_c$ be the species of simple, connected graphs. $\mathscr{G}_c$ has a structure of c-operad $\eta : \mathscr{G}_c(\mathscr{G}_c) \to \mathscr{G}_c$ (see [26, Example 3.19]). For $(\{g_B\}_{B\in\pi}, g_1) \in \mathscr{G}_c(\mathscr{G}_c)[U]$, $\eta(\{g_B\}_{B\in\pi}, g_1) = g$, where $\{x, y\}$ is an edge in g if (i) $\{x, y\}$ is an edge in some graph g_B, $B \in \pi$ or (ii) There exists an edge $\{B_1, B_2\}$ of g_1, $B_1, B_2 \in \pi$, such that $x \in B_1$ and $y \in B_2$ (see Fig. 3).

For a c-operad $(\mathscr{C}, \eta)$, η induces other natural transformation $\tilde{\eta} : E_+(\mathscr{C})(\mathscr{C}) \to E_+(\mathscr{C})$ defined as the composition $\tilde{\eta} = E_+(\eta) \circ \alpha$,

$$E_+(\mathscr{C})(\mathscr{C}) \xrightarrow{\alpha} E_+\bigl(\mathscr{C}(\mathscr{C})\bigr) \xrightarrow{E_+(\eta)} E_+(\mathscr{C}). \tag{150}$$

α being the associativity isomorphism for the substitution of species. The elements of $E(\mathscr{C})(\mathscr{C})[U]$ are pairs of assemblies (a_1, a_2), where $a_1 = \{c_B\}_{B\in\pi}$ is an assembly of $\mathscr{C}$-structures over the set U, and $a_2 \in E_+(\mathscr{C})[\pi]$ is an assembly of $\mathscr{C}$-structures on the set π. More explicitly, if $a_2 = \{\tilde{c}_{B'} \mid B' \in \pi'\}$, π' is the partition induced by the assembly a_2 (a partition on the blocks of π), then a_1 can be written as a union of subassemblies $a_1 = \biguplus_{B'\in\pi'} a_1^{B'}$, $a_1^{B'} = \{c_B \in a_1 \mid B \in B'\}$. The natural transformation $\tilde{\eta}$ is explicitly evaluated as follows

$$\tilde{\eta}(a_1, a_2) = \bigl\{\eta\bigl(a_1^{B'}, \tilde{c}_{B'}\bigr) \bigm| B' \in \pi'\bigr\} = \bigl\{\eta\bigl(\{c_B\}_{B\in B'}, \tilde{c}_{B'}\bigr)\bigr\}_{B'\in\pi'}.$$

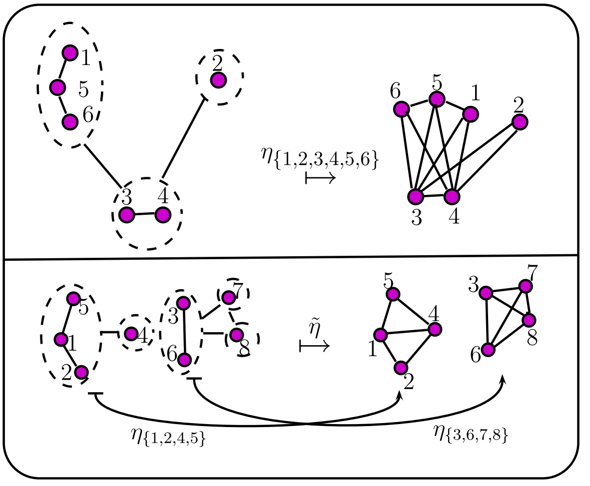

Fig. 3. The natural transformations η and $\tilde{\eta}$ for the operad $\mathscr{G}_c$ of simple and connected graphs.

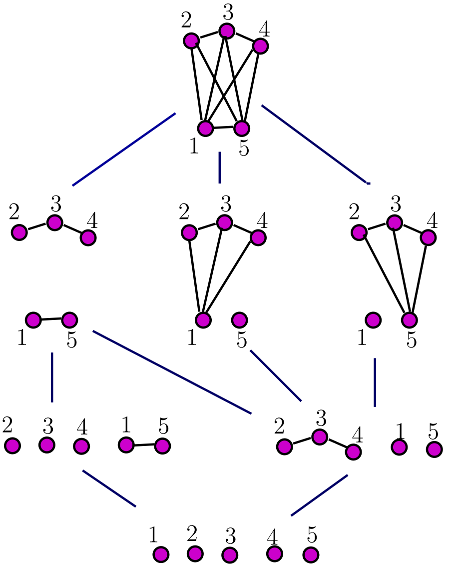

Fig. 4. Subset of an interval of the partially ordered set $\mathscr{P}_{\mathscr{G}_c}[5]$.

$\tilde{\eta}$ inherits the left cancellation law: if $\tilde{\eta}(a_1, a_2') = \tilde{\eta}(a_1, a_2'')$, then $a_2' = a_2''$. We can define a partially ordered set $(E(\mathscr{C})[U], \leqslant_\eta)$, for every finite set U. The partial order $\leqslant_\eta$ defined by

$$a_1 \leqslant_\eta a_2, \quad \text{if there exists } a_2' \in \mathscr{C}[\pi], \text{ such that } \eta\big(a_1, a_2'\big) = a_2, \tag{151}$$

π being the partition associated to the assembly a_1. See Fig. 4 for the partial order induced by the operad of simple, connected graphs $\mathscr{G}_c$, on the simple graphs $\mathscr{G} = E_+(\mathscr{G}_c)$ (assemblies of connected graphs).

The assemblies with only one structure $\{c_U\}$, $c_U \in \mathscr{C}[U]$, are maximal elements of this poset. However, as in the c-monoid case, in general an assembly with more than one element could be maximal. We define the poset $\mathscr{P}_{\mathscr{C}}[U]$ to be the subposet of $(E(\mathscr{C})[U], \leqslant_\eta)$ whose maximal elements are the assemblies with only one structure. Explicitly

$$\mathscr{P}_{\mathscr{C}}[U] := \bigcup_{c_U \in \mathscr{C}[U]} \big[\hat{0}, \{c_U\}\big].$$

The order on $\mathscr{P}_{\mathscr{C}}[U]$ is also functorial. For a bijection $f : U \to U'$, the bijection

$$\mathscr{P}_{\mathscr{C}}[f] : \mathscr{P}_{\mathscr{C}}[U] \to \mathscr{P}_{\mathscr{C}}\big[U'\big], \qquad \mathscr{P}_{\mathscr{C}}[f]a = E(\mathscr{C})[f]a$$

is an order isomorphism.

From the properties of c-operads the following theorem follows (see [26, Theorem 3.4]).

Theorem 54. *The family $\{\mathscr{P}_{\mathscr{C}}[U] \mid U$ a finite set$\}$ satisfies the following properties*:

(1) *$\mathscr{P}_{\mathscr{C}}[U]$ has a $\hat{0}$ equal to the assembly of elements all in $\mathscr{C}_1 = X$.*
(2) *For $a_1 \in \mathscr{P}_{\mathscr{C}}[U]$, the order coideal $\mathcal{C}_{a_1}$ is isomorphic to $\mathscr{P}_{\mathscr{C}}[\pi]$, π being the partition induced by the assembly a_1. Every interval $[a_1, a_2]$ in $\mathscr{P}_{\mathscr{C}}[U]$ is isomorphic to $[\hat{0}, a_2']$ of $\mathscr{P}_{\mathscr{C}}[\pi]$, a_2' being the unique assembly such that $\hat{\eta}(a_1, a_2') = a_2$.*
(3) *If $a_1 = \{c_B \mid B \in \pi\}$, the interval $[\hat{0}, a_1]$ is isomorphic to the direct product*

$$\prod_{B\in\pi} \big[\hat{0}, \{c_B\}\big],$$

where $[\hat{0}, \{c_B\}]$ is an interval of $\mathscr{P}_{\mathscr{C}}[B]$ for each block $B \in \pi$.

Definition 55 *(Möbius substitutional inverse).* Let $\mathscr{C}$ be a c-operad. Define the Möbius (substitutional) inverse $\mathscr{C}^{\langle -1\rangle}$;

$$\mathscr{C}^{\langle -1\rangle}[U] = \big\{\big[\hat{0}, \{c_U\}\big] : \big[\hat{0}, \{c_U\}\big] \text{ an interval of } \mathscr{P}_{\mathscr{C}}[U],\, c_U \in \mathscr{C}[U]\big\},$$

for a bijection $f : U \to U'$, the isomorphism $\mathscr{C}^{\langle -1\rangle}[f] : \coprod_{c\in\mathscr{C}[U]}[\hat{0}, \{c\}] \to \coprod_{c'\in\mathscr{C}[U']}[\hat{0}, \{c'\}]$ is given by

$$\mathscr{C}^{\langle -1\rangle}[f]a_1 = E(\mathscr{C})[f]a_1, \tag{152}$$

where a_1 is an assembly $a_1 \leqslant_\eta \{c_U\}$, for some $c_U \in \mathscr{C}[U]$. It is clear that if $a_1 \in [\hat{0}, \{c_U\}]$, then $\mathscr{C}^{\langle -1\rangle}[f]a_1 \in [\hat{0}, \{\mathscr{C}[f]c_U\}]$.

By Möbius inversion on the poset $\mathscr{P}_{\mathscr{C}}[U]$ we have (see [26, Theorem 3.5]),

$$\text{Möb}\,\mathscr{C}^{\langle -1\rangle}(x) = \big(\mathscr{C}(x)\big)^{\langle -1\rangle}. \tag{153}$$

Let $\mathscr{P}_{\mathscr{C}}[U,\sigma]$ be the poset of assemblies in $\mathscr{P}_{\mathscr{C}}[U]$ fixed by a given permutation $\sigma : U \to U$;

$$\mathscr{P}_{\mathscr{C}}[U,\sigma] = \big\{a \,\big|\, a \in \mathscr{P}_{\mathscr{C}}[U],\ \mathscr{P}_{\mathscr{C}}[\sigma]a = a\big\}.$$

Using similar techniques, by Möbius inversion on the poset $\mathscr{P}_{\mathscr{C}}[U,\sigma]$, and using the formalism of [4] for the combinatorics of plethysm, we obtain

$$\operatorname{Ch}\mathscr{C}^{\langle -1\rangle}(\mathbf{x}) = \left(\operatorname{Ch}\mathscr{C}(\mathbf{x})\right)^{\langle -1\rangle}. \tag{154}$$

Example 56. The species E_+ is a c-operad, there is a unique natural transformation $\eta : E_+(E_+) \to E_+$ sending each element of $E_+(E_+)$ (a partition) into its subjacent set. The tensor species associated to E_+ is the operad $\mathscr{C}$om in [17].

The Möbius species $E_+^{\langle -1\rangle}[U] = ((E_+(E_+))[U], \leqslant_\eta)$ is equal to $\{\Pi[U]\}$, where $\Pi[U]$ is the classical poset of partitions on U ordered by refinement. By Eq. (154),

$$\begin{aligned}\operatorname{Ch} E_+^{\langle -1\rangle}(\mathbf{x}) &= \sum_{n=1}^{\infty}\sum_{\alpha\vdash n}\mu_{\Pi_\alpha[n]}(\hat{0},\hat{1})\frac{p_\alpha(\mathbf{x})}{z_\alpha} = \left(\operatorname{Ch} E_+(\mathbf{x})\right)^{\langle -1\rangle}\\ &= \left(e^{\sum_{n=1}^{\infty}\frac{p_n(\mathbf{x})}{n}} - 1\right)^{\langle -1\rangle} = \ln\left(\prod_{n=1}^{\infty}(1+p_n)^{\mu(n)/n}\right),\end{aligned}$$

from this, we recover Hanlon result about the Möbius function of $\Pi_\sigma[U]$, the poset of partitions fixed by the permutation σ [18, Theorem 4].

Example 57. The c-operads are closed under pointing [26]. The species of pointed sets $E^\bullet[U] = (XD)E[U] = U$ is a c-operad [26, Example 3.13(3)]. It induces the poset of pointed partitions $\mathscr{P}_{E^\bullet}[U]$, with $n = |U|$ the number of maximal elements. $E^\bullet(x) = xe^x$ and $\operatorname{Ch} E^\bullet(\mathbf{x}) = p_1(\mathbf{x})e^{\sum_{n=0}^{\infty}\frac{p_n(\mathbf{x})}{n}}$. We obtain that

$$f(x) = \operatorname{Möb}\left(E^\bullet\right)^{\langle -1\rangle}(x) = \left(xe^x\right)^{\langle -1\rangle}$$

and

$$g(\mathbf{x}) = \operatorname{Ch}\left(E^\bullet\right)^{\langle -1\rangle}(\mathbf{x}) = \left(p_1(\mathbf{x})e^{\sum_{n=0}^{\infty}\frac{p_n(\mathbf{x})}{n}}\right)^{\langle -1\rangle}.$$

Since $f(x)e^{f(x)} = x$ and $g(\mathbf{x})\exp(\sum_{n=1}^{\infty} p_n(\mathbf{x}) * g(\mathbf{x})/n) = p_1(\mathbf{x})$, $f(x)$ is the Lambert function (see [12]) and $g(\mathbf{x})$ its plethystic analog.

By Lagrange inversion in one variable we obtain $|(E^\bullet)^{\langle -1\rangle}[n]|_\mu = (-1)^{n-1}n^{n-1}$. Because all the intervals in $(E^\bullet)^{\langle -1\rangle}[n]$ are isomorphic, and there are n of them, we have $\mu[\hat{0}, \{([n],1)\}] = (-1)^{n-1}n^{n-2}$.

The n-pointed set species $E^{\bullet^n} = (XD)^n E$ is a c-operad, inducing the family of posets of n-pointed partitions. The generating function of $E^{\bullet^n}$ is equal to $(xD)^n e^x = \phi_n(x)e^x$, $\phi_n(x)$ being the exponential polynomial.

7. Koszul duality for operads

7.1. Partial composition

Let $(\mathscr{O}, \eta)$ be a tensor, **g**, or **dg**-operad. For each finite set, the morphism

$$\eta_U : \mathscr{O}(\mathscr{O})[U] = \bigoplus_{\pi\in\Pi[U]}\left(\bigotimes_{B\in\pi}\mathscr{O}[B]\right)\otimes\mathscr{O}[\pi] \to \mathscr{O}[U], \tag{155}$$

decomposes as a direct sum of morphisms η_U^π, $\pi \in \Pi[U]$,

$$\eta_U^\pi : \left(\bigotimes_{B\in\pi}\mathscr{O}[B]\right)\otimes\mathscr{O}[\pi] \to \mathscr{O}[U]. \tag{156}$$

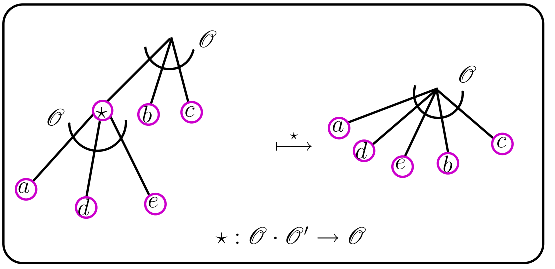

Fig. 5. Graphical representation of the partial composition $\star : \mathscr{O} \cdot \mathscr{O}' \to \mathscr{O}$, for an operad $\mathscr{O}$.

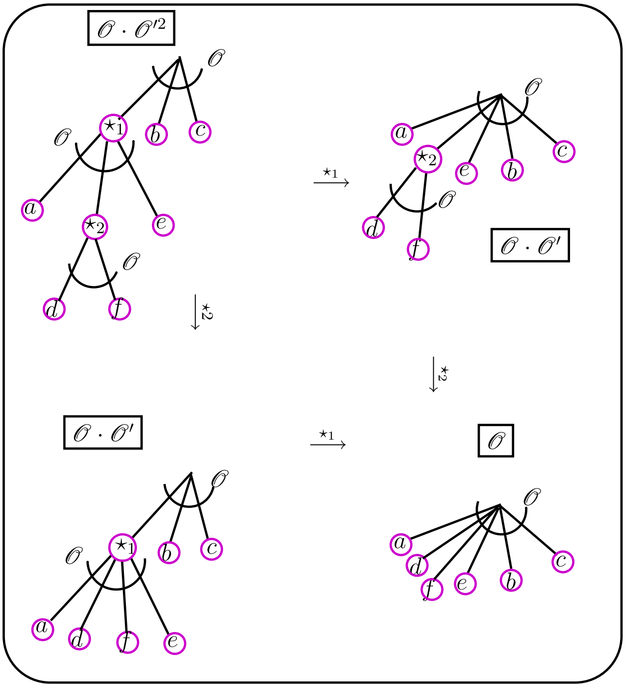

Fig. 6. Graphical representation of the commutation of partial compositions.

Choosing the partition with only one non-singleton block $U_1 \subset U$; $\pi = \{U_1\} \cup \{\{u\} \mid u \in U_2\}$, $U_2 = U \backslash U_1$, the natural transformation η_U^{π} goes from $\mathscr{O}[U_1] \otimes \mathscr{O}[U_2 \uplus \{U_1\}]$ to $\mathscr{O}[U]$. By the definition of derivative, for each proper decomposition $U_1 \uplus U_2 = U$, we have a natural morphism from $\mathscr{O}[U_1] \otimes \mathscr{O}'[U_2]$ to $\mathscr{O}[U]$. Using the definition of product, we obtain a natural transformation (partial composition) from the product $\mathscr{O} \cdot \mathscr{O}'$ to $\mathscr{O}$ (see Fig. 5). The product $\mathscr{O} \cdot \mathscr{O}'$ is clearly isomorphic to the species $\mathscr{F}_{\mathscr{O}}^2$ of $\mathscr{O}$-enriched trees with exactly two internal vertices. By abuse of language we denote this partial composition with the same symbol $\star$ used as 'ghost element' in the definition of derivative. Iterating this procedure we obtain the natural transformations (partial compositions) denoted by $\star_1$ and $\star_2$, from $\mathscr{O} \cdot (\mathscr{O} \cdot \mathscr{O}')' = \mathscr{O}^2 \cdot \mathscr{O}'' + \mathscr{O} \cdot \mathscr{O}'^2$ to $\mathscr{O} \cdot \mathscr{O}'$. The composed of those partial compositions $\star_1 \circ \star_2$ and $\star_2 \circ \star_1$ go from $\mathscr{O}^2 \cdot \mathscr{O}'' + \mathscr{O} \cdot (\mathscr{O}')^2$ to $\mathscr{O}$. The associativity of η gives us the identity $\star_1 \circ \star_2 = \star_2 \circ \star_1$ (see Fig. 6). Conversely, from a partial composition $\star : \mathscr{O} \cdot \mathscr{O}' \to \mathscr{O}$ such that the derived partial compositions $\star_1$ and $\star_2$ commute, we can give $\mathscr{O}$ an operad structure, $\eta : \mathscr{O}(\mathscr{O}) \to \mathscr{O}$.

For a tree t in $\mathscr{F}_{\mathscr{O}}[U]$ and a pair of internal vertices v and v', such that v is a son of v', a partial composition can be performed through v and the resulting tree will be denoted by $v(t) \in \mathscr{F}_{\mathscr{O}}[U]$.

Definition 58 *(Quadratic operads).* Let G be a tensor species of the form $G = X + G_{2^+}$. The species of enriched Schröder trees $\mathscr{F}_G$ is the free operad generated by G. Let R be a subspecies of $\mathscr{F}_G^{\underline{2}}$. Define $\mathscr{R}_{\mathscr{O}} = \langle\langle R \rangle\rangle$ to be the operad ideal generated by R in $\mathscr{F}_G$. The operad $\mathscr{O} = \mathscr{F}_G/\mathscr{R}_{\mathscr{O}}$ will be called the quadratic operad generated by G with quadratic relations in R, and denoted by $\mathscr{O} = \mathcal{O}(G, R)$. There is a natural graduation on $\mathscr{O}$ given by $\mathscr{O}^{\underline{g}} = \mathscr{O}^{\underline{0}} + \mathscr{O}^{\underline{1}} + \sum_{k=2}^{\infty} \mathscr{F}_G^{\underline{k}}/\mathscr{R}_{\mathscr{O}}^{\underline{k}}$, where $\mathscr{O}^{\underline{0}} = X$ and $\mathscr{O}^{\underline{1}} = G_{2^+}$. The *quadratic dual cooperad* of $\mathscr{O}$ (see [16]) is defined by

$$\mathscr{O}^{\mathrm{i}} := \mathcal{O}\big(G^*, R^{\perp}\big)^*. \tag{157}$$

When G is concentrated in cardinality 2, the *quadratic dual* of $\mathscr{O}$ (see [17]) is defined by

$$\mathscr{O}^{!} := \mathcal{O}\big(G^* \odot \Lambda_2, \widetilde{R}^{\perp}\big), \tag{158}$$

where $\widetilde{R}$ is the kernel of the natural transformation $\star^{\underline{2}} : \mathscr{F}^{\underline{2}}_{G\odot\Lambda_2} \to \mathscr{O}_3 \odot \Lambda_3$, $\star$ being the partial composition on the operad $\mathscr{O} \odot \Lambda$.

Remark 59. Observe that if $\mathscr{O} = \mathcal{O}(G, R)$, R is the kernel of

$$\star : G_{2^+} \cdot G'_{2^+} \to \mathscr{O}^{\underline{2}}.$$

Hence, $\mathscr{O}$ is quadratic if and only if $\mathscr{O} = \mathscr{F}_G/\langle\langle \mathrm{Ker}(\star)\rangle\rangle$, $G_{2^+} = \mathscr{O}^{\underline{1}}$.

7.2. Bar construction for operads

We follow here B. Fresse (see [16]) on his generalization of the original Ginzburg–Kapranov definition of Koszul operads. Let $\mathscr{O} = \mathcal{O}(G, R)$ be a quadratic operad and $\mathscr{O}^{\underline{g}}$ its corresponding graded operad. We are going to construct a **dg**-species by defining a differential on the inverse $\mathscr{O}^{\underline{g}\langle -1\rangle} = \mathscr{F}_{\leftharpoondown\mathscr{O}^{\underline{g}}}$. Explicitly, using Eqs. (49) and (66)

$$\mathscr{F}_{\leftharpoondown\mathscr{O}}[U] = \bigoplus_{t\in\mathscr{F}_E[U]} \bigotimes_{v\in \mathrm{Iv}(t)} \leftharpoondown\mathscr{O}[\pi_v] = \bigoplus_{t\in\mathscr{F}_E[U]} s^{-|\mathrm{Iv}(t)|}\Bigg(\bigotimes_{v\in\mathrm{Iv}(t)} \mathscr{O}[\pi_v]\Bigg) \otimes \Lambda\big[\mathrm{Iv}(t)\big]. \tag{159}$$

Hence, a decomposable element of $\mathscr{F}^{\underline{k}}_{\leftharpoondown\mathscr{O}^{\underline{g}}}[U]$ can be written in the form $v_1 \wedge v_2 \wedge \cdots \wedge v_{r-1} \wedge v_r \otimes t$ for some $r \geqslant 0$, where t is an $\mathscr{O}$-enriched Schröder tree $t \in \mathscr{F}^{\underline{r}}_{\mathscr{O}}[U]$ and $\{v_1, v_2, \ldots, v_r\}$ is the set of internal vertices of t in any order such that the root is indexed as last element v_r. Define $\tilde{d} : \mathscr{F}^{\underline{k}}_{\leftharpoondown\mathscr{O}^{\underline{g}}} \to \mathscr{F}^{\underline{k+1}}_{\leftharpoondown\mathscr{O}^{\underline{g}}}$, $k = 0, 1, 2, \ldots,$

$$\tilde{d}(v_1 \wedge v_2 \wedge \cdots \wedge v_{r-1} \wedge v_r \otimes t) = \sum_{i=1}^{r-1} (-1)^{i-1} v_1 \wedge v_2 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_r \otimes v_i(t). \tag{160}$$

Clearly $\tilde{d}^2 = 0$, and this will be called the bar construction for operads and denoted by $\mathscr{B}\mathrm{ar}(\mathscr{O})$; $\mathscr{B}\mathrm{ar}(\mathscr{O}) = (\mathscr{O}^{\underline{g}\langle -1\rangle}, d)$. It is easy to see that $\mathscr{B}\mathrm{ar}(\mathscr{O})^{\underline{0}} = \mathscr{F}_G$, and that the cohomology $H^0\,\mathscr{B}\mathrm{ar}(\mathscr{O})$ is isomorphic to the quadratic dual cooperad $\mathscr{O}^{\mathrm{i}}$.

Definition 60. A quadratic operad $\mathscr{O}$ is called Koszul if the **g**-species $\mathcal{H}\mathscr{B}\mathrm{ar}(\mathscr{O})$ is concentrated in degree zero.

Proposition 61. *If $\mathscr{O}$ is Koszul, then*

$$\mathscr{O}^{\mathrm{i}}(x) = \left(\mathscr{O}^{\underline{g}}(x)\right)^{\langle -1\rangle}, \tag{161}$$

$$\mathrm{Ch}\,\mathscr{O}^{\mathrm{i}}(\mathbf{x}) = \left(\mathrm{Ch}\,\mathscr{O}^{\underline{g}}(\mathbf{x})\right)^{\langle -1\rangle}. \tag{162}$$

Proof. Similar to the proof of Proposition 20. □

7.3. The operads of M-enriched and small trees

We now consider the species $\mathscr{A}$, of rooted trees where all the vertices (internal and leaves) are labelled. It is the solution of the implicit equation $\mathscr{A} = XE(\mathscr{A})$ (see [4]).

More generally, for an arbitrary species M, $|M[\emptyset]| = 1$, the species $\mathscr{A}_M$ of M-enriched trees is defined as the solution of the implicit equation

$$\mathscr{A}_M = XM(\mathscr{A}_M). \tag{163}$$

The species $\mathscr{A}_M$ is explicitly defined as follows

$$\mathscr{A}_M[U] = \bigoplus_{t\in\mathscr{A}[U]} \bigotimes_{u\in U} M\left[t^{-1}(u)\right], \tag{164}$$

where for each vertex u of the rooted tree $t \in \mathscr{A}[U]$, $t^{-1}(u)$ denotes the fiber (set of sons) of u in t. An M-enriched tree t_U^M is a decomposable element of $\mathscr{A}_M[U]$. From Eq. (164) each one of it is of the form $t_U^M = (t_U, \bigotimes_{u\in U} m_u)$, where $t_U \in \mathscr{A}[U]$ and for each $u \in U$, $m_u \in M[t^{-1}(u)]$.

The (set) species of rooted trees $\mathscr{A}$ has a c-operad structure $\eta : \mathscr{A}(\mathscr{A}) \to \mathscr{A}$. An element of $\mathscr{A}(\mathscr{A})[U]$ is a pair (a, t') where $a = \{t_B\}_{B\in\pi}$ is a forest of rooted trees and t' is a rooted tree on the blocks of the partition π. The tree $t = \eta(a, t')$ will have all the edges in the forest a plus a few more defined as follows: for every pair of trees t_{B_1} and t_{B_2} such that B_1 and B_2 form an edge in t', insert an edge between the roots of t_{B_1} and t_{B_2}. The operadic structure of $\mathscr{A} = \mathscr{A}_E$ can be generalized to the species of rooted trees enriched with a monoid (M, ν), $\nu : M.M \to M$ (see [25], and [26] for a set-operad version of this construction). For that end, it is enough to describe an appropriate partial composition $\star : \mathscr{A}_M \cdot \mathscr{A}_M' \to \mathscr{A}_M$. Let $t_{U_1}^M \otimes t_{U_2}^M, U_1 \uplus U_2 = U$ be an element of $(\mathscr{A}_M \cdot \mathscr{A}_M')[U]$, $t_{U_1}^M \in \mathscr{A}_M[U_1]$ and $t_{U_2}^M \in \mathscr{A}_M'[U_2] = \mathscr{A}_M[U_2 \uplus \{\star\}]$ being M-enriched trees. We are going to describe the tree $t_U^M = \star(t_{U_1}^M \otimes t_{U_2}^M) \in \mathscr{A}_M[U]$. Assume first that the ghost element of $t_{U_2}^M$ is the root. Then t_U^M is obtained by imbedding the $t_{U_1}^M$ into $t_{U_2}^M$ using the following procedure. Place the tree $t_{U_1}^M$ into $t_{U_2}^M$ by replacing the (ghost) root of it by the root r of $t_{U_1}^M$ and enriching its fiber $t_U^{-1}(r) = t_{U_1}^{-1}(r) \uplus t_{U_2}^{-1}(\star)$ with the vector $\nu(m_r \otimes m_\star)$, $m_r \in M[t_{U_1}^{-1}(r)]$ and $m_\star \in M[t_{U_2}^{-1}(\star)]$ being the structures enriching the fibers of the roots of t_{U_1} and t_{U_2} respectively (see Fig. 7). When the ghost element of $t_{U_2}^M$ is not its root, imbed $t_{U_1}^M$ in $t_{U_2}^M$ by using the previous procedure with the subtree of $t_{U_2}^M$ formed by the descendants of $\star$ ($\star$ will be the root of it). By the associativity of ν we have the commutation of the partial compositions.

An M-enriched tree will be called *small* if all the vertices different from the root are its sons. Denote by $\mathscr{T}_M$ the species of M-enriched small trees. $\mathscr{T}_M$ is clearly isomorphic to XM.

By the product rule for the derivative (see [21]), we have that $\mathscr{T}_M\mathscr{T}_M' = XM(M + XM') = XM^2 + X^2MM'$. Hence, if M is a monoid (M, ν), $\mathscr{T}_M$ has a natural operad structure given by the partial composition $\star = I_X.\nu + 0$:

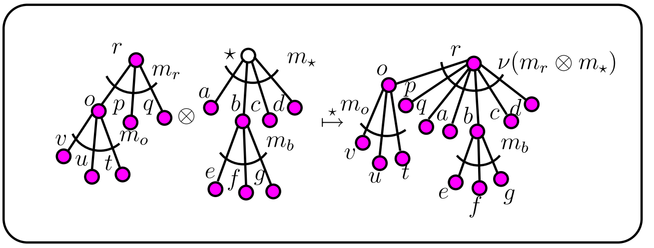

Fig. 7. The partial composition for the operad of M-enriched rooted trees.

$$\star : XM^2 + X^2MM' \to XM. \tag{165}$$

The commutation of partial compositions follows from the associativity of the product ν. The proof of the following proposition is easy, and left to the reader.

Proposition 62. *The operad $\mathscr{T}_M$ is quadratic if and only if M is a quadratic monoid. Moreover, we have that if $M = \mathcal{M}(F, R)$, then $\mathscr{T}_M = \mathcal{O}(XF, XR + X^2FF')$ and conversely.*

Theorem 63. *Let M be a species* (*of any kind*) *of the form $M = 1 + M_+$. Then, the species $\mathscr{F}_{XM}$ and $\mathscr{A}_{\mathbb{L}(M_+)}$ are isomorphic. Moreover, if M is a quadratic monoid, we have the following isomorphism of* **dg**-*tensor species*

$$\mathscr{B}\mathrm{ar}(\mathscr{T}_M) = \mathscr{A}_{\mathcal{B}\mathrm{ar}(M)}. \tag{166}$$

Proof. An XM-enriched tree t in $\mathscr{F}_{XM}[U]$ is a rooted tree, where each internal node $v \in \mathrm{Iv}(t)$ is decorated with an element of

$$M_+\big[\pi_v - \{v'\}\big] \equiv \mathbb{K}\{v'\} \otimes M_+\big[\pi_v - \{v'\}\big] \subseteq (XM_+)[\pi_v] = \bigoplus_{v'' \in \pi_v} \mathbb{K}\{v''\} \otimes M_+\big[\pi_v - \{v''\}\big],$$

π_v standing for the set of sons of v. So, at each internal vertex v, a preferred son $v' = p(v)$ is chosen and an element m_v is placed on the rest of them ($m_v \in M[\pi_v - \{p(v)\}]$). By starting at the root of t, we can construct a unique path from the root to a leaf (the *distinguished leaf* of t), by letting the successor of an internal node be its preferred son. We shall call this path the *main spine* of t. For each internal vertex v in the main spine, and for each $w \in \pi_v - \{p(v)\}$, the descendants of w form a tree in $\mathscr{F}_{XM}[U_w]$. The XM-enriched tree t is then an element of the species $X[\mathbb{L}(M) \circ \mathscr{F}_{XM}][U]$ (see Fig. 9). Since the M-enriched trees generate the vector space $\mathscr{F}_{XM}[U]$, $\mathscr{F}_{XM}$ satisfies the implicit equation

$$\mathscr{F}_{XM} = X\big[\mathbb{L}(M_+) \circ \mathscr{F}_{XM}\big]. \tag{167}$$

This is the same implicit equation defining the species $\mathscr{A}_{\mathbb{L}(M_+)}$. By the theorem of implicit equations for species ([21], see also [5]), straightforwardly extended to this context, we obtain the result. As a consequence of that we obtain

$$\big(\mathscr{T}_M^{\mathrm{g}}\big)^{\langle -1\rangle} = \mathscr{F}_{-XM^{\mathrm{g}}} = \mathscr{F}_{X(-M^{\mathrm{g}})} = \mathscr{A}_{\mathbb{L}(-M^{\mathrm{g}})} = \mathscr{A}_{(M^{\mathrm{g}})^{-1}}. \tag{168}$$

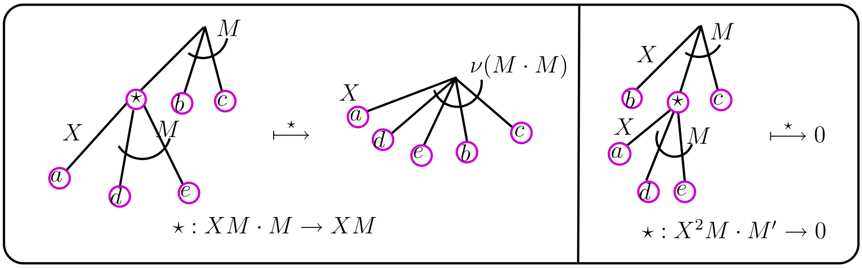

Fig. 8. The partial composition for the operad of small trees.

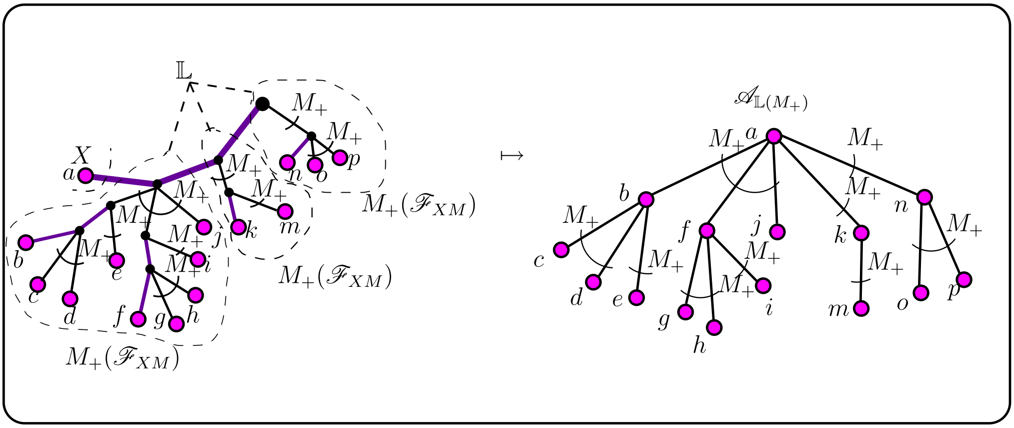

Fig. 9. The isomorphism $\mathscr{F}_{XM} = X\cdot[\mathbb{L}(M_+)\circ\mathscr{F}_{XM}] = \mathscr{A}_{\mathbb{L}(M_+)}$.

Eq. (166) is equivalent to the implicit equation

$$\mathscr{B}ar(\mathscr{T}_M) = X\,\mathcal{B}ar(M)\big(\mathscr{B}ar(\mathscr{T}_M)\big). \tag{169}$$

We rewrite it as follows

$$(\mathscr{F}_{\leftharpoonup XM^{g}}, \tilde{d}) = X\cdot\big[\big(\mathbb{L}\big(\leftharpoonup M_+^{g}\big), d\big)\big((\mathscr{F}_{\leftharpoonup XM^{g}}, \tilde{d})\big)\big], \tag{170}$$

where $\tilde{d}$ is as in Eq. (160) and d is as in (99). By Eq. (168) we only have to prove that the differentials match. To that end, choose a Schröder tree $t\in\mathscr{F}_{\leftharpoonup XM^{g}}[U]$, and assume that t has k internal vertices. Order them by indexing the root as last element v_k and the internal vertices on the main spine $\{v_1, v_2, \dots, v_j\}$ enumerated as follows: $v_1 = p(v_k)$ is the preferred son of the root, and $v_{i+1} = p(v_i)$, $i = 1, 2, \dots, j-1$ (see Fig. 10). Apply the same procedure recursively on each subtree attached to the (non-preferred) sons of the vertices on the main spine. By the definition of partial composition on small trees (see Eq. (165) and Fig. 8) it is easy to see that the sum defining $\tilde{d}(t)$ restricted to the first j terms (the vertices on the main spine) is equal to d applied to $m_{v_1}\otimes m_{v_2}\otimes\cdots\otimes m_{v_j}$. Denote by $\pi = \{v_{j+1}, v_{j+2}, \dots, v_s\} = \biguplus_{i=2}^{j}\tilde{\pi}_{v_i}$, $\tilde{\pi}_{v_i} = \pi_{v_i} - \{v_{i+1}\}$, $i = 1, \dots, j$, the set of sons of the vertices on the main spine that are outside of it, and by t_{v_i}, the corresponding subtree of the descendants of v_i in t, $i = j+1, j+2, \dots, s$. We have

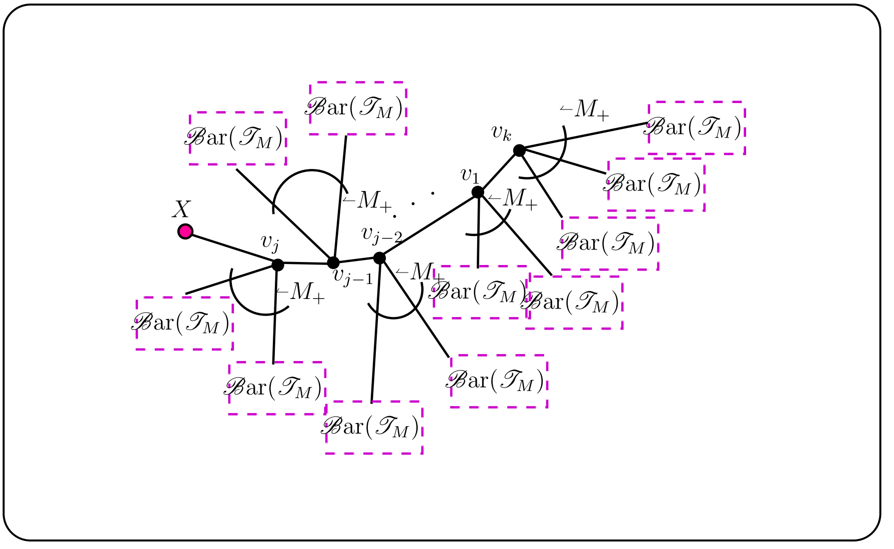

Fig. 10. Graphical representation of the equation $\mathscr{B}\mathrm{ar}(\mathscr{T}_M) = X \cdot \mathcal{B}\mathrm{ar}(M)(\mathscr{B}\mathrm{ar}(\mathscr{T}_M))$.

$$\tilde{d}(t) = d(m_{v_1} \otimes m_{v_2} \otimes \cdots \otimes m_{v_j}) \otimes \bigotimes_{i=j+1}^{s} t_{v_i} \tag{171}$$

$$+ \sum_{i=j+1}^{s} \pm m_{v_1} \otimes m_{v_2} \otimes \cdots \otimes m_{v_j} \otimes t_{v_{j+1}} \otimes \cdots \otimes \tilde{d}(t_{v_i}) \otimes \cdots \otimes t_{v_s}. \tag{172}$$

Then, we get

$$(\mathscr{F}_{\leftharpoonup X M^{\mathrm{g}}}, \tilde{d})[U] = \bigoplus_{u \in U} \bigoplus_{\pi \in \Pi[U - \{u\}]} \big(\mathbb{L}\big(\leftharpoonup M_+^{\mathrm{g}}\big), d\big)[\pi] \otimes \bigotimes_{B \in \pi} (\mathscr{F}_{\leftharpoonup X M^{\mathrm{g}}}, \tilde{d})[B] \tag{173}$$

$$= X \cdot \big[\big(\mathbb{L}\big(\leftharpoonup M_+^{\mathrm{g}}\big), d\big)\big((\mathscr{F}_{\leftharpoonup X M^{\mathrm{g}}}, \tilde{d})\big)\big][U]. \qquad \square \tag{174}$$

Corollary 64. *Let M be a quadratic monoid. Then we have*

$$\mathscr{T}_M^{¡} = \mathscr{A}_{M^{¡}}. \tag{175}$$

Moreover, M is a Koszul monoid if and only if $\mathscr{T}_M$ is a Koszul operad.

Proof. Let G be a **dg**-tensor species of the form $G = 1 + G_+$. From the properties of the functor $\mathcal{H}$ we have that

$$\mathcal{H}\mathscr{A}_G = \mathcal{H}X \cdot (\mathcal{H}G \circ \mathcal{H}\mathscr{A}_G) = X \cdot (\mathcal{H}G \circ \mathcal{H}\mathscr{A}_G), \tag{176}$$

hence, $\mathcal{H}\mathscr{A}_G = \mathscr{A}_{\mathcal{H}G}$.

From Eq. (166) and the explicit form for the species of enriched rooted trees (Eq. (163)),

$$\mathcal{H}\mathscr{B}\mathrm{ar}(\mathscr{T}_M) = \mathscr{A}_{\mathcal{H}\mathcal{B}\mathrm{ar}(M)}, \tag{177}$$

$$\mathcal{H}\mathscr{B}\mathrm{ar}(\mathscr{T}_M)[U] = \bigoplus_{t\in\mathscr{A}[U]}\bigotimes_{u\in U}\big(\mathcal{H}\mathcal{B}\mathrm{ar}(M)\big)\big[t^{-1}u\big]. \tag{178}$$

Then, by the definition of tensor product in the category $\mathrm{gVec}_{\mathbb{K}}$

$$H^i\mathscr{B}\mathrm{ar}(\mathscr{T}_M)[U] = \bigoplus_{t\in\mathscr{A}[U]}\bigoplus_{\sum_{u\in U} j_u=i}\left(\bigotimes_{u\in U} H^{j_u}\mathcal{B}\mathrm{ar}(M)\big[t^{-1}u\big]\right). \tag{179}$$

From this we obtain that $H^0\mathscr{B}\mathrm{ar}(\mathscr{T}_M) = \mathscr{A}_{H^0\mathcal{B}\mathrm{ar}(M)}$ and hence Eq. (175). Moreover if $\mathcal{H}\mathcal{B}\mathrm{ar}(M)$ is concentrated in degree zero, $\mathcal{H}\mathscr{B}\mathrm{ar}(\mathscr{T}_M)$ is also. Conversely, if $\mathcal{H}\mathscr{B}\mathrm{ar}(\mathscr{T}_M)$ is concentrated in degree zero, by restricting the direct sum in (179) to the set $\mathscr{T}_E[U]$ of small trees, we obtain that for $i\neq 0$, $X\cdot(H^i\mathcal{B}\mathrm{ar}(M)) = 0$. Then $H^i\mathcal{B}\mathrm{ar}(M) = 0$. □

Corollary 65. *The monoid M is Koszul if and only if $\mathscr{A}_M$ and $\mathscr{A}_{M^{!\cdot}}$ are both Koszul operads.*

Proof. By duality according to Fresse (see [16, Lemmas 5.2.9 and 5.2.10]), $\mathscr{T}_M$ is a Koszul operad if and only if the operad $(\mathscr{T}_M^{\mathsf{i}})^* = (\mathscr{A}_{M^{\mathsf{i}\cdot}})^* = \mathscr{A}_{M^{!\cdot}}$ is so. The result follows from Corollary 64 and from the fact that M and $M^{!\cdot}$ are simultaneously Koszul. □

Observe that if M is generated by a species F_1 concentrated in cardinality 1, then $\mathscr{T}_M$ is generated by XF_1, concentrated in cardinality 2. The quadratic dual of $\mathscr{T}_M$, according to Ginzburg and Kapranov [17], is equal to $(\mathscr{A}_{M^{\mathsf{i}\cdot}})^*\odot\Lambda = \mathscr{A}_{M^{!\cdot}}\odot\Lambda$.

We now restate Vallette result (see [40, Theorem 9]), in this context and using the terminology of this article.

Theorem 66. *Let $\mathscr{C}$ be a quadratic c-operad generated by a species concentrated in some cardinality k, $\mathscr{C} = \mathcal{O}(G_k, R)$. Then, $\mathscr{C}$ is Koszul if and only if the Möbius species $\mathscr{C}^{\langle -1\rangle}$ is Cohen–Macaulay.*

If $\mathscr{C}$ is as in the previous theorem, we obtain

$$\text{Möb}\,\mathscr{C}^{\langle -1\rangle}(x) = \big(\mathscr{C}^{\mathsf{i}}\big)^{\underline{g}}(x), \tag{180}$$

$$\mathrm{Ch}\,\mathscr{C}^{\langle -1\rangle}(\mathbf{x}) = \big(\mathrm{Ch}\,\mathscr{C}^{\mathsf{i}}\big)^{\underline{g}}(\mathbf{x}) \tag{181}$$

we have also

$$\sum_{c\in\mathscr{C}^{\underline{k}}[n]}\mu\big(\hat{0},\{c\}\big) = (-1)^k\dim\big(\mathscr{C}^{\mathsf{i}}\big)^{\underline{k}}[n], \tag{182}$$

$$\sum_{c\in\mathscr{C}^{\underline{k}}[n],\ \mathscr{C}[\sigma]c=c}\mu\big(\big[\hat{0},\{c\}\big]_\sigma\big) = (-1)^k\,\mathrm{tr}\big(\mathscr{C}^{\mathsf{i}}\big)^{\underline{k}}[\sigma]. \tag{183}$$

It is easy to check that if M is a c-monoid, the operad $\mathscr{A}_M$ is a c-operad. The following corollary is trivial from the previous theory.

Corollary 67. *Let $M = \mathcal{M}(F_k, R)$ be a quadratic c-monoid generated by a species F_k concentrated in cardinality k. Then, the following statements are equivalent*

(1) *The Möbius species* M^{-1} *is Cohen–Macaulay.*
(2) M *is Koszul.*
(3) *The Möbius species* $\mathscr{A}_M^{\langle -1\rangle}$ *is Cohen–Macaulay.*
(4) *The operad* $\mathscr{A}_M$ *is Koszul.*

Example 68. By the previous corollaries we obtain the following.

- Since E is a Koszul monoid, the operads $\mathscr{T}_E$, $\mathscr{T}_\Lambda$, and $\mathscr{A} = (\mathscr{T}_\Lambda^{¡})^* = \mathscr{T}_E^!$ are Koszul. We also have that $\mathscr{A}_\Lambda = (\mathscr{T}_E^{¡})^*$ is Koszul.
- By Proposition 27, the monoid Cosh $:= E_{(2)}$ is Koszul. Then, $\mathscr{T}_{\mathrm{Cosh}}$ and $\mathscr{A}_{\mathrm{Cosh}}$, respectively the species of small trees and rooted trees where each vertex has an even number of sons, are Koszul.
- By Proposition 29, $E \underline{\circ} E$ and $E \underline{\bullet} E$ are Koszul monoids. The operads $\mathscr{T}_{E\underline{\circ}E}$, $\mathscr{T}_{E\underline{\bullet}E}$, $\mathscr{A}_{E\underline{\circ}E}$ and $\mathscr{A}_{E\underline{\bullet}E}$ are Koszul.
- Since the c-monoids E, $E_{(k)}$, $E_{(k_1)} \underline{\circ} E_{(k_2)}$, $E_{(k_1)} \underline{\circ} E_{(k_2)} \underline{\circ} E_{(k_3)}, \ldots,$ are Koszul, the Möbius species $\mathscr{A}^{\langle -1\rangle}$, $\mathscr{A}_{E_{(k)}}^{\langle -1\rangle}$, $\mathscr{A}_{E_{(k_1)}\underline{\circ}E_{(k_2)}}^{\langle -1\rangle}$, $\mathscr{A}_{E_{(k_1)}\underline{\circ}E_{(k_2)}\underline{\circ}E_{(k_3)}}^{\langle -1\rangle}, \ldots,$ are Cohen–Macaulay.

Koszulness for associative algebras, and hence for monoids in species, is a property that is closed under many operations (see [3]). Segre and Manin product are two among them. So, Corollaries 64 and 65 give us a wide class of Koszul operads.

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