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On n-Skein Isomorphisms of Graphs

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H. Whitney [*Amer. J. Math.* **54** (1932), 150–168] proved that edge isomorphisms between connected graphs with at least five vertices are induced by isomorphisms and that circuit isomorphisms between 3-connected graphs are induced by isomorphisms. R. Halin and H. A. Jung [*J. London Math. Soc.* **42** (1967), 254–256] generalized these results by showing that for $n \geqslant 2$, n-skein isomorphisms between $(n+1)$-connected graphs are induced by isomorphisms. In this paper we show that for $n \geqslant 2$, n-skein isomorphisms between 3-connected graphs having $(n+1)$-skeins are induced by isomorphisms.

1. INTRODUCTION

A mapping $\sigma: E(G) \to E(G^*)$ between the edge sets of the graphs G and G^* induces a mapping between the subgraphs of G and G^*: for a subgraph H of G having edges, the image, denoted by H', is the unique subgraph of G^* with $E(H') = \sigma(E(H))$ and having no isolated vertices (i.e., vertices of degree 0).

In [1] the following general question was raised. Suppose that $\sigma: E(G) \to E(G^*)$ induces a bijection between subgraphs of a certain type. Which conditions ensure that σ itself is induced by an isomorphism from G onto G^*?

The general question was motivated by two papers of Whitney [6, 7]. In [6], he showed (among other things) that, with minor exceptions, each *edge isomorphism* σ (i.e., σ and σ^{-1} preserve the adjacencies of edges) is induced by an isomorphism and that each circuit isomorphism σ is induced by an isomorphism if G and G^* are 3-connected. As a consequence of a result in [7] only the 3-connectedness of G needs to be hypothesized to get the circuit

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isomorphism result (for a further improvement of this result see [5]). Halin and Jung [1] generalized these results to arbitrary n-skeins where an *n-skein* is a graph spanned by n paths joining two vertices and having no other vertices in common. They called a bijection $\sigma: E(G) \to E(G^*)$ an *n-skein isomorphism* if σ induces a bijection between the two collections of n-skeins, and they showed that an n-skein isomorphism is induced by an isomorphism if G and G^* are $(n+1)$-connected $(n \geqslant 2)$.

Our main result, Theorem 2, will be that an n-skein isomorphism is induced by an isomorphism if G is 3-connected and contains at least one $(n+1)$-skein $(n \geqslant 2)$. On the way to this goal it is shown that for $n \geqslant 2$ each n-skein isomorphism is an $(n+1)$-skein isomorphism (cf. Theorem 1).

Obviously we cannot eliminate the hypothesis of 3-connectedness in Theorem 2. The following examples illustrate that σ can be quite arbitrary if G contains no $(n+1)$-skein.

Let $G_{n,m}$ be a graph on the globe formed by n arcs $A_1, A_2, ..., A_n$ of longitude and m circuits $C_1, C_2, ..., C_m$ of latitude: the vertices are the points of intersection and the edges are the corresponding segments. It is easily seen that $G_{n,m}$ contains exactly one n-skein if $n \geqslant 5$. Therefore each bijection of $E(G_{n,m})$ which maps the set of all longitudinal edges onto itself is an n-skein isomorphism $(n \geqslant 5)$.

From $G_{n,m}$ we construct $G_{n,m,k}$ by adding all "slanted" edges $[A_i \cap C_j, A_{i+1} \cap C_{j+t}]$ whenever $1 \leqslant i \leqslant n$, $1 \leqslant t \leqslant k$ and $j + t \leqslant m$ (put $A_{n+1} = A_1$). Now assume $m \geqslant 1$, $k \geqslant 1$ and $n > 2k + 4$ (the maximum degree of a vertex in any C_i). Then the set E of all edges which do not belong to any n-skein of $G_{n,m,k}$ includes all edges of $C_1 \cup C_m$. An arbitrary bijection on E together with the identity on $E(G_{n,m,k}) - E$ defines an n-skein isomorphism of $G_{n,m,k}$. It is easily seen that this graph is $(n+4)/2$-connected.

The graphs in this paper are allowed to be infinite and to have multiple edges.

2. NOTATION AND THEOREM 1

In all that follows, an integer $n \geqslant 2$, graphs G and G^*, and an n-skein isomorphism $\sigma: E(G) \to E(G^*)$ are fixed.

On several occasions we will consider a subgraph H of G which is either an $(n+1)$-skein or an $(n+1)$-skein together with a path joining vertices of this $(n+1)$-skein. Although we do not exploit the generality it seems convenient to introduce the following concepts for arbitrary graphs H.

A path P in H is called a *special path of H* if each inner vertex of P has degree 2 in H while at least one endvertex of P has degree at least 3 in H. If both endvertices of P have degree at least 3 in H we call P a *constituent path of H*. Note that this characterizes the *special vertices of H* (special paths of length zero) as those with degree at least 3 in H. A path P connecting vertices v_1 and v_2 of H and having no other vertex in common with H is called an *H-jumper*. P is called a *crossover H-jumper* if, in addition, v_1 and v_2 are inner vertices of different constituent paths of H.

For vertices x and y on a path P let $P[x,y]$ denote the unique subpath of P with endvertices x and y. The (possibly empty) subpath of $P[x,y]$ comprising only the inner vertices of $P[x,y]$ is denoted by $P(x,y)$.

THEOREM 1. *Let σ be an n-skein isomorphism between graphs G and G^* with $n \geqslant 2$. Then σ is an $(n+1)$-skein isomorphism between G and G^*. Moreover σ maps each constituent path of an $(n+1)$-skein S onto a constituent path of S'.*

Proof. Let S be an $(n+1)$-skein in G having constituent paths $P_1, P_2,..., P_{n+1}$. Further let S_i be the n-skein obtained from S by deleting edges and inner vertices of P_i. By hypothesis S_i' is an n-skein.

(I) We first observe that each edge e' of P_i' is contained in some (S_i')-jumper $Q' = Q'(e')$ with $E(Q') \subseteq E(P_i')$: for pick any $j \neq i$ and any edge $f' = [x,y]$ in S' such that $f' \notin E(P_i') \cup E(P_j')$. Then some path in S_j' joins x to y and contains e'. Since x and y are also vertices of S_i' some subpath Q' of that path has the desired property.

We complete the proof by considering two cases; the first giving the desired conclusion and the second leading to a contradiction.

Case 1. Suppose S' has a vertex x with degree at least $n+1$ in S'. If no edge of P_i' is incident with x, then x also has degree at least $n+1$ in S_i'—which is impossible. Hence x is incident with some edge e_i' of P_i' $(i = 1, 2,..., n+1)$. Consider $Q_i' = Q'(e_i')$ as defined in (I). Since e_i' is incident with x which has degree (at least) n in S_i' the graph $S_i' \cup Q_i'$ contains more than one n-skein, and hence the preimage $S_i \cup Q_i$ does also. We infer that $Q_i' = P_i'$ and that x is an endvertex of P_i'.

Let y_i be the second endvertex of $Q_i' = P_i'$ $(i = 1, 2,..., n+1)$. Now y_i is not an inner vertex of any Q_j': for then S_j' is not an n-skein. Likewise $y_i \neq y_j$ is not possible. It follows that $y_1 = y_2 = \cdots = y_{n+1}$ and that S' is an $(n+1)$-skein with constituent paths $P_1',..., P_{n+1}'$.

Case 2. Suppose that each vertex x of S' has degree at most n in S'. Thus for each x, there is some P_i' with no edges incident with x: thus the degree of x is the same in S' and in S_i', and hence is 2 or n. This excludes the case $n = 2$ since S is not a 2-skein.

Choose $e_i' \in E(P_i')$ and determine $Q_i' = Q'(e_i')$ according to (I) $(i = 1, 2,..., n+1)$. The endvertices x_i and y_i of Q_i' have degree n in S'. Thus x_i is incident with an edge of P_i' and an edge of P_j' for some $j \neq i$. So x_i has degree 2 in S_i' and S_j' and hence has degree $n-2$ in P_i' and P_j'. Therefore

$n \geqslant 2(n-2)$ which means $n \leqslant 4$. We consider the remaining two cases separately.

($n=3$). In this case the graph $S_i' \cup Q_i'$ contains more than one n-skein. Therefore, as in Case 1, $E(Q_i') = E(P_i')$ $(i=1,2,3,4)$ and $S_1'(S_2')$ has constituent paths Q_2', Q_3' and Q_4' (Q_1', Q_3' and Q_4'). It follows that S' is a 4-skein with constituent paths Q_1', Q_2', Q_3' and Q_4', a contradiction (S' has no vertices of degree greater than 3).

($n=4$). Recall that each endvertex of Q_i' has degree 4 in S' and degree $n-2=2$ in P_i' and in some P_j' with $j \neq i$. We may assume that x_1 has degree 2 in P_2' and y_1 has degree 2 in P_2' or P_3'. One of x_3, y_3 is different from y_1, say $x_3 \neq y_1$. Now P_4' or P_5', say P_4', has no edge incident with x_3. Hence in S_4' the three different vertices x_1, y_1, x_3 have degree 4, a contradiction. This completes the proof of Theorem 1.

Note that a 1-skein isomorphism maps multiple edges into multiple edges. So, by virtue of Whitney's theorem (and its extension to infinite graphs by Jung [3]), Theorem 1 also holds in the case $n=1$, if G contains no components with less than five vertices.

3. The Main Result

Theorem 1 furnishes us with the first step in showing that σ is induced by an isomorphism. From it we see that edges of a constituent path in S are mapped into only one constituent path in S'; however, no restriction on the order of occurrence of the image edges within the path was obtained. In Lemmas 1 and 2 to follow, we show that a similar situation holds for jumpers. These results are sharpened in Lemma 3 where we find that the order of occurrence is preserved but the possibility of the path being "turned around" remains. Lemma 4 eliminates that possibility. The remaining part of the proof mainly involves showing that σ is an edge isomorphism.

In the following sequence of lemmas S will always be an $(n+1)$-skein in G with constituent paths $P_1, P_2, \ldots, P_{n+1}$ and special vertices a and b. Further $\mathscr{H}_1$ will denote the set of all $(n+1)$-skeins in G and $\mathscr{H}_2$ will denote the set of graphs H where $H = S \cup Q$ with $S \in \mathscr{H}_1$ and Q a crossover S-jumper. For a path P in G we will let $\mathscr{H}_i(P)$ denote the set of graphs in $\mathscr{H}_i$ that contain P as a special path $(i=1,2)$.

Lemma 1. *Let Q be a crossover S-jumper between P_i and P_j. Then Q' is a crossover S'-jumper between P_i' and P_j'; moreover σ maps each constituent path of $S \cup Q$ onto a constituent path of $S' \cup Q'$.*

Proof. By Theorem 1, S' is an $(n+1)$-skein whose constituent paths are the images of the constituent paths of S. For notational convenience, assume

$i=1$, $j=2$. Now let $T \subset S \cup Q$ be the n-skein containing Q, P_i $(3 \leqslant i \leqslant n+1)$, $P_1[a, q_1]$, and $P_2[q_2, b]$, where q_1 in P_1 and q_2 in P_2 are the endvertices of Q. So T is missing at least one edge from each of P_1 and P_2, and T' is an n-skein containing Q' and P_i' $(3 \leqslant i \leqslant n+1)$. Since Q' does not contain an n-skein, there is a subgraph R' of Q' that is an S'-jumper. The preimage of $S' \cup R'$ contains an n-skein not contained in S and we conclude that $R' = Q'$.

Since T' contains some but not all edges of P_i' $(i=1, 2)$, there is at least one pair of adjacent edges in P_i' with only one member of the pair in T'. Since the common vertex of such an adjacent pair is an inner vertex of P_i' it must be an endvertex of Q'. Hence there is only one such pair in each of P_1' and P_2' and the lemma follows.

LEMMA 2. *Let $H \in \mathscr{H}_1 \cup \mathscr{H}_2$ and let $J = J[j_1, j_2]$ be an H-jumper in G with j_1 an inner vertex of a constituent path R of H. Then J' is an H'-jumper. If j_2 is not on R, then σ maps constituent paths of $H \cup J$ onto constituent paths of $H' \cup J'$. If j_2 is on R, then J' and the image of $R[j_1, j_2]$ are paths with the same endvertices.*

Proof. As in the proof of Theorem 1 with P_i', we see that J' is an H'-jumper. By Lemma 1, R' is a constituent path of H'. Let T be an n-skein in $H \cup J$ that contains J and some but not all edges of R (there are several different possibilities to check here depending on H and J but such a T always exists). Thus there is at least one pair of adjacent edges from R' with only one member of the pair in T'. The common vertex of such a pair is of course an inner vertex of R' and so must be an endvertex of J'. We consider three cases.

Case 1. Let j_2 be an inner vertex of a constituent path R_1 of H with $R_1 \neq R$. Then, as above, the other endvertex of J' is an inner vertex of R_1'. Thus there can be no other pair of adjacent edges from R' or R_1' with only one member of the pair in T'. The lemma follows in this case.

Case 2. Let j_2 be a special vertex of H that is not on R. By applying the part of the lemma proved in Case 1 to σ^{-1}, H', and J' we see that J' does not have an endvertex that is an inner vertex of a constituent path of H' other than R'. Moreover there exists an n-skein T_1 in H containing J such that $E(R) - E(T) = E(T_1) \cap E(R)$. So not both endvertices of J' are in R' since T' and T_1' both contain J' and edges of $E(H') - E(R')$. Thus one endvertex of J' is a special vertex of H' not on R'. The lemma now follows in this case.

Case 3. Let j_2 be a vertex of R. Applying the parts of the lemma proved above to σ^{-1}, H', and J' we conclude that J' has both endvertices on R'; call them x and y. But the edges of $R'[x, y]$ are the elements of

$E(R')-E(T')$ which is $\sigma(E(R)-E(T))$. Hence σ maps $R[j_1,j_2]$ onto $R'[x,y]$ as claimed in the lemma.

LEMMA 3. *Let G be a 3-connected graph, P a path in G and $H\in\mathscr{H}=\mathscr{H}_1(P)\cup\mathscr{H}_2(P)$. Then there exists an isomorphism from P onto a special path of H' which induces σ on P. (Of course this special path is P' if P has positive length.)*

Proof. We proceed by induction on the length of P, denoted by $m(P)$, where H varies over $\mathscr{H}$. The assertion in the case $m(P)=0$ is merely that H' has special vertices—a trivial consequence of Theorem 1. So let P have vertices $p_0, p_1,..., p_m$ $(m>0)$ and be a subpath of the constituent path $R=R[p_0,r]$ of H. By the induction hypothesis there is an isomorphism of $P[p_0,p_{m-1}]$ onto a special path in H'. If $m=1$ we can choose this path (of length zero) to be one of the endvertices of R' and if $m\geqslant 2$, then, again by the induction assumption, $\sigma([p_0,p_1])$ is also a special path in H'. In either case we may assume that the image of $P[p_0,p_{m-1}]$ has vertices $w_0, w_1,..., w_{m-1}$ where w_0 is a special vertex of H on R' and where $\sigma([p_i,p_{i+1}])=[w_i,w_{i+1}]$ for all i with $0\leqslant i<m-1$. By Lemma 1, R' is a constituent path of H'. Let the successor of w_{m-1} on R' be w_m. If $p_m=r$, then R' has edges $[w_0,w_1]$, $[w_1,w_2]$,...,$[w_{m-2},w_{m-1}]$, $[w_{m-1},w_m]$. It follows that $\sigma([p_{m-1},r])=[w_{m-1},w_m]$ and the lemma follows.

Let $p_m\neq r$. Since G is 3-connected there exists an H-jumper $J=J[j_1,j_2]$ with j_1 on $R(p_{m-1},r)$ and j_2 not on $R[p_{m-1},r]$. Such a jumper is called (P,H)-compatible and the length of $R[p_m,j_1]$ is denoted by $h(P,H,J)$. We may assume that the lemma holds for all $H^*\in\mathscr{H}$ having a (P,H^*)-compatible jumper J^* with $h(P,H^*,J^*)<h(P,H,J)$. We consider two cases depending on the value of $h(P,H,J)$.

Case 1. $h(P,H,J)=0$, i.e., $j_1=p_m$. If j_2 is not on R, Lemma 2 yields that the image of $R[p_0,j_1]$ is a constituent path of $H'\cup J'$ and hence a special path of H'. If $m>1$, that path will be $[w_0,w_1,...,w_m]$ with $[w_0,w_1]$ as a special edge of H' and hence we have $\sigma([p_{m-1},p_m])=[w_{m-1},w_m]$. Note that $\sigma([p_0,p_1])$ need not be $[w_0,w_1]$ in the case $m=1$, but in either case P' is a special path of H' and, on P, σ is induced by an isomorphism.

If j_2 is on R, then, by construction, j_2 is p_k for some k, $0\leqslant k<m-1$. But, by Lemma 2, the image of $R[p_k,p_m]$ is a subpath of R' and so has vertices w_k, w_{k+1},..., w_{m-1}, j where j is also on J'. Since w_k is adjacent with at most w_{k-1} and w_{k+1} in R' it follows that $j=w_m$. Hence P' is a special path of H' and, on P, σ is induced by an isomorphism.

Case 2. $h(P,H,J)\geqslant 1$. Since G is 3-connected there exists an H-jumper $J_1=J_1[x,y]$ such that x is on $R(p_{m-1},j_1)$ while y is not on $R[p_{m-1},j_1]$. If y is not on $R[j_1,r]$, the path J_1 is (P,H)-compatible and $h(P,H,J_1)<h(P,H,J)$. Thus the claim follows in this case. So let y be on

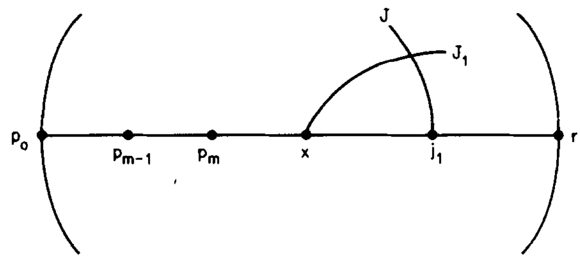

FIGURE 1

$R[j_1, r]$. If J_1 and J have a vertex in common we construct an H-jumper J_2 from $J_1 \cup J$ (see Fig. 1) such that J_2 is (P, H)-compatible and $h(P, H, J_2) < h(P, H, J)$. If J and J_1 are disjoint we construct H^* from H by replacing $R[x, y]$ with J_1. Then $J^* = R[x, j_1] \cup J$ is an H^*-jumper and is (P, H^*)-compatible; moreover, we have $h(P, H^*, J^*) < h(P, H, J)$. In either subcase the claim follows from our assumption.

LEMMA 4. *Let G be 3-connected and let $H = S \cup R$, where R is an S-jumper. Then there exists an isomorphism from H onto H' that induces σ on H.*

Proof. We first prove that, on S, σ is induced by an isomorphism. According to Lemma 3, there exist isomorphisms τ_i from P_i onto P_i' which induce σ on P_i $(1 \leqslant i \leqslant n+1)$. For a crossover S-jumper $Q = Q[q_i, q_j]$ between P_i and P_j we consider the n-skein T in $S \cup Q$ that contains Q, $P_i[a, q_i]$ and $P_j[q_2, b]$. If $\tau_i(a) \neq \tau_j(a)$, then $\tau_i(a) = \tau_j(b)$ and σ maps $P_i[a, q_i]$ and $P_j[q_j, b]$ into incident paths. On the other hand, the image of $Q \cup P_i[a, q_i] \cup P_j[q_j, b]$ is a path of T' with Q' in the middle—a contradiction. Hence $\tau_i(a) = \tau_j(a)$.

We call P_i and P_j related if there is a crossover S-jumper between P_i and P_j. Since $\{a, b\}$ is not a cutset of G, this relation on the set of all P_i with $|E(P_i)| \geqslant 2$ is connected. Therefore the union τ of all τ_i $(|E(P_i)| \geqslant 2)$ is an isomorphism of S onto S' that induces σ on S.

Let $R = R[r_1, r_2]$. If r_1 and r_2 are both on the same P_i, then application of the above to the two $(n+1)$-skeins in $S \cup R$ yields the claim. In the remaining cases R is a crossover S-jumper, say with $r_i \in P_i$ $(i = 1, 2)$. By Lemma 3, there exists an isomorphism τ^* that induces σ on R. Thus $\sigma(e)$ is incident with either P_1' or P_2' if e is the edge of R that is incident with P_2. The proof will be completed by showing that $\sigma(e)$ is incident with P_2'. Obviously we can assume $|E(R)| \geqslant 2$ and hence we can find an H-jumper $J = J[j_1, j_2]$ with j_1 on $R(r_1, r_2)$ and with j_2 not on R. Without loss of generality we may assume that j_2 is not an inner vertex of P_1. Thus, the path $J_1 = J \cup R[j_1, r_2]$ is an S-jumper with no endvertex on $P_1(a, b)$. Hence by

Lemmas 1 and 2, J_1' has no endvertex on $P_1'(\tau(a), \tau(b))$. But $\sigma(e)$ is in J_1' and is incident with an inner vertex of either P_1' or P_2', so it must be P_2'.

THEOREM 2. *Let σ be an n-skein isomorphism between G and G^* where $n \geqslant 2$, where G is 3-connected and contains at least one $(n+1)$-skein, and where G^* has no isolated vertices. Then σ is induced by an isomorphism of G onto G^*.*

Proof. We will first prove that σ is an edge isomorphism. So let $e_1, e_2 \in E(G)$.

Case 1. There exists some $(n+1)$-skein S containing e_1. Then $e_2 \in E(S)$ or, by 3-connectedness, we can find an S-jumper containing e_2. By virtue of Lemma 4, edges e_1 and e_2 are adjacent in G if and only if $\sigma(e_1)$ and $\sigma(e_2)$ are adjacent in G^*.

Case 2. Neither e_1 nor e_2 is contained in any $(n+1)$-skein. Let S be an $(n+1)$-skein in G. Then there are S-jumpers R_i containing e_i $(i = 1, 2)$. By our assumption in this case, R_1 and R_2 are both crossover S-jumpers. Thus R_2 contains an $(S \cup R_1)$-jumper J that contains e_2 and so there is an S-jumper $J_2 \subset R_1 \cup J$ that contains e_2. By Lemma 4, we now have isomorphisms τ_1 and τ_2 from $S \cup R_1$ onto $S' \cup R_1'$ and from $S \cup J_2$ onto $S' \cup J_2'$ that induce σ on $S \cup R_1$ and $S \cup J_2$, respectively. Clearly, the union of τ_1 and τ_2 is an isomorphism between $S \cup R_1 \cup J_2$ and $S' \cup R_1' \cup J_2'$.

We have shown that σ is an edge isomorphism of G onto G^*. So G^* must be connected. But, by Whitney's theorem (including Jung's [3] extension to the infinite case) edge isomorphisms between connected graphs without multiple edges are induced by isomorphisms with exactly four small exceptions. The exceptions are not pertinent here since none of those edge isomorphisms are n-skein isomorphisms for $n \geqslant 2$. Observe that the preservation of multiple edges by σ is a consequence of Lemma 2. This completes the proof of the theorem.

In [2], Hemminger studied edge isomorphisms on graphs having multiple edges. While much could be said, they are far from being induced by isomorphisms in general. However, by an argument like that just used we have the following:

Let τ be an edge isomorphism from G onto G^* where G and G^* are graphs without isolated vertices. If each edge of G is contained in an n-skein of G and if τ is an n-skein isomorphism for some $n \geqslant 2$, then τ is induced by an isomorphism of G onto G^*.

In [4], Mader proved that a finite graph without multiple edges contains an m-skein with special vertices a and b where m is the minimum of the degrees of a and b.

Combining this with our result we have:

COROLLARY. *Let G and G^* be 3-connected graphs without multiple edges and let σ be an n-skein isomorphism between them, $n \geqslant 2$. If the minimum degree of G and G^* is at least $n+1$, then σ is induced by an isomorphism.*

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