Asymptotic property of linear Volterra difference systems

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Abstract

By using the resolvent matrix and the comparison principle, we investigate the asymptotic behavior of linear Volterra difference systems.

Keywords: Linear Volterra difference system; Resolvent matrix; Asymptotic equilibrium; Asymptotic equivalence

1. Introduction and definitions

This paper deals with the asymptotic behavior of linear Volterra difference system

\[ x(n+1) = A(n)x(n) + \sum_{s=n_0}^{n} B(n,s)x(s), \quad x(n_0) = x_0, \] (1.1)

and its perturbation

\[ y(n+1) = A(n)y(n) + \sum_{s=n_0}^{n} B(n,s)y(s) + g(n), \quad y(n_0) = x_0, \] (1.2)

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where \( A(n) \) and \( B(n, s) \) are \( m \times m \) matrix functions on \( \mathbb{N}(n_0) \) and \( \mathbb{N}(n_0) \times \mathbb{N}(n_0) \), respectively, 
\( \mathbb{N}(n_0) = \{ n_0, n_0 + 1, \ldots, n_0 + k, \ldots \} \), \( n_0 \) is a nonnegative integer, and \( g(n) \) is a vector function on \( \mathbb{N}(n_0) \).

Two systems of differential or difference equations are said to be asymptotically equivalent if, corresponding to each solution of one system, there exists a solution of the other system such that the difference between these two solutions tends to zero. If we know that two systems are asymptotically equivalent, and if we also know the asymptotic behavior of the solutions of one of the systems, then it is clear that we can obtain information about the asymptotic behavior of the solutions of the other systems [1,3,12].

Many authors studied the asymptotic behavior of difference systems [7,12,13,15,16]. Pinto [15] investigated the asymptotic equivalence between two difference systems by means of the concept of dichotomy. Also, Medina and Pinto [13] studied this problem via a global domination of the fundamental matrix of the linear system instead of the dichotomy condition. Moreover, Medina [12] established asymptotic equivalence by using the general discrete inequality combined with the Schauder’s fixed point theorem.

For the asymptotic behavior and the stability of Volterra difference systems, see [4–9,17]. Zouyousefain and Leela in [17] developed a useful method of finding a linear difference system to a given linear Volterra system:

\[
\Delta y(n) = y(n + 1) - y(n) = A(n)y(n) + \sum_{s=n_0}^{n-1} B(n, s)y(s) + g(n).
\]

(1.3)

As a modification of Theorem 2.1 in [17] Choi et al. [6] obtained one asymptotic equivalence property between two systems (1.1) and (1.2). Also, they proved that (1.1) and (1.2) are asymptotically equivalent with respect to a nonnegative function \( \mu : \mathbb{N}(n_0) \to \mathbb{R} \) in [5].

In this paper we characterize asymptotic equilibrium of linear Volterra system (1.1) by means of the resolvent matrix solution which is a modification of Proposition 1 in [10]. Also, we extend this result to the perturbation (1.2) of (1.1). With the results about asymptotic equilibrium we investigate asymptotic equivalence between (1.1) and (1.2), and between two linear Volterra systems, respectively.

Consider the nonlinear difference system

\[
x(n + 1) = f(n, x(n)),
\]

(1.4)

where \( f : \mathbb{N}(n_0) \times \mathbb{R}^m \to \mathbb{R}^m \) and \( \mathbb{R}^m \) is the \( m \)-dimensional Euclidean space with any convenient norm \( | \cdot | \). Let \( f_x = \frac{\partial f}{\partial x} \) exists and be continuous and invertible on \( \mathbb{N}(n_0) \times \mathbb{R}^m \), and \( f(n, 0) = 0 \). Denote \( x(n) = x(n, n_0, x_0) \) by the solution of system (1.4) with \( x(n_0, n_0, x_0) = x_0 \). System (1.4) is said to have asymptotic equilibrium if there exist a single \( \xi \in \mathbb{R}^m \) and \( r > 0 \) such that any solution \( x(n) \) of (1.4) with \( |x_0| < r \) satisfies

\[
x(n) = \xi + o(1) \quad \text{as } n \to \infty
\]

(1.5)

and for every \( \xi \in \mathbb{R}^m \), there exists a solution of (1.4) such that it satisfies (1.5). Two difference systems (1.4) and

\[
y(n + 1) = g(n, y(n))
\]

(1.6)

are said to be asymptotically equivalent if, for every solution \( x(n) \) of (1.4), there exists a solution \( y(n) \) of (1.6) such that

\[
x(n) = y(n) + o(1) \quad \text{as } n \to \infty
\]

(1.7)
and conversely, for every solution $y(n)$ of (1.6), there exists a solution $x(n)$ of (1.4) such that the asymptotic relationship (1.7) holds.

### 2. Asymptotic property of linear Volterra difference systems

Consider the linear Volterra difference system

$$x(n + 1) = A(n)x(n) + \sum_{s=n_0}^{n} B(n, s)x(s), \quad x(n_0) = x_0, \quad (2.1)$$

and its perturbation

$$y(n + 1) = A(n)y(n) + \sum_{s=n_0}^{n} B(n, s)y(s) + g(n), \quad y(n_0) = x_0, \quad (2.2)$$

where $A(n)$ and $B(n, s)$ are $m \times m$ matrix functions on $\mathbb{N}(n_0)$ and $\mathbb{N}(n_0) \times \mathbb{N}(n_0)$, respectively and $g(n)$ is a vector function on $\mathbb{N}(n_0)$.

We need the following well-known discrete inequality which is the discrete analogue of [14, Theorem 2.1].

**Lemma 2.1.** [2, Corollary 15.10] Let $u(n)$ and $a(n)$ be nonnegative functions defined on $\mathbb{N}(n_0)$ such that

$$u(n) \leq k + \sum_{s=n_0}^{n-1} \left[ a(s)u(s) + \sum_{\sigma=n_0}^{s-1} b(s, \sigma)u(\sigma) \right], \quad n \in \mathbb{N}(n_0), \quad (2.3)$$

where $k$ is a nonnegative constant and $b(s, \sigma) \geq 0, s \geq \sigma \geq n_0$. Then

$$u(n) \leq k \prod_{s=n_0}^{n-1} \left( 1 + a(s) + \sum_{\sigma=n_0}^{s-1} b(s, \sigma) \right), \quad n \geq n_0. \quad (2.4)$$

The following lemma states a result corresponding to Fubini’s theorem, which can be proved by the induction.

**Lemma 2.2.** [17] Let $L(n, s)$ and $K(n, s)$ be $m \times m$ matrices defined for $s, n \geq n_0$ such that $L$ and $K$ are zero matrices for $s, n \leq n_0$. Then the relation

$$\sum_{s=n_0}^{n-1} L(n, s + 1) \sum_{\sigma=n_0}^{s-1} K(s, \sigma)x(\sigma) = \sum_{s=n_0}^{n-1} \sum_{\sigma=s+1}^{n-1} L(n, \sigma + 1)K(\sigma, s)x(s)$$

holds, where $x : \mathbb{N}(n_0) \to \mathbb{R}^m$ is a vector function.

We define the resolvent matrix $R(n, m)$ of (2.1) as the unique solution of the matrix difference equation

$$R(n, m) = R(n, m + 1)A(m) + \sum_{r=m}^{n-1} R(n, r + 1)B(r, m), \quad n - 1 \geq m \geq n_0, \quad (2.5)$$

with $R(m, m) = I$ (see [10]). It is straightforward to show the existence and uniqueness of $R(n, m)$. Using the resolvent matrix $R(n, m)$, we can establish the following variation of constants formula.
Lemma 2.3. The unique solution \( y(n, n_0, y_0) \) of (2.2) satisfying \( y(n_0) = y_0 \) is given by
\[
y(n, n_0, y_0) = R(n, n_0)y_0 + \sum_{s=n_0}^{n-1} R(n, s + 1)g(s),
\]
where \( R(n, m) \) is the unique solution of the matrix difference equation (2.5).

Proof. Letting \( p(s) = R(n, s)y(s) \), \( n - 1 \geq s \geq n_0 \), we have
\[
\Delta p(s) = R(n, s + 1)y(s + 1) - R(n, s)y(s) \\
= \left[ R(n, s + 1)A(s) - R(n, s) \right]y(s) + R(n, s + 1) \sum_{r=n_0}^{s} B(s, r)y(r) \\
+ R(n, s + 1)g(s).
\]
(2.7)
Summing both sides of (2.7) from \( n - 1 \) to \( n_0 \), we obtain
\[
p(n) = y(n) = R(n, n_0)y_0 + \sum_{s=n_0}^{n-1} R(n, s + 1)g(s) \\
+ \sum_{s=n_0}^{n-1} \left[ R(n, s + 1)A(s) - R(n, s) \right]y(s) \\
+ \sum_{s=n_0}^{n-1} \sum_{r=n_0}^{s-1} B(s, r)y(r) + \sum_{s=n_0}^{n-1} R(n, s + 1)B(s, s)y(s) \\
= R(n, n_0)y_0 + \sum_{s=n_0}^{n-1} R(n, s + 1)g(s) \\
+ \sum_{s=n_0}^{n-1} \left[ R(n, s + 1)A(s) - R(n, s) + \sum_{r=s}^{n-1} R(n, r + 1)B(r, s) \right]y(s)
\]
by Lemma 2.2. From (2.5) we have
\[
y(n) = R(n, n_0)y_0 + \sum_{s=n_0}^{n-1} R(n, s + 1)g(s).
\]
This completes the proof. \( \Box \)

Example 2.4. [6] We consider the linear Volterra difference equation
\[
x(n + 1) = A(n)x(n) + \sum_{s=n_0}^{n} B(n, s)x(s) = 2x(n) + \sum_{s=n_0}^{n} 2^{n-s}x(s),
\]
(2.8)
where \( A(n) = 2 \) and \( B(n, s) = 2^{n-s} \). Then any solution \( x(n, n_0, x_0) \) of (2.8) through the initial point \( x(n_0, n_0, x_0) = x_0 \) is given by
\[
x(n, n_0, x_0) = \frac{x_0}{3} \left[ 1 + 2 \cdot 4^{n-n_0} \right], \quad n \in \mathbb{N}(n_0).
\]
Thus the resolvent matrix solution $R(n, m)$ with $R(m, m) = 1$ of the difference equation

$$R(n, m) = R(n, m + 1)A(n) + \sum_{r=m}^{n-1} R(n, r + 1)B(r, m)$$

$$= R(n, m + 1)2 + \sum_{r=m}^{n-1} R(n, r + 1)2^{r-m}, \quad n - 1 \geq m \geq n_0,$$

given by

$$R(n, m) = \frac{1}{3}[1 + 2 \cdot 4^{n-m}].$$

It follows from the simple calculation that

$$R(n, m + 1)A(n) + \sum_{r=m}^{n-1} R(n, r + 1)B(r, m)$$

$$= \frac{2}{3}[1 + 2 \cdot 4^{n-m-1}] + \sum_{r=m}^{n-1} \frac{1}{3}[1 + 2 \cdot 4^{n-r-1}]2^{r-m}$$

$$= \frac{1}{3}[2 + 4^{n-m}] + \frac{1}{3}[2^{n-m} - 1] + \frac{2^{2n-m-1}}{3}[2^{1-m} - 2^{1-n}]$$

$$= \frac{1}{3}[1 + 2 \cdot 4^{n-m}]$$

$$= R(n, m).$$

In fact, we note that the fundamental matrix of (2.8) is given by

$$\frac{\partial x(n, n_0, x_0)}{\partial x_0} = \Phi(n, n_0) = \frac{1}{3}[1 + 2 \cdot 4^{n-n_0}]$$

and

$$\Phi(n, n_0) = R(n, n_0).$$

**Remark 2.5.** In the special case when $f(n, x) = A(n)x(n)$ and $g(n, s, x) = B(n, s)x(s)$ in the nonlinear Volterra difference system

$$x(n + 1) = f(n, x(n)) + \sum_{s=n_0}^{n} g(n, s, x(s)),$$

we note that the resolvent matrix $R(n, m)$ for Eq. (2.5) is closely related to the fundamental matrix $\Phi(n, n_0)$. By the uniqueness of solution, it is easy to see that $R(n, n_0) = \Phi(n, n_0)$, and $R(n, m) = \Phi(n - m)$ for the equation of convolution type such as $B(n, s) = B(n - s)$.

**Lemma 2.6.** (2.1) has asymptotic equilibrium if and only if $\lim_{n \to \infty} R(n, n_0)$ exists and is invertible, where $R(n, n_0)$ is the resolvent matrix solution of (2.5) for $n \geq n_0$.

**Proof.** It suffices to show that the necessary condition holds. Suppose that (2.1) has asymptotic equilibrium. Then $\lim_{n \to \infty} R(n, n_0) = R_{\infty}(n_0)$ exists for each $n_0 \geq 0$. 


Let \( E_i = (0, \ldots, 1, \ldots, 0)^T \) be the \( i \)th unit vector in \( \mathbb{R}^m \) for each \( i = 1, 2, \ldots, m \). Then there exist the solutions \( x(n, n_0, x_{0i}) \) of (2.1) such that
\[
\lim_{n \to \infty} x(n, n_0, x_{0i}) = \lim_{n \to \infty} R(n, n_0)x_{0i} = R_\infty(n_0)x_{0i} = E_i, \quad i = 1, 2, \ldots, m.
\]

It follows that
\[
R_\infty(n_0)[x_{01} \cdots x_{0m}] = I,
\]
where \( [x_{01} \cdots x_{0m}] \) is the inverse matrix of \( R_\infty(n_0) \) and \( I \) is the identity matrix. Thus \( R_\infty(n_0) \) is invertible.

**Corollary 2.7.** If (2.1) has asymptotic equilibrium, then there exists a positive constant \( M > 0 \) such that
\[
|R(n, m)| \leq M, \quad n \geq m \geq n_0 \geq 0.
\]

Now, we characterize asymptotic equilibrium by means of the resolvent matrix solution.

**Theorem 2.8.** Assume that \((|I + A(n)| + \sum_{s=n_0}^{n} |B(n, s)|) \in \ell_1(\mathbb{N}(n_0))\). Then (2.1) has asymptotic equilibrium.

**Proof.** Since the solution of (2.1) is given by \( x(n, n_0, x_0) = R(n, n_0)x_0 \) for each \( x_0 \in \mathbb{R}^m \), \( R(n, n_0) \) satisfies the matrix difference equation
\[
R(n+1, n_0) = A(n)R(n, n_0) + \sum_{s=n_0}^{n} B(n, s)R(s, n_0)
\]
(2.9)

and is given by
\[
R(n, n_0) = I + \sum_{s=n_0}^{n-1} \left[ (I + A(s))R(s, n_0) + \sum_{\sigma=n_0}^{s} B(s, \sigma)R(\sigma, n_0) \right]. \quad (2.10)
\]

Letting \( |R(n, n_0)| = u(n) \) and
\[
v(n) = 1 + \sum_{s=n_0}^{n-1} \left[ |I + A(s)||R(s, n_0)| + \sum_{\sigma=n_0}^{s} |B(s, \sigma)||R(\sigma, n_0)| \right],
\]
we have
\[
v(n) = 1 + \sum_{s=n_0}^{n-1} \left[ |I + A(s)||u(s)| + \sum_{\sigma=n_0}^{s} |B(s, \sigma)||u(\sigma)| \right]
\leq 1 + \sum_{s=n_0}^{n-1} \left[ |I + A(s)||v(s)| + \sum_{\sigma=n_0}^{s} |B(s, \sigma)||v(\sigma)| \right].
\]

From Lemma 2.1 and the assumption we obtain
\begin{align*}
v(n) & \leq \prod_{s=n_0}^{n-1} \left[ 1 + |I + A(s)| + \sum_{\sigma=n_0}^{s} |B(s, \sigma)| \right] \\
& \leq \exp \sum_{s=n_0}^{n-1} \left[ |I + A(s)| + \sum_{\sigma=n_0}^{s} |B(s, \sigma)| \right] \\
& \leq \exp \sum_{s=n_0}^{\infty} \left[ |I + A(s)| + \sum_{s=n_0}^{\infty} |B(s, \sigma)| \right] < \infty.
\end{align*}

Then it is easy to see that \( u(n) \leq v(n) \) for each \( n \geq n_0 \) and \( v(n) \) is increasing and bounded. Furthermore, for any \( n > n_1 > n_0 \), we have

\[
|R(n, n_0) - R(n_1, n_0)| \leq \sum_{s=n_1}^{n-1} \left[ |I + A(s)||R(s, n_0)| + \sum_{\sigma=n_0}^{s} |B(s, \sigma)||R(\sigma, n_0)| \right] \\
= \sum_{s=n_1}^{n-1} \left[ |I + A(s)|u(s) + \sum_{\sigma=n_0}^{s} |B(s, \sigma)|u(\sigma) \right] \\
= v(n) - v(n_1).
\]

This implies that, given an \( \epsilon > 0 \), we can choose an \( n_1 > 0 \) sufficiently large so that

\[
|R(n, n_0) - R(n_1, n_0)| < \epsilon \quad \text{for all } n > n_1,
\]

since \( v(n) \) has the Cauchy property. Hence \( R(n, n_0) \) converges to a constant \( m \times m \) matrix \( R_{\infty}(n_0) \) as \( n \to \infty \).

Since \( \sum_{s=0}^{\infty} [|I + A(s)| + \sum_{\sigma=n_0}^{s} |(B(s, \sigma))|] \) is finite, we can choose \( n_0 > 0 \) sufficiently large so that

\[
\sum_{s=n_0}^{\infty} |I + A(s)| + \sum_{\sigma=n_0}^{s} |B(s, \sigma)| < \frac{1}{M},
\]

where \( M \) is the bounded number for \( R(n, n_0) \). Letting

\[
P(n, n_0) = \sum_{s=n_0}^{n-1} \left( I + A(s) \right) R(s, n_0) + \sum_{\sigma=n_0}^{s} B(s, \sigma)R(\sigma, n_0)
\]

we have

\[
\lim_{n \to \infty} |P(n, n_0)| \leq \lim_{n \to \infty} \left[ \sum_{s=n_0}^{n-1} \left( |I + A(s)||R(s, n_0)| + \sum_{\sigma=n_0}^{s} |B(s, \sigma)||R(\sigma, n_0)| \right) \right] \\
\leq M \lim_{n \to \infty} \left[ \sum_{s=n_0}^{n-1} \left( |I + A(s)| + \sum_{\sigma=n_0}^{s} |B(s, \sigma)| \right) \right] \\
= M \sum_{s=n_0}^{\infty} \left[ |I + A(s)| + \sum_{\sigma=n_0}^{s} |B(s, \sigma)| \right] < 1.
\]

We see that \( \lim_{n \to \infty} R(n, n_0) = R_{\infty} \) is invertible (see [11]). Then (2.1) has asymptotic equilibrium by Lemma 2.6. \( \square \)
Example 2.9. To illustrate Theorem 2.8, we consider the linear Volterra difference equation

\[ x(n + 1) = a(n)x(n) + \sum_{s=n_0}^{n} b(n, s)x(s) \]

\[ = (2^{-n} - 1)x(n) + \sum_{s=n_0}^{n} 2^{-(n+s)}x(s), \quad x(n_0) = x_0, \quad n_0 \geq 0, \]  \hspace{1cm} (2.11)

where \( a(n) = 2^{-n} - 1 \) and \( b(n, s) = 2^{-(n+s)} \) satisfy the assumption of Theorem 2.8. Then (2.11) has asymptotic equilibrium.

Theorem 2.10. Suppose that there exists a positive constant \( \alpha \) with \( |\det R(n, n_0)| \geq \alpha > 0 \) for each \( n \geq n_0 \) and \( \lim_{n \to \infty} R(n, n_0) = R_\infty \) exists. Then (2.1) has asymptotic equilibrium.

Proof. We easily have

\[ \lim_{n \to \infty} |\det R(n, n_0)| = |\det \lim_{n \to \infty} R(n, n_0)| = |\det R_\infty| \geq \alpha > 0. \]

From the invertibility of \( R_\infty \) and Lemma 2.6, (2.1) has asymptotic equilibrium. \( \square \)

Example 2.11. We consider the linear Volterra difference equation

\[ x(n + 1) = A(n)x(n) + \sum_{s=n_0}^{n} B(n, s)x(s) \]

\[ = -x(n) + \sum_{s=n_0}^{n} \left( \frac{1}{2} \right)^{n-s} x(s), \quad x(n_0) = x_0, \quad n_0 \geq 0, \]  \hspace{1cm} (2.12)

whose solution is given by \( x(n, n_0, x_0) = \frac{x_0}{3}[1 + 2(-1/2)^{(n-n_0)}] \). This solution can be obtained by using the properties of the Z-transformation of Volterra difference equation of convolution type [10]. Thus the resolvent solution \( R(n, n_0) \) of (2.5) is given by \( R(n, n_0) = (1/3)[1 + 2(-1/2)^{(n-n_0)}] \) and \( \lim_{n \to \infty} R(n, n_0) = 1/3 \neq 0 \). It follows from Lemma 2.6 that (2.12) has asymptotic equilibrium.

Theorem 2.12. Assume that (2.1) has asymptotic equilibrium and \( |g(n)| \in \ell_1(\mathbb{N}(n_0)) \). Then (2.2) also has asymptotic equilibrium.

Proof. Let \( y(n) \) be the solution of (2.2). Then the solution \( y(n) \) of (2.2) is given by

\[ y(n) = R(n, n_0)x_0 + \sum_{s=n_0}^{n-1} R(n, s + 1)g(s), \]

by (2.6) in Lemma 2.3. Note that \( r(n) = \sum_{s=n_0}^{n-1} R(n, s + 1)g(s) \) has the Cauchy property from the boundedness of \( R(n, s + 1) \) and \( \sum_{s=n_0}^{\infty} g(n) < \infty \). It follows that \( y(n) \) converges to a vector in \( \mathbb{R}^m \).

Conversely, let \( \xi \) be any vector in \( \mathbb{R}^m \). Since \( r(n) = \sum_{s=n_0}^{n-1} R(n, s + 1)g(n) \) has the Cauchy property, \( \lim_{n \to \infty} r(n) = r_\infty \) exists. Thus there exists a solution \( y(n) \) of (2.2) with the initial value \( y_0 = R_\infty^{-1}(\xi - r_\infty) \) such that the asymptotic relationship holds:
y(n) = R(n, n_0)y_0 + \sum_{s=n_0}^{n-1} R(n, s + 1)g(s)

= R(n, n_0)R_\infty^{-1}(\xi - r_\infty) + r_\infty - \sum_{s=n}^{\infty} R(n, s + 1)g(s)

= \xi + o(1) \quad \text{as } n \to \infty,

since \int_n^{\infty} R(n, s + 1)g(s) \to 0 \text{ as } n \to \infty. \text{ This completes the proof.} \quad \square

**Theorem 2.13.** Assume that (2.1) has asymptotic equilibrium and \(|g(n)| \in \ell_1(\mathbb{N}(n_0))\). Then (2.1) and (2.2) are asymptotically equivalent.

**Proof.** Let \(x(n, n_0, x_0)\) be any solution of (2.1). Then there exists a solution \(y(n)\) of (2.2) with the initial value \(y_0 = x_0 - R_\infty^{-1}r_\infty\) such that the asymptotic relationship

\[
x(n, n_0, x_0) = R(n, n_0)x_0
= y(n, n_0, y_0) - \sum_{s=n_0}^{n-1} R(n, s + 1)g(s) + R(n, n_0)R_\infty^{-1}r_\infty
= y(n, n_0, y_0) + o(1) \quad \text{as } n \to \infty,
\]

where \(r_\infty = \sum_{s=n_0}^{\infty} R(\infty, s + 1)g(s)\), holds.

Conversely, let \(y(n)\) be any solution of (2.2). Then there exists a solution \(x(n, n_0, x_0)\) of (2.1) with the initial \(x_0 = y_0 + R_\infty^{-1}r_\infty\) satisfying

\[
y(n, n_0, y_0) = R(n, n_0)y_0 + \sum_{s=n_0}^{n-1} R(n, s + 1)g(s)
= x(n, n_0, x_0) + \sum_{s=n_0}^{n-1} R(n, s + 1)g(s) - R(n, n_0)R_\infty^{-1}r_\infty
= x(n) + o(1) \quad \text{as } n \to \infty.
\]

This completes the proof. \quad \square

**Example 2.14.** To illustrate Theorem 2.13, we consider two linear Volterra difference equations

\[
x(n + 1) = -x(n) + \sum_{s=n_0}^{n} \left(\frac{1}{2}\right)^{n-s} x(s), \quad x(n_0) = x_0 \quad (2.12)
\]

and

\[
y(n + 1) = -y(n) + \sum_{s=n_0}^{n} \left(\frac{1}{2}\right)^{n-s} y(s) + \alpha^n, \quad y(n_0) = y_0, \quad (2.13)
\]

where \(A(n) = -1, B(n, s) = (1/2)^{n-s}\) and \(g(n) = \alpha^n\) with the constant \(|\alpha| < 1\) and \(\alpha \neq -1/2\).

Then (2.12) and (2.13) are asymptotically equivalent.
**Proof.** Note that the solutions \( x(n) \) and \( y(n) \) of (2.12) and (2.13) are given by

\[
x(n, n_0, x_0) = \frac{1}{3} \left[ 1 + 2 \left( -\frac{1}{2} \right)^{(n-n_0)} \right] x_0
\]

and

\[
y(n, n_0, y_0) = \frac{1}{3} \left[ 1 + 2 \left( -\frac{1}{2} \right)^{(n-n_0)} \right] y_0 + \frac{1}{3} \left[ \frac{\alpha^n - \alpha^{n_0}}{(\alpha - 1)} + \frac{4\alpha^n + 2(-1/2)^{n-n_0-1}}{(2\alpha + 1)} \right],
\]

respectively. Note that (2.13) has asymptotic equilibrium. Furthermore, putting \( y_0 = x_0 + \frac{\alpha^{n_0}}{\alpha - 1} \), we see that (2.12) and (2.13) are asymptotically equivalent. \( \square \)

### 3. Asymptotic equivalence between two linear Volterra difference systems

We consider two linear Volterra difference systems

\[
x(n + 1) = A(n)x(n) + \sum_{s=n_0}^{n} B(n, s)x(s), \quad x(n_0) = x_0,
\]

(3.1)

and

\[
y(n + 1) = C(n)y(n) + \sum_{s=n_0}^{n} D(n, s)y(s), \quad y(n_0) = y_0,
\]

(3.2)

where \( A(n) \) and \( C(n) \) are \( m \times m \) matrix functions on \( \mathbb{N}(n_0) \), and \( B(n, s) \) and \( D(n, s) \) are \( m \times m \) matrix functions on \( \mathbb{N}(n_0) \times \mathbb{N}(n_0) \) for \( n \geq s \geq n_0 \geq 0 \).

**Lemma 3.1.** [2, Theorem 15.20] Let \( u(n) \) and \( a(n) \) be functions defined on \( \mathbb{N}(n_0) \) such that

\[
u(n) \leq a(n) + \sum_{s=n_0}^{n-1} Q(n, s, u(s)), \quad n \in \mathbb{N}(n_0),
\]

where \( Q(n, s, u) \) is a function defined on \( \mathbb{N}(n_0) \times \mathbb{N}(n_0) \times \mathbb{R} \) which is nondecreasing with respect to each \( u \in \mathbb{R} \). Then

\[
u(n) \leq w(n), \quad n \in \mathbb{N}(n_0),
\]

where \( w(n) \) is the solution of the equation

\[
w(n) = a(n) + \sum_{s=n_0}^{n-1} Q(n, s, w(s)), \quad n \in \mathbb{N}(n_0).
\]

**Theorem 3.2.** Assume that \( |A(n) - C(n)| \in \ell_1(\mathbb{N}(n_0)) \) and \( |B(n, s) - D(n, s)| \in \ell_1(\mathbb{N}(n_0)) \). (3.1) has asymptotic equilibrium if and only if (3.2) also has asymptotic equilibrium.

**Proof.** Suppose that (3.1) has asymptotic equilibrium. Then (3.2) can be written as

\[
y(n + 1) = A(n)y(n) + \sum_{s=n_0}^{n} B(n, s)y(s) + g(n, y(n)), \quad y(n_0) = y_0,
\]
where
\[ g(n, y(n)) = [C(n) - A(n)]y(n) + \sum_{s=n_0}^{n} [B(n, s) - D(n, s)]y(s). \]

Let \( y(n) \) be any solution of (3.2) with the initial value \( y(n_0) = y_0 \). Then by the variation of constants formula in Lemma 2.3, we obtain
\[ y(n) = R(n, n_0)y_0 + \sum_{s=n_0}^{n-1} R(n, s+1)g(s, y(s)), \tag{3.3} \]
where \( R(n, s) \) is the resolvent solution of matrix difference equation (2.5).

It follows from (3.3) that
\[ S(n, n_0)y_0 = R(n, n_0)y_0 + \sum_{s=n_0}^{n-1} R(n, s+1)(C(s) - A(s))S(s, n_0)y_0 \]
\[ + \sum_{s=n_0}^{n-1} R(n, s+1) \left[ \sum_{\sigma=n_0}^{s} [B(s, \sigma) - D(s, \sigma)]S(\sigma, n_0)y_0 \right], \]
where \( S(n, m) \) is the unique solution of the matrix difference equation
\[ S(n, m) = S(n, m+1)C(m) + \sum_{r=m}^{n-1} S(n, r+1)D(r, m), \quad n-1 \geq m \geq n_0, \tag{3.4} \]
with \( S(m, m) = I \), which is the resolvent matrix of (3.2).

Also, it follows from the boundedness of \( R(n, m) \) that
\[ \left| S(n, n_0) \right| \leq \left| R(n, n_0) \right| + \sum_{s=n_0}^{n-1} \left| R(n, s+1) \right| \left[ |C(s) - A(s)| |S(s, n_0)| \right] \]
\[ + \sum_{\sigma=n_0}^{s} |D(s, \sigma) - B(s, \sigma)| |S(\sigma, n_0)| \]
\[ \leq M + M \sum_{s=n_0}^{n-1} \left[ |C(s) - A(s)| |S(s, n_0)| \right] \]
\[ + \sum_{\sigma=n_0}^{s} |D(s, \sigma) - B(s, \sigma)| |S(\sigma, n_0)|. \]

Putting \( |S(n, n_0)| = u(n) \), we obtain
\[ u(n) \leq M \exp \sum_{s=n_0}^{n-1} M \left[ |C(s) - A(s)| + \sum_{\sigma=n_0}^{s} |D(s, \sigma) - B(s, \sigma)| \right] \]
\[ \leq M \exp \sum_{s=n_0}^{\infty} M \left[ |C(s) - A(s)| + \sum_{\sigma=n_0}^{s} |D(s, \sigma) - B(s, \sigma)| \right] < \infty, \]
by Lemma 2.1 and the assumption. Thus \( \lim_{n \to \infty} S(n, n_0) = S(\infty)(n_0) \) exists for each fixed \( n_0 \in \mathbb{N}(n_0) \).

Also, we obtain the following relationship between \( R(n, n_0) \) and \( S(n, n_0) \):

\[
S(n, n_0) = R(n, n_0) + \sum_{s=n_0}^{n-1} R(n, s+1) \left[ C(s) - A(s) \right] S(s, n_0) + \sum_{s=n_0}^{n-1} \sum_{\sigma=n_0}^{s} \left[ D(s, \sigma) - B(s, \sigma) \right] S(\sigma, n_0)
\]

\[
= R(n, n_0) + R(\infty)P(n, n_0),
\]

(3.5)

where

\[
P(n, n_0) = R(\infty)^{-1} \sum_{s=n_0}^{n-1} R(n, s+1) \left[ C(s) - A(s) \right] S(s, n_0)
\]

\[
+ R(\infty)^{-1} \sum_{s=n_0}^{n-1} \sum_{\sigma=n_0}^{s} \left[ D(s, \sigma) - B(s, \sigma) \right] S(\sigma, n_0).
\]

Since both \( R(n, m) \) and \( S(n, m) \) are bounded, \( |C(n) - A(n)| \in \ell_1(\mathbb{N}(n_0)) \) and \( |D(n, s) - B(n, s)| \in \ell_1(\mathbb{N}(n_0)) \), \( P(n, n_0) \) has the Cauchy property. Thus \( \lim_{n \to \infty} P(n, n_0) = P(\infty)(n_0) \) exists for each \( n_0 \geq 0 \). So we can choose \( n_0 > 0 \) sufficiently large so that \( |P(\infty)(n_0)| < 1 \). Then we obtain from (3.5)

\[
S(\infty) = \lim_{n \to \infty} S(n, n_0) = R(\infty) \left[ I + P(\infty)(n_0) \right].
\]

It follows from \( |P(\infty)(n_0)| < 1 \) that \( I + P(\infty)(n_0) \) is invertible and \( S(\infty) \) is also invertible. Hence (3.2) has asymptotic equilibrium by Lemma 2.6.

By the same manner we can obtain the converse. \( \square \)

Now, we obtain the asymptotic equivalence between two linear Volterra difference systems (3.1) and (3.2).

**Theorem 3.3.** In addition to the assumptions of Theorem 3.2, suppose that (3.1) has asymptotic equilibrium. Then (3.1) and (3.2) are asymptotically equivalent.

**Proof.** We see that (3.2) has asymptotic equilibrium by Theorem 3.2. Let \( x(n, n_0, x_0) \) be any solution of (3.1). Then \( \lim_{n \to \infty} x(n, n_0, x_0) = x(\infty) \) exists. Thus there exists a solution \( y(n, n_0, y_0) \) of (3.2) such that \( \lim_{n \to \infty} y(n, n_0, y_0) = x(\infty) \) and the asymptotic relationship holds

\[
x(n, n_0, x_0) = y(n, n_0, y_0) + o(1) \quad \text{as} \quad n \to \infty.
\]

The converse asymptotic relationship can be obtained similarly. \( \square \)

**Example 3.4.** We consider the linear Volterra difference equations

\[
x(n+1) = A(n)x(n) + \sum_{s=n_0}^{n} B(n, s)x(s)
\]
\[ x(n) = -x(n) + \sum_{s=n_0}^{n-1} \left( \frac{1}{2} \right)^{n-s} x(s), \quad x(n_0) = x_0, \quad (3.6) \]

and

\[ y(n + 1) = C(n)y(n) + \sum_{s=n_0}^{n} D(n, s)y(s) \]

\[ = (-1 + \alpha^n) y(n) + \sum_{s=n_0}^{n} \left[ \left( \frac{1}{2} \right)^{n-s} + \beta^n \alpha^{n-s} \right] y(s), \quad y(n_0) = y_0, \quad (3.7) \]

where \( C(n) = -1 + \alpha^n \) with the constant \(|\alpha| < 1\) and \( D(n, s) = (1/2)^{n-s} + \beta^n \alpha^{n-s} \) with the constant \(|\beta| < 1\). Then (3.6) has asymptotic equilibrium by Example 2.11. Since \( \sum_{n=0}^{\infty} |A(n) - C(n)| = 1/(1 - \alpha) \), we have

\[
\sum_{n=0}^{\infty} \left[ \sum_{s=0}^{n} |B(n, s) - D(n, s)| \right] \leq \sum_{n=0}^{\infty} \beta^n \alpha^n \left[ \frac{\alpha^n - \alpha}{1 - \alpha} \right] \leq \frac{1}{1 - \alpha} \left( \frac{1}{1 - \beta} - \frac{\alpha}{1 - \alpha \beta} \right) < \infty.
\]

Hence (3.6) and (3.7) are asymptotically equivalent by Theorem 3.3.

References