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Connection coefficients on an interval and wavelet solutions of Burgers equation

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Abstract

A definition of connection coefficients is introduced and techniques of computation are presented. We use semi-implicit time difference scheme to solve Burgers equation by applying the evaluations of connection coefficients in calculating the integrals of the variational form. Comparisons of accuracy and robustness of numerical solutions are mentioned in the examples. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The use of multiresolution techniques and wavelets has become increasingly popular in the development of numerical schemes for the solution of partial differential equations [1,8,10,11,13,15,16]. The multilevel and approximation properties of the scaling function of the multiresolution analysis provide computational efficiency and accuracy of numerical solutions of partial differential equations. In this paper, we shall solve Burgers equation to illustrate this claim. Burgers equation is a useful model for many physically interesting problems, particularly those of a fluid-flow nature, in which either shocks or viscous dissipation is significant in part of the region [3]. For many combinations of initial and boundary conditions, an exact solution of Burgers equation is available. Burgers equation is one of the simplest nonlinear partial differential equations for which it is possible to obtain exact solution, it behaves as an elliptic, parabolic or hyperbolic partial differential equation. Therefore, Burgers equation has been used widely as a model equation for testing and comparing computational techniques.

Numerical techniques for the solution of differential equations usually fall into the following classes: finite-difference, finite element and spectral methods. Sometimes the latter two methods are

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considered as subsets of the method of weighted residuals. Galerkin method is one of the method of weighted residual. In this paper we use semi-implicit time differencing scheme which is to combine Galerkin method and finite-difference method in solving PDEs. This was first mentioned in [8,10]. Here we use different techniques to calculate the integrals in the variational form of the given equation. Our method in computing the integrals is based on the evaluation of appropriate connection coefficients. It gives us an efficient way to do the computation. To convert a differential equation into a system of algebraic equations, we use an interpolation formula to approximate the coefficient functions of the given differential equation.

We organize our paper as follows. In Section 2, we review coiflet interpolation formula and its approximation theorem. Section 3 provides definition and theory of connection coefficients which differs from that considered in [9]. As mentioned in [9], the calculation of connection coefficients consists of two parts. First, the connection coefficients are shown to satisfy a set of homogeneous equations that are derived from repeated use of the refinement equation. This procedure gives rise to an algebraic eigenvalue problem. In fact, the connection coefficients are not uniquely determined by this algebraic constraint. We need additional constraints which are inhomogeneous equations. These are derived from moment conditions. Our approach will not only handle the boundary conditions of the PDEs naturally but also provide a robust computational technique. In Section 4, we use semi-implicit time differencing scheme to solve several Burgers equations by using the interpolation formula of Section 2 and connection coefficients introduced in Section 3. It turns out that our methods provide a robust and accurate alternative to conventional methods.

2. Interpolations

We recall a generalized convergence theorem in R^2 which is also true in R^n. A similar interpolation formula using multi-scaling functions has been considered in [12]. In fact, there is a close relationship between accuracy of scaling function and approximation order of scaling function interpolation [12].

Let $\phi, \psi \in C^1$ be, respectively, the scaling and wavelet function of the orthonormal multiresolution analysis with compact support [4,5,14]. Construction of wavelet functions can start from the building of scaling function, $\phi(x)$, and a set of related coefficients, $\{a_k\}_{k\in Z}$, which satisfy the *two-scale relation* or *refinement equation*,

$$\phi(x) = \sum_k a_k \phi(2x-k).$$

The wavelet function is

$$\psi(x) = \sum_k b_k \phi(2x-k),$$

where $b_k = (-1)^k a_{-k+1}$. Let c and $\{M_\ell\}$ denote the moments of ϕ as follows:

$$c := \int x\phi(x)\,\mathrm{d}x = \frac{1}{2}\sum_{k=N_1}^{N_2} k a_k, \tag{2.1}$$

the first moment of the scaling function $\phi(x)$, and

$$M_l := \int x^l \phi(x)\,\mathrm{d}x, \quad l = 1,2,\ldots,L-1. \tag{2.2}$$

Where the scaling function ϕ and wavelet ψ are of compact support $[N_1, N_2]$, and $\{a_{N_1}, \dots, a_{N_2}\}$ satisfies

$$\sum_{k=N_1}^{N_2} a_k = 2,$$

$$\sum_{k=N_1}^{N_2} a_k a_{k+2\ell} = 2\delta_{0\ell}, \quad \ell \in Z,$$

$$\sum_{k=N_1}^{N_2} (-1)^k k^m a_k = 0, \quad m = 0, 1, \dots, L-1.$$

Theorem 2.1 (Lin and Zhou [13]). (*Here we state a special case of* [13], *namely*, $c=0$.) *Assume the function* $f \in C^k(\bar{\Omega})$, *where* Ω *is a bounded open set in* R^2, $k \geqslant L \geqslant 2$. *Let, for* $j \in Z$,

$$f^j(x,y) := \frac{1}{2^j} \sum_{(p,q) \in \Lambda} f\left(\frac{p}{2^j}, \frac{q}{2^j}\right) \phi_p^j(x)\phi_q^j(y) \quad (x,y) \in \Omega, \tag{2.3}$$

where the index set

$$\Lambda = \{(p,q) | (\mathrm{supp}(\phi_p^j) \otimes \mathrm{supp}(\phi_q^j)) \cap \Omega \neq \emptyset\}. \tag{2.4}$$

$$\phi_k^j(x) = 2^{\frac{j}{2}} \phi(2^j x - k), \tag{2.5}$$

$$\psi_k^j(x) = 2^{\frac{j}{2}} \psi(2^j x - k). \tag{2.6}$$

In addition the first moment vanishes, i.e.,

$$\int x\phi(x)\,\mathrm{d}x = 0. \tag{2.7}$$

Then

$$\|f - f^j\|_{L^2(\Omega)} \leqslant C \|f^{(L)}\|_\infty \left(\frac{1}{2^j}\right)^L, \tag{2.8}$$

$$\|f - f^j\|_{H^1(\Omega)} \leqslant C \|f^{(L)}\|_\infty \left(\frac{1}{2^j}\right)^{L-1}, \tag{2.9}$$

where C is a constant depending only on L and diameter of Ω.

$$\|f^{(L)}\|_\infty := \max_{(x,y) \in \Omega,\ m=0,1,\dots,L} \left| \frac{\partial^L f}{\partial x^m \partial y^{L-m}}(x,y) \right|. \tag{2.10}$$

Daubechies wavelets of order 3 and coiflets satisfy the conditions in Theorem 2.1. We will use the interpolation formula (2.3) in Section 4.

3. Connection coefficients

In this section we introduce the definition of connection coefficients on the interval $(0,1)$ which enhance the precision and speed of the computation for the solutions of linear and nonlinear differential equations. In Section 4 we will use these connection coefficients and the interpolation formula (2.3) to solve nonlinear partial differential equations — Burgers equations.

3.1. The connection coefficients on (0, 1)

Definition 3.1. Let ϕ be a scaling function with a finite support as described in Section 2. For $j \geqslant 0$,

$$\Lambda^j = \{p|\operatorname{supp}(\phi_p^j) \cap (0,1) \neq \emptyset\}. \tag{3.1}$$

For $p, m, k \in \Lambda^j$, we define the connection coefficients on $(0,1)$ at level j as

$$\Gamma_{p,m}^j := \int_0^{2^j} \phi(x-p)\phi(x-m)\,\mathrm{d}x, \tag{3.2}$$

$$\Omega_{p,m}^j := \int_0^{2^j} \phi'(x-p)\phi'(x-m)\,\mathrm{d}x, \tag{3.3}$$

$$\gamma_{p,m,k}^j := \int_0^{2^j} \phi'(x-p)\phi(x-m)\phi(x-k)\,\mathrm{d}x. \tag{3.4}$$

3.2. Properties of the connection coefficients

The basic properties are

$$\Gamma_{p,m}^j = \Gamma_{m,p}^j, \tag{3.5}$$

$$\Omega_{p,m}^j = \Omega_{m,p}^j, \tag{3.6}$$

$$\gamma_{p,m,k}^j = \gamma_{p,k,m}^j, \tag{3.7}$$

$$\gamma_{p,m,k}^j + \gamma_{m,k,p}^j + \gamma_{k,p,m}^j = \phi(x-p)\phi(x-m)\phi(x-k)|_0^{2^j}. \tag{3.8}$$

For $\phi(x)$ we have the dilation equation

$$\phi(x) = \sum_k a_k \phi(2x-k). \tag{3.9}$$

So for $\phi'(x)$ we have

$$\phi'(x) = \sum_k 2a_k \phi'(2x-k). \tag{3.10}$$

Use these equations we derive the following relations between the connection coefficients at two consecutive levels:

$$\Gamma_{p,m}^j = \int_0^{2^j} \phi(x-p)\phi(x-m)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \sum_{i,\ell} a_i a_\ell \int_0^{2^j} \phi(2x-2p-i)\phi(2x-2m-\ell)\,\mathrm{d}x \\
&= \frac{1}{2}\sum_{i,\ell} a_i a_\ell \int_0^{2^{j+1}} \phi(x-2p-i)\phi(x-2m-\ell)\,\mathrm{d}x \\
&= \frac{1}{2}\sum_{i,\ell} a_i a_\ell \Gamma^{j+1}_{2p+i,2m+\ell}
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
\Gamma^{j+1}_{p,m} &= \int_0^{2^{j+1}} \phi(x-p)\phi(x-m)\,\mathrm{d}x \\
&= \int_0^{2^j} \phi(x-p)\phi(x-m)\,\mathrm{d}x + \int_{2^j}^{2^{j+1}} \phi(x-p)\phi(x-m)\,\mathrm{d}x \\
&= \Gamma^j_{p,m} + \Gamma^j_{p-2^j,m-2^j}.
\end{aligned}
\tag{3.12}
$$

Similarly,

$$
\Omega^{j+1}_{p,m} = \Omega^j_{p,m} + \Omega^j_{p-2^j,m-2^j},
\tag{3.13}
$$

$$
\gamma^{j+1}_{p,m,k} = \gamma^j_{p,m,k} + \gamma^j_{p-2^j,m-2^j,k-2^j}.
\tag{3.14}
$$

Relations (3.12)–(3.14) show that the calculations of the connection coefficients on $(0,1)$ are essentially at level 0. These are the key ingredients for the fast algorithm of the calculations of the connection coefficients.

3.3. Exact calculations of the connection coefficients

From the dilation equation (3.9) we have

$$
\begin{aligned}
\Gamma^j_{p,m} &= \int_0^{2^j} \phi(x-p)\phi(x-m)\,\mathrm{d}x \\
&= \int_0^{2^j} \sum_i a_i\phi(2x-2p-i)\sum_\ell a_\ell \int_0^{2^j} \phi(2x-2m-\ell)\,\mathrm{d}x \\
&= \frac{1}{2}\sum_{i,\ell} a_i a_\ell \int_0^{2^{j+1}} \phi(x-2p-i)\phi(x-2m-\ell)\,\mathrm{d}x \\
&= \frac{1}{2}\sum_{i,\ell} a_i a_\ell \left\{ \int_0^{2^j} \phi(x-2p-i)\phi(x-2m-\ell)\,\mathrm{d}x \right. \\
&\qquad \left. + \int_{2^j}^{2^{j+1}} \phi(x-2p-i)\phi(x-2m-\ell)\,\mathrm{d}x \right\}
\end{aligned}
$$

$$= \frac{1}{2}\sum_{i,\ell} a_i a_\ell \left\{ \Gamma^j_{2p+i,2m+\ell} + \int_0^{2^j} \phi(x+2^j-2p-i)\phi(x+2^j-2m-\ell)\,\mathrm{d}x \right\}$$

$$= \frac{1}{2}\sum_{i,\ell} a_i a_\ell \{ \Gamma^j_{2p+i,2m+\ell} + \Gamma^j_{2p+i-2^j,2m+\ell-2^j} \}. \quad (3.15)$$

Similarly by (3.3), (3.4) and (3.10) we have

$$\Omega^j_{p,m} = 2\sum_{i,\ell} a_i a_\ell \{ \Omega^j_{2p+i,2m+\ell} + \Omega^j_{2p+i-2^j,2m+\ell-2^j} \}, \quad (3.16)$$

$$\gamma^j_{p,m,k} = \sum_{i,\ell,t} a_i a_\ell a_t \{ \gamma^j_{2p+i,2m+\ell,2k+t} + \gamma^j_{2p+i-2^j,2m+\ell-2^j,2k+t-2^j} \}. \quad (3.17)$$

Each of these is a system of linear homogeneous equations. To determine the unique solution, we need to create some linearly independent inhomogeneous equations using the moment equations,

$$x^i = \sum_p (c+p)^i \phi(x-p), \quad i = 0,1,\ldots,L-1, \quad (3.18)$$

where c is defined in (2.1).

For example, to evaluate $\{\Omega^j_{p,m}\}$, we use

$$ix^{i-1} = \sum_p (c+p)^i \phi'(x-p), \quad i = 1,\ldots,L-1, \quad (3.19)$$

to obtain

$$\int_0^{2^j} i^2 x^{2i-2}\,\mathrm{d}x = \sum_{p,m} (c+p)^i (c+m)^i \int_0^{2^j} \phi'(x-p)\phi'(x-m)\,\mathrm{d}x \quad (3.20)$$

or

$$\frac{i^2}{2i-1} 2^{2(2i-1)j} = \sum_{p,m} (c+p)^i (c+m)^i \Omega^j_{p,m}, \quad i = 1,\ldots,L-1. \quad (3.21)$$

In this way, we can have more equations which will give rise to the solution of unknowns. In general, it is an open problem, namely, how many independent inhomogeneous equations are needed to find the unique solution and the problem of whether a solution exists. For the cases of Daubechies wavelet of order 3 and coiflets of order 4, we are able to find $\{\Gamma^j_{p,m}\}$, $\{\Omega^j_{p,m}\}$ and $\{\gamma^j_{p,m,k}\}$. In these cases there is a unique solution. The dimension of the solution space of (3.15) is 1 and the dimension of each of the solution spaces of (3.16) and (3.17) is 2.

For the case of higher derivatives and more terms, we have the following results. We now use the following shorthand for differentiation of a function:

$$\phi^{d_i}_m := \frac{\mathrm{d}^{d_i}\phi_m(x)}{\mathrm{d}x^{d_i}}. \quad (3.22)$$

Define 2-term connection coefficients as follows:

$$\Lambda^{j,d_1d_2}_{lm} := \int_0^{2^j} \phi^{d_1}_l(x)\phi^{d_2}_m(x)\,\mathrm{d}x. \quad (3.23)$$

Let us assume that ϕ is d-times differentiable. Substitute (3.9) into (3.23) and simplify it, we have the following *scaling equations*.

Theorem 3.2.

$$A\Lambda^{d_1d_2} = \frac{1}{2^{d-1}}\Lambda^{d_1d_2} \tag{3.24}$$

where $d := d_1 + d_2$ *and*

$$A_{l,m;p,q} := a_{p-2l}a_{q-2m} + a_{p-2l+2^j,q-2m+2^j}.$$

Let $B^{c_1c_2} = \phi_l^{c_1-1}(x)\phi_m^{c_2}(x)|_0^{2^j}$, we then have,

$$\Lambda^{c_1c_2} = \sum_0^{c_1}(-1)^i B^{c_1-i,c_2+i} + (-1)_1^c\Lambda^{0,d}. \tag{3.25}$$

This implies the following statement.

Theorem 3.3. *The space spanned by* $\{\Lambda^{d_1d_2} \mid d_1 + d_2 = d\}$ *is of dimension* 1.

We next consider moment equations which can be used to uniquely determine the solution of Λ.

Let K be the largest k such that x^k can be expressed as a locally finite series of translations of the scaling functions. Namely, for $0 \leqslant k \leqslant K$,

$$x^k = \sum c_l\phi_l(x), \tag{3.26}$$

where the coefficients are

$$c_l = M_l^k = \int_{-\infty}^{\infty} x^k\phi_l(x)\,\mathrm{d}x. \tag{3.27}$$

Differentiate (3.26) k times, we obtain

$$k! = \sum_i M_i^k\phi_i^k(x). \tag{3.28}$$

Further differentiation gives rise to

$$0 = \sum_i M_i^k\phi_i^j(x), \quad j > k. \tag{3.29}$$

Multiplying both sides of the above 2 equations by $\phi_m^{d_1}(x)$ and integrating over $(0, 2^j)$, we have

Theorem 3.4.

$$\sum_l M_l^k\Lambda_{lm}^{j,d_1d_2} = \begin{cases} k!\int_0^{2^j}\phi_m^{d_1}(x)\,\mathrm{d}x & \text{for } k = d_2, \\ 0 & \text{for } k < d_2. \end{cases} \tag{3.30}$$

Similarly, multiplying both sides of the equations (3.28) amd (3.29) by $\phi_l^{d_1}(x)$ and integrating over $(0, 2^j)$, we have

Theorem 3.5.

$$\sum_m M_m^k\Lambda_{lm}^{j,d_1d_2} = \begin{cases} k!\int_0^{2^j}\phi_l^{d_2}(x)\,\mathrm{d}x & \text{for } k = d_1, \\ 0 & \text{for } k < d_1. \end{cases} \tag{3.31}$$

Table 1
The coefficients for coiflets of order 4

n	a_n
−4	0.011587596739
−3	−0.029320137980
−2	−0.047639590310
−1	0.273021046535
0	0.574682393857
1	0.294867193696
2	−0.054085607092
3	−0.042026480461
4	0.016744410163
5	0.003967883613
6	−0.001289203356
7	−0.000509505399

In a similar fashion, depending on the need in solving differential equations or other applications, one can derive 3 or more terms connection coefficients.

4. Burgers equation

In this section we use the connection coefficients (3.2)–(3.4) and the interpolation formula (2.3) to solve Burgers equations. We use $j=6$ and scaling functions are coiflets of order 4 (see Table 1) in the following three problems.

4.1. The first problem

We consider the following initial-boundary value problem:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2}=0, \tag{4.1}$$

where Re is the Reynolds number. The solution will be sought in the region $-1\leqslant x\leqslant 1$ for $t\geqslant 0$. Initial and boundary conditions are taken to be

$$u_0(x)=u(x,0)=\begin{cases}1 & \text{if } -1\leqslant x\leqslant 0,\\ 0 & \text{if } 0<x\leqslant 1,\end{cases} \tag{4.2}$$

$$u(-1,t)=1, \quad u(1,t)=0. \tag{4.3}$$

We apply the following semi-implicit scheme in the time direction

$$u^0=u_0, \tag{4.4}$$

$$\frac{u^{k+1}-u^k}{\Delta t}+u^k\frac{\partial u^{k+1}}{\partial x}=v\frac{\partial^2 u^{k+1}}{\partial x^2} \quad \text{on } (0,1),\ \ k\geqslant 0, \tag{4.5}$$

$$u^{k+1}(-1)=0, \quad u^{k+1}(1)=1, \tag{4.6}$$

where $u^k(x) := u(x, k\,\Delta t)$, $v = 1/\mathrm{Re}$ and the time discretization step $\Delta t = 10^{-3}$. At each time step (4.4)–(4.6) provides a boundary value problem for an elliptic ordinary differential equation about u^{k+1}.

To solve (4.4)–(4.6), let $w = u^{k+1}$, then we have

$$-w'' + \frac{1}{v}u^k w' + \frac{1}{v\,\Delta t}w = \frac{1}{v\,\Delta t}u^k, \tag{4.7}$$

$$w(-1) = 0, \quad w(1) = 1. \tag{4.8}$$

We convert the above equation to its variational form and approach the problem by reducing the Dirichlet condition to Neumann condition and use interpolation formula to approximate the coefficient function.

To do this, we consider the following setting:

$$a(u, v) = \int_0^1 (u'v' + gu'v + cuv)\,\mathrm{d}x, \tag{4.9}$$

$$L(v) = \langle f, v\rangle, \quad \text{for every } v \in H_0^1(0,1) \times R, \tag{4.10}$$

where $g(x) = (1/v)u^k(x)$, $c = 1/v\,\Delta t$, $f = cu^k$.

By completing square and choosing suitable constants, one can check the V-ellipticity of the bilinear form $a(u, v)$. This implies the existence and uniqueness of the solution at each stage in solving the above equation [7].

Define u_0, u_1, u_2 as the solution of the following variational problem in $H^1(0,1) \times R$, respectively,

$$\begin{aligned} &u_0 \in H^1(0,1),\\ &a(u_0, v) = L(v) \quad \text{for all } v \in H^1(0,1), \end{aligned} \tag{4.11}$$

$$\begin{aligned} &u_1 \in H^1(0,1),\\ &a(u_1, v) = v(0) \quad \text{for all } v \in H^1(0,1), \end{aligned} \tag{4.12}$$

$$\begin{aligned} &u_2 \in H^1(0,1),\\ &a(u_2, v) = v(1) \quad \text{for all } v \in H^1(0,1). \end{aligned} \tag{4.13}$$

To the original boundary value problem, we associate the following problem:

$$a(u, v) = L(v) + \lambda_1 v(0) + \lambda_2 v(1) \quad \text{for every } v \in H^1(0,1), \quad u(0) = 1,\ u(1) = 0. \tag{4.14}$$

The function u in the above equation necessarily satisfies

$$u = u_0 + \lambda_1 u_1 + \lambda_2 u_2, \tag{4.15}$$

which implies that λ_1, λ_2 satisfies

$$\begin{cases} u_1(0)\lambda_1 + u_2(0)\lambda_2 = 1 - u_0(0) \\ u_1(1)\lambda_1 + u_2(1)\lambda_2 = -u_0(1). \end{cases} \tag{4.16}$$

Table 2
Solutions of Burgers equation (4.1)–(4.3)

x	Approx. u_a	Exact u_e	Approx. u_w
−1.0	1.0000	1.0000	1.0000
−0.9	0.9956	1.0000	1.0000
−0.8	1.0456	1.0000	1.0000
−0.7	1.0672	1.0000	1.0000
−0.6	1.0402	1.0000	1.0000
−0.5	0.9831	1.0000	1.0000
−0.4	0.9303	1.0000	1.0000
−0.3	0.9128	1.0000	1.0000
−0.2	0.9444	1.0000	1.0000
−0.1	1.0159	1.0000	1.0000
0	1.0963	1.0000	1.0000
0.1	1.1411	1.0000	1.0000
0.2	1.1057	1.0000	1.0000
0.3	0.9613	0.9998	0.9995
0.4	0.7099	0.9714	0.9723
0.5	0.3933	0.1861	0.2034
0.6	0.0905	0.0015	0.0004
0.7	−0.1017	0.0000	0.0000
0.8	−0.1154	0.0000	0.0000
0.9	0.0091	0.0000	0.0000
1.0	0.0000	0.0000	0.0000
	$\|u_a - u_e\|_{rms} = 0.1049$		$\|u_w - u_e\|_{rms} = 0.0377$

The solutions of the Neumann problems (4.11)–(4.13) will give rise to the solution of the associated problem (4.14).

The exact solution of Eqs. (4.1)–(4.3) is

$$u_e = \frac{\int_{-\infty}^{\infty} [(x-\xi)/t] \exp\{-0.5\,\mathrm{Re}\,F\}\,\mathrm{d}\xi}{\int_{-\infty}^{\infty} \exp\{-0.5\,\mathrm{Re}\,F\}\,\mathrm{d}\xi}, \tag{4.17}$$

where

$$F(\xi; x, t) = \int_0^{\xi} u_0(\xi')\,\mathrm{d}\xi' + \frac{0.5(x-\xi)^2}{t}. \tag{4.18}$$

In [6] the traditional Galerkin method is used to solve this problem for various Reynolds number Re. Here we solve it for $\mathrm{Re} = 100$ at $t = 0.92$. The approximate solution is denoted by u_w. We list the results in Table 2, and compare them with the traditional Galerkin method solution u_a and exact solution u_e. We also compute the discrete rms error which is defined by

$$\|u_a - u_e\|_{rms} = \frac{[\sum_{l=1}^{L} (u_a - u_e)_l^2]^{1/2}}{L^{1/2}}. \tag{4.19}$$

Table 3
Numerical solutions of Burgers equation (4.20)–(4.22)

x	$t=0.00$	$t=0.13$	$t=0.25$
−1.000	0.0000	0.0000	0.0000
−0.900	0.3090	0.2226	0.1736
−0.800	0.5878	0.4369	0.3441
−0.700	0.8090	0.6338	0.5082
−0.600	0.9511	0.8024	0.6616
−0.500	1.0000	0.9285	0.7984
−0.400	0.9511	0.9926	0.9099
−0.300	0.8090	0.9665	0.9806
−0.200	0.5878	0.8106	0.9779
−0.100	0.3090	0.4828	0.8063
0.000	0.0000	0.0000	−0.0019
0.100	−0.3090	−0.4828	−0.8068
0.200	−0.5878	−0.8106	−0.9780
0.300	−0.8090	−0.9665	−0.9806
0.400	−0.9511	−0.9926	−0.9099
0.500	−1.0000	−0.9285	−0.7984
0.600	−0.9511	−0.8024	−0.6616
0.700	−0.8090	−0.6338	−0.5082
0.800	−0.5878	−0.4369	−0.3441
0.900	−0.3090	−0.2226	−0.1736
1.000	0.0000	0.0000	0.0000

4.2. The second problem

We consider the following Burgers equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2} \tag{4.20}$$

for $t > 0,\ 0 < x < 1$ with the initial condition

$$u(x,0) = -\sin(\pi x), \tag{4.21}$$

and boundary equations

$$u(1,t) = u(-1,t) = 0 \tag{4.22}$$

whose analytic solution is known [2]. Here $v = 10^{-2}/\pi$. We use the same method as problem 1 and list the results in Tables 3 and 4.

One of the special features of our methods is the multilevel iterations algorithms which provide more efficient calculations. As we compare our results of problem 1 with [6] and problem 2 with [15], our method is relatively more robust and accurate. The above two problems are Dirichlet boundary value problems. We next consider mixed boundary conditions.

Table 4
Exact solution of Burgers equation (4.20)–(4.22)

x	$t=0.00$	$t=0.13$	$t=0.25$
−1.00	0.0000	0.0000	0.0000
−0.90	0.3090	0.2226	0.1736
−0.80	0.5878	0.4369	0.3441
−0.70	0.8090	0.6338	0.5082
−0.60	0.9511	0.8024	0.6616
−0.50	1.0000	0.9286	0.7985
−0.40	0.9511	0.9927	0.9100
−0.30	0.8090	0.9656	0.9808
−0.20	0.5878	0.8103	0.9777
−0.10	0.3090	0.4829	0.8073
0.00	0.0000	0.0000	0.0000
0.10	−0.3090	−0.4829	−0.8073
0.20	−0.5878	−0.8103	−0.9777
0.30	−0.8090	−0.9656	−0.9808
0.40	−0.9511	−0.9927	−0.9100
0.50	−1.0000	−0.9286	−0.7985
0.60	−0.9511	−0.8024	−0.6616
0.70	−0.8090	−0.6338	−0.5082
0.80	−0.5878	−0.4369	−0.3441
0.90	−0.3090	−0.2226	−0.1736
1.00	0.0000	0.0000	0.0000

4.3. The third problem

We consider the following mixed initial boundary-value problem

$$\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2} \tag{4.23}$$

for $t > 0$, $0 < x < 1$ with the initial condition

$$u(x,0) = u_0(x) \tag{4.24}$$

and boundary equations

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad u(1,t) = 1, \tag{4.25}$$

where $v = 2 \times 10^{-3}$ and $u_0(x) = \mathrm{e}^{-8(1-x)}$. We apply the following semi-implicit scheme in the time direction

$$u^0 = u_0, \tag{4.26}$$

$$\frac{u^{k+1} - u^k}{\Delta t} - u^k\frac{\partial u^{k+1}}{\partial x} = v\frac{\partial^2 u^{k+1}}{\partial x^2} \quad \text{on } (0,1), \quad k \geqslant 0, \tag{4.27}$$

$$\frac{\partial u^{k+1}(0)}{\partial x} = 0, \quad u^{k+1}(1) = 1, \tag{4.28}$$

where $u^k(x) := u(x, k\,\Delta t)$ and the time discretization step $\Delta t = 10^{-3}$. At each time step (4.26)–(4.28) provides a mixed boundary-value problem for an elliptic ordinary differential equation about u^{k+1}.

To solve (4.26)–(4.28), let $w = u^{k+1}$, then we have

$$-w'' - \frac{1}{v}u^k w' + \frac{1}{v\,\Delta t}w = \frac{1}{v\,\Delta t}u^k, \tag{4.29}$$

$$w'(0) = 0, \quad w(1) = 1. \tag{4.30}$$

In order to reduce the Dirichlet condition to Neumann condition, we consider the following setting:

$$a(u, v) = \int_0^1 (u'v' + gu'v + cuv)\,\mathrm{d}x, \tag{4.31}$$

$$L(v) = \langle f, v\rangle \quad \text{for every } v \in H_0^1(0,1) \times R, \tag{4.32}$$

where $g(x) = (1/v)u^k(x)$, $c = 1/v\,\Delta t$, $f = cu^k$.

Let u_0, u_1 satisfy the following variational problem in $H^1(0,1) \times R$, respectively,

$$u_0 \in H^1(0,1), \quad a(u_0, v) = L(v) \quad \text{for all } v \in H^1(0,1), \tag{4.33}$$

$$u_1 \in H^1(0,1), \quad a(u_1, v) = v(1) \quad \text{for all } v \in H^1(0,1). \tag{4.34}$$

Corresponding to the original mixed boundary value problem, we have the following associated problem:

$$a(u, v) = L(v) + \lambda_1 v(1) \quad \text{for every } v \in H^1(0,1), \quad u(1) = 1. \tag{4.35}$$

The function u in the above equation necessarily satisfies

$$u = u_0 + \lambda_1 u_1, \tag{4.36}$$

which implies that λ_1 satisfies

$$u(1) = u_0(1) + \lambda_1 u_1(1) = 1. \tag{4.37}$$

Hence,

$$\lambda_1 = \frac{1 - u_0(1)}{u_1(1)}. \tag{4.38}$$

The solutions of the Neumann problems (4.33), (4.34) will give rise to the solution of the associated problem (4.35). Fig. 1 shows the solution at times 0, 0.03 and 0.18, illustrating the development of a quasi shock starting at $t = 0.03$ and fully developed at $t = 0.18$.

In this paper we have been exploring the wavelet interpolation based approximations for the numerical solutions of nonlinear problems. For this class of problems, we have seen wavelets compare favorably with traditional methods. Our method seems to be promising for higher-dimensional cases. We plan to solve Navier–Stokes equations along these lines.

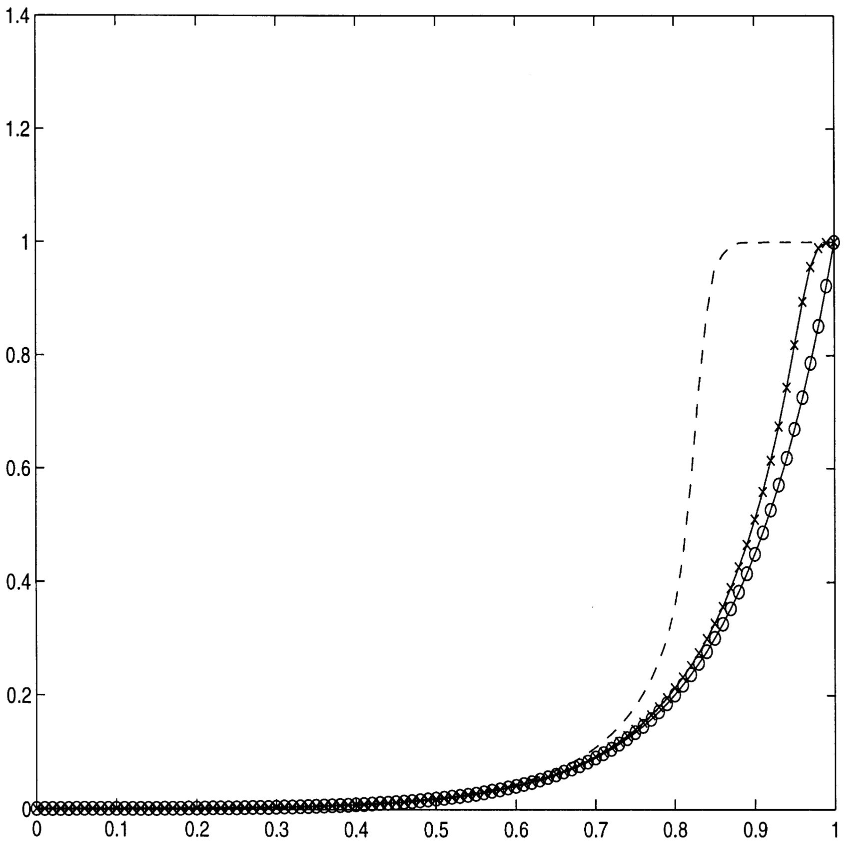

Fig. 1. The solution of Burgers equation (4.23)–(4.25) at $t=0\,(\circ)$, $t=0.3\,(\times)$, $t=0.18\,(-)$.

Appendix

In this appendix we will discuss the convergence and error analysis of the methods used in Section 4. More generally, we consider the following boundary value problems in terms of variational formulation.

Find $u \in V$ such that

$$a(u,v)=L(v) \quad \text{for every } v \in V. \tag{A.1}$$

In (A.1), V, a(.,.), L are as follows:
(1) V is a Hilbert Space with scalar product (.,.) and associated norm $\|.\|$.
(2) $a: V \times V \to R$ is a bilinear form, continuous and V-elliptic over $V \times V$
(3) $L: \to R$ is linear and continuous.

If properties (1)–(3) hold it follows from the Lax–Milgram Theorem that (A.1) has a unique solution.

In fact, applying the Riesz Representation Theorem, there exists $A \in \mathrm{Isom}(V, V^*)$, uniquely defined, such that

$$a(v,w)=\langle Av, w\rangle \quad \text{for every } v,w \in V. \tag{A.2}$$

And hence,

$$|a(v,w)| \leqslant \|A\|\|v\|\|w\| \quad \text{for every } v,w \in V. \tag{A.3}$$

Consider a family $\{V_n\}_n$ of closed subspaces of V and $V_1 \subset V_2 \subset \cdots V_{n-1} \subset V_n \cdots$. In V_n it is quite natural to approximate the problem (A.1) by the following setting.

Find $u_n \in V_n$ such that

$$a(u_n, v) = L(v) \quad \text{for every } v \in V_n. \tag{A.4}$$

Problem (A.4) has a unique solution by the Lax–Milgram Theorem. On the other hand, we have the following approximation property:

$$\|u_n - u\| \leqslant \|A\|\|A^{-1}\|\|v - u\| \quad \text{for every } v \in V_n. \tag{A.5}$$

If the bilinear form $a(.,.)$ is symmetric then the above inequality will give rise to

$$\|u_n - u\| \leqslant [\|A\|\|A^{-1}\|]^{1/2}\|v - u\| \quad \text{for every } v \in V_n. \tag{A.6}$$

Therefore, we have the following convergence result,

$$\lim_{n\to\infty} \|u_n - u\| = 0, \tag{A.7}$$

provided,

$$\lim_{n\to\infty} (\inf \|v - u\|\colon v \in V_n) = 0. \tag{A.8}$$

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