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# On Some Types of Maximal Abelian Subalgebras

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Let $H$ be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on $H$. It is known that there are two kinds of maximal abelian subalgebras in $B(H)$, to one of which there exists a unique faithful normal projection of norm one from $B(H)$ and to the other any projection of norm one is singular. Any maximal abelian subalgebra $A$ contains a projection $e$ such that $Ae$ is a maximal abelian subalgebra of $B(eH)$ of the first kind and $A(1-e)$ is the one of the second kind in $B((1-e)H)$. This will be generalized to an arbitrary von Neumann algebra together with the existence problem of those kinds of maximal abelian subalgebras.

## INTRODUCTION

Let $B(H)$ be the algebra of all bounded linear operators on a Hilbert space $H$. It is known that there are two kinds of maximal abelian subalgebras in $B(H)$, the one generated by minimal projections and the other which does not contain minimal projections. They are called maximal abelian subalgebras of discrete type and of continuous type. One difference between these two kinds of maximal abelian subalgebras is illustrated by the fact that there is a unique faithful $\sigma$-weakly continuous projection of norm one onto a discrete maximal abelian subalgebra, and, for a continuous maximal abelian subalgebra, no norm-one projection is $\sigma$-weakly continuous, and each vanishes on $\sigma$-weakly dense ideal $C(H)$. Each maximal abelian subalgebra $A$ contains a projection $e$ such that $Ae$ is a discrete maximal abelian subalgebra in $eB(H)e = B(eH)$ and $A(1-e)$ is a continuous one in $(1-e)B(H)(1-e)$.

The purpose of this paper is to show that the above phenomena holds in all von Neumann algebras. We shall extend the notion of

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discrete and continuous maximal abelian subalgebras to arbitrary von Neumann algebras as smooth and completely nonsmooth algebras, by referring to the existence of $\sigma$-weakly continuous projection of norm one. We prove that (Corollary 3.1) any maximal abelian subalgebra $A$ in a von Neumann algebra $M$ contains a projection $e$ such that $Ae$ is a smooth maximal abelian subalgebra with respect to $eMe$ and $A(1 - e)$ is a completely nonsmooth maximal abelian subalgebra with respect to $(1 - e)M(1 - e)$. It will be shown that $M$ is finite if and only if every maximal abelian subalgebra is smooth.

1. Since we shall be mainly concerned with projections of norm one throughout this paper, we recall some results about them [15, 16, 19]. A projection $\pi$ of norm one in a von Neumann algebra $M$ to its von Neumann subalgebra $N$ is necessarily positive and two-sided $N$-module mapping [i.e., $\pi(axb) = a\pi(x)b$ for $a, b \in N$]. Conversely a projection $\pi$ from $M$ to $N$ is of norm one if it is positive. Let $M_*$ and $N_*$ be the space of all $\sigma$-weakly continuous linear functional on $M$ and $N$. We denote by $M_*{}^\perp$ and $N_*{}^\perp$ the singular parts of $M^*$ and $N^*$. A positive functional $\varphi$ is singular if and only if for any nonzero projection $e$ there is a nonzero projection $f$ majorized by $e$ with $\langle f, \varphi \rangle = 0$ ([11]). The projection $\pi$ is called singular if ${}^t\pi(N_*) \subset M_*{}^\perp$. On the other hand, we have always ${}^t\pi(N_*{}^\perp) \subset M_*{}^\perp$. Every projection $\pi$ of norm one can be uniquely decomposed as

$$\pi = \pi_n + \pi_s ,$$

where $\pi_n$ is $\sigma$-weakly continuous, $\pi_s$ is singular, and both are positive $N$-module mappings. We call $\pi_n$ the normal component of $\pi$, and $\pi_s$ the singular component of $\pi$. We emphasize that a singular mapping is merely not $\sigma$-weakly continuous but is far from the $\sigma$-weak continuity.

It can be generally proved that if the direct summands of $M$ are simpler than those of $N$, every projection to $N$ is singular. Thus, roughly speaking, if the structure of $M$ is simpler than that of $N$ all projections to $N$ should be singular. This principle is still valid within the algebras of type I. We have

PROPOSITION 1.1. *Let $M$ be an atomic von Neumann algebra and $N$ a von Neumann subalgebra which has no minimal projections. Then every projection of norm one from $M$ to $N$ is singular.*

*Proof.* Let $\pi$ be a projection to $N$ and $\varphi$ be a pure state of $N$. Since $N$ contains no minimal projections, $\varphi$ is a singular state, i.e., $\varphi \in N_*{}^\perp$.

Hence ${}^t\pi(\varphi) \in M_*{}^\perp$. Let $e$ be a minimal projection of $M$. Then the characterization of singular states cited above shows that

$$\langle e, {}^t\pi(\varphi)\rangle = \langle \pi(e), \varphi\rangle = 0.$$

Therefore $\pi(e) = 0$, and this implies $\pi_n(e) = 0$. Hence $\pi_n(M) = 0$ and $\pi = \pi_s$.

A consequence of this proposition is the following result which is somewhat known. If there exists a normal projection from an atomic von Neumann algebra $M$ to its subalgebra $N$, then $N$ is also an atomic von Neumann algebra.

In case $M = B(H)$, $B(H)_*^\perp$ is known to be $C(H)^0$, the polar of the algebra of compact operators on $H$. Hence a projection $\pi$ from $B(H)$ to a continuous maximal abelian subalgebra $A$ satisfies ${}^t\pi(A_*) \subset C(H)^0$, and $\pi(C(H)) = 0$. This is the situation for a continuous maximal abelian subalgebra of $B(H)$.

As for a discrete maximal abelian subalgebras it can be shown, generally, that if $N$ is an atomic von Neumann subalgebra there always exists a normal projection to $N$.

With these results we shall define

DEFINITION. A maximal abelian subalgebra $A$ in a von Neumann algebra $M$ is called smooth with respect to $M$ if there exists a $\sigma$-weakly continuous projection of norm one. If every projection of norm one to $A$ is singular we call $A$ completely nonsmooth with respect to $M$.

We first investigate the situation for a smooth maximal abelian subalgebra. Let $G(A)$ be the group of automorphisms of $M$ induced by unitary operators in $A$. We write $\sigma_u$ for an element of $G(A)$, where $\sigma_u(x) = uxu^*$. Then the fixed algebra of $G(A)$ is $A$ itself. For $x \in M$, we denote by $\overline{co}(x)$ the weak closure of the convex hull of

$$\{\sigma(x) \mid \sigma \in G(A)\}.$$

Then the fixed-point theorem of Kakutani–Markov says that

$$\overline{co}(x) \cap A \neq \varnothing.$$

Let $\pi$ be a normal projection to $A$. Then for $\sigma_u \in G(A)$ we have

$$\pi(\sigma_u(x)) = \pi(uxu^*) = u\pi(x)\, u^* = \pi(x).$$

Hence $\pi$ is $G(A)$-invariant and this implies that the above intersection must reduce to one point, $\pi(x)$, i.e.,

$$\overline{co}(x) \cap A = \{\pi(x)\}.$$

Thus $\pi$ is unique. Furthermore, if we consider the set

$$\{{}^t\pi(\varphi) \mid \varphi \text{ is a normal state of } A\}$$

it is the set of $G(A)$-invariant normal states which is clearly faithful on the fixed algebra $A$. Hence it is faithful on $M$ as well (cf. [6]). Therefore $\pi$ is faithful, and $M$ is $G(A)$-finite. Thus we have

PROPOSITION 1.2. *If there exists a normal projection to a maximal abelian subalgebra, the projection is faithful and is unique among normal projections.*

The uniqueness of normal projections is a known fact but the faithfulness does not seem to have been mentioned in the literature.

On the other hand a completely nonsmooth maximal abelian subalgebra may be characterized in terms of invariant normal states in the following way.

PROPOSITION 1.3. *A maximal abelian subalgebra $A$ is completely nonsmooth if and only if there exist no $G(A)$-invariant normal states.*

*Proof.* Suppose there is an invariant normal state $\varphi$. Then its support $e$ belongs to $A$ and $Ae$ is a maximal abelian subalgebra of $eMe$. As $\varphi$ is also $G(Ae)$-invariant, $eMe$ is $G(Ae)$-finite; and there is a faithful normal projection $\pi_1$ from $eMe$ to $Ae$ [6, Theorem 1]. Let $\pi_2$ be an arbitrary projection from $(1-e)M(1-e)$ to $A(1-e)$. Since $A(1-e)$ has the extension property, as a Banach space [3], such a norm-one projection exists. Writing the map $\pi$ as

$$\pi(x) = \pi_1(exe) + \pi_2((1-e)x(1-e))$$

for $x \in M$, one easily sees that it is a projection of norm one from $M$ to $A$ whose normal component is not zero.

Conversely let $\pi$ be a projection to $A$ for which $\pi_n \neq 0$. There is a normal state $\varphi$ of $A$ with ${}^t\pi_n(\varphi) \neq 0$. As $\pi_n$ is also $G(A)$-invariant, ${}^t\pi_n(\varphi)$ is a non-zero $G(A)$-invariant normal functional on $M$. This completes the proof.

As we have mentioned above, the unique faithful normal projection $\pi$ to a maximal abelian subalgebra is given in a form

$$\overline{co}(x) \cap A = \{\pi(x)\}.$$

It would be interesting to know whether the converse of this fact is true or not. Although we do not know the answer, by Proposition 2.2 it will suffice to know whether the projection given by

$$x \to \{y\} = \overline{co}(x) \cap A$$

is faithful or not.

2. In order to show the decomposition theorem cited in the introduction, in a more general form, we need

LEMMA 2.1. *Let $M$ be a von Neumann algebra and $N$ be its von Neumann subalgebra. Let $\pi$ be a projection to $N$ and $z$ be the support projection of $\pi_n(1)$ in $N$. Then there exists a normal projection from $zMz$ to $Nz$.*

*Proof.* We note first that $\pi_n(1)$ is a central element of $N$ and $z$ is a central projection. Let $z = \sum_\alpha z_\alpha$ be the decomposition into orthogonal central projections such that $\pi_n(1)z_\alpha = \pi_n(z_\alpha)$ is invertible in $Nz_\alpha$. Let $k_\alpha$ be the inverse of $\pi_n(z_\alpha)$ in $Nz_\alpha$. We define a mapping

$$\pi_\alpha \,;\, z_\alpha M z_\alpha \to N z_\alpha$$

by $\pi_\alpha(z_\alpha\, x\, z_\alpha) = \pi_n(z_\alpha\, x\, z_\alpha)k_\alpha$. This is a positive normal projection to $Nz_\alpha$, hence it is of norm one. Then, for an element $x \in zMz$, we write

$$\pi^n(x) = \sum_\alpha \pi_\alpha(z_\alpha\, x\, z_\alpha),$$

where the sum is taken in $\sigma$-weak topology. It may be easily seen that $\pi^n$ is a normal projection of norm one to $Nz$.

THEOREM 2.1. *Let $M$ be von Neumann algebra and $A$ a commutative subalgebra, then $A$ contains a projection $e$ such that there exists a normal projection from $eMe$ to $Ae$ and all projections of norm one from $(1 - e)M(1 - e)$ to $A(1 - e)$ are singular.*

*Proof.* Let $\{e_\alpha\}$ be a maximal family of orthogonal projections in $A$ such that there is a normal projection $\pi_\alpha$ from $e_\alpha M e_\alpha$ to $Ae_\alpha$. Put $e = \sum_\alpha e_\alpha$, then, as in the proof of the above lemma, we get a normal projection $\pi$ from $eMe$ to $Ae$ by defining

$$\pi(x) = \sum_\alpha \pi_\alpha(e_\alpha x e_\alpha).$$

Let $\pi$ be an arbitrary projection from $(1 - e)M(1 - e)$ to $A(1 - e)$ and suppose $\pi_n \neq 0$. Then, by Lemma 2.1, we get a nonzero projection $f \in A(1-e)$ such that there is a normal projection from $fMf$ to $Af$, which contradicts to the maximality of $\{e_\alpha\}$. Hence $\pi_n = 0$, i.e., $\pi$ is singular.

COROLLARY 2.1. *Let $A$ be a maximal abelian subalgebra, then $A$ contains a projection $e$ such that $Ae$ is a smooth maximal abelian subalgebra with respect to $eMe$ and $A(1-e)$ is a completely nonsmooth maximal abelian subalgebra with respect to $(1-e)M(1-e)$.*

Lemma 2.1 suggests that the situation which admits a faithful projection of norm one could be the same as the one which admits a faithful normal projection. In fact, we can show the following:

PROPOSITION 2.2. *With the notations as above, if there exists a faithful projection of norm one from $M$ to $N$, there also exists a faithful normal projection to $N$.*

*Proof.* Let $\pi$ be a faithful projection to $N$. By Lemma 2.1, it suffices to show that $\pi_n$ is also faithful. Since, for any projection $e$ in $N$, $\pi$ induces a projection from $eMe$ to $eNe$, we may assume that $N$ is countably decomposable. Take a positive element $a \in M$, then we can find a spectral projection $e$ and a positive number $\lambda$ with $a \geqslant \lambda e$. Let $\varphi$ be a faithful normal state of $N$. Since ${}^t\pi_s(\varphi)$ is a positive singular functional (assuming $\pi_s \neq 0$), there exists a nonzero projection $f \leqslant e$ such that

$$\langle f, {}^t\pi_s(\varphi)\rangle = \langle \pi_s(f), \varphi\rangle = 0.$$

Hence $\pi_s(f) = 0$. Therefore

$$\pi_n(a) \geqslant \lambda\pi_n(e) \geqslant \lambda\pi_n(f) = \lambda\pi(f) > 0.$$

3. We shall discuss here the types of product projections. Although we need only the case of maximal abelian subalgebras in the next section, we state our results in general form for the benefit of another use. We recall here Theorem 4 in [18] (cf. also [2]). Let $M_1$ and $M_2$ be von Neumann algebras and let $N_1$ and $N_2$ be their von Neumann subalgebras. Let $\pi_1$ and $\pi_2$ be projections from $M_1$ and $M_2$ to $N_1$ and $N_2$ respectively. Then the above theorem says that, without referring to $\sigma$-weak continuity of projections, we can construct a projection

$$\pi : M_1 \otimes B(K) \cap B(H) \otimes M_2 \to N_1 \otimes B(K) \cap B(H) \otimes N_2$$

($H$ and $K$ are underlying Hilbert spaces of $M_1$ and $M_2$) such that

$$\pi(a \otimes b) = \pi_1(a) \otimes \pi_2(b)$$

for $a \in M_1$ and $b \in M_2$. Since we have now the general commutation theorem for tensor products of arbitrary von Neumann algebras proved by Tomita theory [12, 13], we can say that $\pi$ is a projection of norm one from $M_1 \otimes M_2$ to $N_1 \otimes N_2$. Let us call such a projection a product projection of $\pi_1$ and $\pi_2$. Product projections might be generally not unique.

LEMMA 3.1. *Let*

$$\pi_1 = \pi_n{}^1 + \pi_s{}^1, \qquad \pi_2 = \pi_n{}^2 + \pi_s{}^2$$

*be canonical decompositions of* $\pi_1$ *and* $\pi_2$, *then there is a unique normal mapping* $\Phi$ *from* $M_1 \otimes M_2$ *to* $N_1 \otimes N_2$ *such that*

$$\Phi(a \otimes b) = \pi_n{}^1(a) \otimes \pi_n{}^2(b).$$

*We denote this mapping by* $\Phi = \pi_n{}^1 \otimes \pi_n{}^2$.

*Proof.* Let $M_1 \odot M_2$ and $N_1 \odot N_2$ be algebraic tensor product of $M_1$ and $M_2$, and, $N_1$ and $N_2$. Define a mapping $\Phi$ as

$$\Phi\left(\sum_{i=1} a_i \otimes b_i\right) = \sum_{i=1}^{n} \pi_n{}^1(a_i) \otimes \pi_n{}^2(b_i).$$

We assert first that $\Phi$ is norm continuous and it can be extended to the $C^*$-tensor product $M_1 \otimes_\alpha M_2$. Let $\varphi$ and $\psi$ be states of $N_1$ and $N_2$, respectively, then

$$\left\langle \Phi\left(\sum_{i=1}^{n} a_i \otimes b_i\right), \varphi \otimes \psi \right\rangle = \left\langle \sum_{i=1}^{n} a_i \otimes b_i\, ,\, {}^t\pi_n{}^1(\varphi) \otimes {}^t\pi_n{}^2(\psi) \right\rangle.$$

Hence, putting $X = \sum_{i=1}^{n} a_i \otimes b_i$, we have

$$\begin{aligned} &\langle \Phi((X - \Phi(X))^*(X - \Phi(X))), \varphi \otimes \psi \rangle \\ &\quad = \langle (X - \Phi(X))^*(X - \Phi(X)), {}^t\pi_n{}^1(\varphi) \otimes {}^t\pi_n{}^2(\psi) \rangle \geqslant 0. \end{aligned}$$

Here the first member is easily seen to be

$$\begin{aligned} &\langle X^*X, {}^t\pi_n{}^1(\varphi) \otimes {}^t\pi_n{}^2(\psi) \rangle \\ &\quad - \langle \Phi(X)^*\Phi(X)(2(1 \otimes 1) - \pi_n{}^1(1) \otimes \pi_n{}^2(1)), \varphi \otimes \psi \rangle. \end{aligned}$$

Therefore

$$\begin{aligned}&\langle \Phi(X)^*\Phi(X),\ {}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi)\rangle \\ &\quad = \langle \Phi(X)^*\Phi(X),\ \varphi \otimes \psi\rangle \\ &\quad \leqslant \langle \Phi(X)^*\Phi(X)(2(1 \otimes 1) - \pi_n{}^1(1) \otimes \pi_n{}^2(1)),\ \varphi \otimes \psi\rangle \\ &\quad \leqslant \langle X^*X,\ {}^t\pi_n{}^1(\varphi) \otimes {}^t\pi_n{}^2(\psi)\rangle \\ &\quad \leqslant \langle X^*X,\ {}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi)\rangle.\end{aligned}$$

Thus, we have

$$\begin{aligned}\| \Phi(X)\|^2 &= \sup \frac{\langle Y^*\Phi(X)^*\Phi(X)Y,\ \varphi \otimes \psi\rangle}{\langle Y^*Y,\ \varphi \otimes \psi\rangle} \\ &= \sup \frac{\langle \Phi(XY)^*\Phi(XY),\ {}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi)}{\langle Y^*Y,\ {}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi)\rangle} \\ &\leqslant \sup \frac{\langle Y^*X^*XY,\ {}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi)\rangle}{\langle Y^*Y,\ {}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi)\rangle} \\ &\leqslant \| X \|^2,\end{aligned}$$

where sup is taken over all states on $N_1$ and $N_2$ and over all elements $Y_s{}'$ in $N_1 \odot N_2$ such as $\langle Y^*Y, \varphi \otimes \psi\rangle \neq 0$. Hence $\Phi$ is norm continuous and $\| \Phi \| \leqslant 1$. We denote the extension of $\Phi$ to $M_1 \otimes_\alpha M_2$ by the same $\Phi$. Let us consider the transpose of $\Phi$, say ${}^t\Phi$. We note first that

$${}^t\Phi(\varphi \otimes \psi) = {}^t\pi_n{}^1(\varphi) \otimes {}^t\pi_n{}^2(\psi).$$

Now, by Kaplansky's density theorem,

$$(N_1 \otimes N_2)_* = N_{1*} \otimes_{\alpha'} N_{2*} \qquad \text{and} \qquad (M_1 \otimes M_2)_* = M_{1*} \otimes_{\alpha'} M_{2*}$$

are isometric to their restrictions to $N_1 \otimes_\alpha N_2$ and $M_1 \otimes_\alpha M_2$. Hence if we define a mapping $\rho$ by

$$\begin{aligned}\rho : N_{1*} \otimes_{\alpha'} N_{2*} &\longrightarrow N_{1*} \otimes_{\alpha'} N_{2*} \mid N_1 \otimes_\alpha N_2 \\ &\xrightarrow{{}^t\Phi} M_{1*} \otimes_{\alpha'} M_{2*} \mid M_1 \otimes_{\alpha'} M_2 \longrightarrow M_{1*} \otimes_{\alpha'} M_2,\end{aligned}$$

the transpose of $\rho$ will give the $\sigma$-weakly continuous extension of $\Phi$ to $M_1 \otimes M_2$. This completes the proof.

There is an another proof of the above lemma. The existence of $\Phi$ on $M_1 \otimes_\alpha M_2$ depends mainly on the complete positivity of $\pi_n{}^1$ and

$\pi_n{}^2$. Therefore, once we show the complete positivity of $\pi_n{}^1$ and $\pi_n{}^2$ we can make use of the form of completely positive mappings by Stinespring [20], which implies easily the existence of $\Phi | M_1 \otimes_\alpha M_2$. However, we prefer here the above elementary proof.

We remark that $\alpha$-norm is not a uniform cross norm [7].

THEOREM 3.1. *With the same notations as above, if one of $\pi_1$ and $\pi_2$ is singular all product projections associated to them are singular. The converse is also true.*

*Proof.* Let $\pi$ be a product projection and let $\varphi$ and $\psi$ be normal states of $N_1$ and $N_2$. Since $\pi(\alpha \otimes b) = \pi_1(a) \otimes \pi_2(b)$, we have

$$^t\pi(\varphi \otimes \psi) \mid M_1 \otimes_\alpha M_2 = {}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi).$$

Suppose here $\pi_1$ is singular, then $^t\pi_1(\varphi)$ is a singular state. Furthermore, the above identity shows that $^t\pi(\varphi \otimes \psi)$ may be considered as a state extension of the singular state

$$^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi) \mid M_1 \otimes 1.$$

Hence $^t\pi(\varphi \otimes \psi)$ is singular (cf. [16; Lemma 5.1]). As $(N_1 \otimes N_2)_*$ is the uniform closure of linear combinations of $\varphi \otimes \psi$, one can see that

$$^t\pi((N_1 \otimes N_2)_*) \subset (M_1 \otimes M_2)_*{}^\perp.$$

Thus, $\pi$ is singular.

Conversely assume that all product projections are singular. Then, in particular, the projection $\pi$ constructed in [18, Theorem 4] is also singular. We use the notations in the proof of the above theorem. So, let $\{e_i \mid i \in I\}$ be the set of minimal projections corresponding to the basis in $K$, the underlying space of $M_2$ and $N_2$. Let $J$ be a finite subset of $I$. Put

$$e_J = \sum_{i \in J} e_i \qquad \text{and} \qquad \tilde{e}_J = 1 \otimes e_J.$$

Then, for a positive element $a \in M_1 \otimes B(K)$,

$$\pi_J{}^1(a) \geqslant \pi_n{}^1 \otimes 1_J(\tilde{e}_J a \tilde{e}_J) = \pi_n{}^1 \otimes 1(\tilde{e}_J a \tilde{e}_J),$$

where $\pi_n{}^1 \otimes 1_J$ and $\pi_n{}^1 \otimes 1$ mean the normal product mappings of $\pi_n{}^1$ with the identity mappings on $B(e_J K)$ and $B(K)$. We take the operator Banach limit (cf. [9]) of both members in the above inequality.

Since the operator Banach limit preserves the order, we get

$$\begin{aligned}\lim_J(\pi_J{}^1(a) - \pi_n{}^1 \otimes 1(\tilde{e}_J a \tilde{e}_J)) &= \lim_J \pi_J{}^1(a) - \{\sigma\text{-weak limit } \pi_n{}^1 \otimes 1(\tilde{e}_J a \tilde{e}_J)\} \\ &= \pi^1(a) - \pi_n{}^1 \otimes 1(a) \geqslant 0.\end{aligned}$$

Hence

$$\pi^1 \geqslant \pi_n{}^1 \otimes 1 \quad \text{on} \quad M_1 \otimes B(K),$$

and similarly

$$\pi^2 \geqslant 1 \otimes \pi_n{}^2 \quad \text{on} \quad B(H) \otimes M_2 .$$

Here $\pi$ is defined on $M_1 \otimes M_2$ as $\pi^1\pi^2$. On the other hand,

$$1 \otimes \pi_n{}^2(M_1 \odot M_2) \subset M_1 \odot N_2 .$$

Hence

$$1 \otimes \pi_n{}^2(M_1 \otimes M_2) \subset M_1 \otimes N_2 .$$

Thus, for a positive element $a$ in $M_1 \otimes M_2$, we have

$$\begin{aligned}\pi(a) = \pi^1\pi^2(a) &\geqslant (\pi_n{}^1 \otimes 1)\, \pi^2(a) \\ &\geqslant (\pi_n{}^1 \otimes 1)(1 \otimes \pi_n{}^2)(a) = \pi_n{}^1 \otimes \pi_n{}^2(a),\end{aligned}$$

i.e.,

$$\pi \geqslant \pi_n{}^1 \otimes \pi_n{}^2 \quad \text{on} \quad M_1 \otimes M_2 .$$

As $\pi$ is singular, $\pi_n{}^1 \otimes \pi_n{}^2 = 0$ and one of $\pi_n{}^1$ or $\pi_n{}^2$ must be zero [16]. This completes the proof.

**4.** Let $M$ and $N$ be von Neumann algebras and $M \otimes N$ their tensor product. For $\varphi \in M_*$ and $\psi \in N_*$ there associate $\sigma$-weakly continuous mappings $L_\psi$ and $R_\varphi$ from $M \otimes N$ to $M$ and $N$, taking their values on $M \odot N$ as

$$R_\varphi\left(\sum_{i=1}^n a_i \otimes b_i\right) = \sum_{i=1}^n \langle a_i , \varphi\rangle\, b_i ,$$

$$L_\psi\left(\sum_{i=1}^n a_i \otimes b_i\right) = \sum_{i=1}^n \langle b_i , \psi\rangle\, a_i .$$

We call them right and left Fubini mappings in $M \otimes N$ (cf. [18, 19]).

Let $A$ and $B$ be maximal abelian subalgebras of $M$ and $N$. It is known that $A \otimes B$ is a maximal abelian subalgebra of $M \otimes N$.

PROPOSITION 4.1. *$A \otimes B$ is smooth if and only if both $A$ and $B$ are smooth maximal abelian subalgebras.*

*Proof.* The "if" part is a consequence of Lemma 3.1, where $\pi_s^1 = \pi_s^2 = 0$ (or cf. [17]). Let $\pi$ be a normal projection to $A \otimes B$ and $\varphi$ be a normal state of $N$. Writing

$$\pi_1(x) = L_\varphi \pi(x \otimes 1) \qquad \text{for} \qquad x \in M.$$

We see that it is a normal projection to $A$. Similarly by using right Fubini mapping we get a normal projection from $N$ to $B$.

PROPOSITION 4.2. *$A \otimes B$ is completely nonsmooth if and only if one of $A$ or $B$ is completely nonsmooth.*

*Proof.* If $A \otimes B$ is completely nonsmooth, then all product projections to $A \otimes B$ associated to those projections to $A$ and $B$ are singular. Hence by Theorem 3.1 one of $A$ or $B$ must be completely nonsmooth. Conversely assume that $A$ is completely nonsmooth and let $\pi$ be a projection to $A \otimes B$. Let $\varphi$ be a normal state of $B$, then the mapping

$$x \in M \to L_\varphi \pi(x \otimes 1)$$

is a singular projection to $A$, and, for a positive element $a \in M$, we have

$$0 \leqslant L_\varphi \pi_n(a \otimes 1) \leqslant L_\varphi \pi(a \otimes 1).$$

Since $L_\varphi \pi_n(x \otimes 1)$ is a normal mapping, by the property of a singular mapping,

$$L_\varphi \pi_n(x \otimes 1) = 0$$

for all $x \in M$. Hence $L_\varphi \pi_n(1 \otimes 1) = 0$. As $\varphi$ is an arbitrary normal state, we have $\pi_n(1 \otimes 1) = 0$. Therefore for any $x \in M$

$$\begin{aligned}\pi_n(x)^*\pi_n(x) &\leqslant \pi_n(x)^*\pi_n(x)(2 - \pi_n(1 \otimes 1)) \\ &\leqslant \pi_n(x^*x) \leqslant \| x \|^2 \pi_n(1 \otimes 1) = 0\end{aligned}$$

and $\pi_n = 0$.

When $M$ is a finite von Neumann algebra it is known that all maximal abelian subalgebras are smooth in our sense. We shall show that the converse of this result is also true. Namely,

THEOREM 4.3.[1] *In a properly infinite von Neumann algebra $M$ there always exists a completely nonsmooth maximal abelian subalgebra. Hence, if all maximal abelian subalgebras in $M$ are smooth, $M$ is finite.*

*Proof.* By the structure of properly infinite algebras we may assume that

$$M = N \otimes B(H),$$

where $H$ is a separable Hilbert space. Let $A$ be an arbitrary maximal abelian subalgebra of $N$ and $B$ be a maximal abelian subalgebra of continuous type in $B(H)$. $B$ is completely nonsmooth, hence, by Proposition 4.2, $A \otimes B$ is a completely nonsmooth maximal abelian subalgebra in $M$.

In a semifinite von Neumann algebra we get the following characterization of smooth maximal abelian subalgebras which is just an application of Størmer's recent result [8, Theorem 2.b].

PROPOSITION 4.4. *Let $M$ be a semifinite von Neumann algebra, then a maximal abelian subalgebra is smooth if and only if it is generated by finite projections.*

Thus, in this case, we know that the existence of a normal projection to a maximal abelian subalgebra is just a contribution of finiteness. However, in an algebra of type III we do not know what things may contribute to such existence, or whether there always exists a smooth maximal ablian subalgebra. We can find, of course, an example of type-III algebra which contains a smooth maximal abelian subalgebra (for example, in a factor of type III constructed by the method of crossed product as we can see in Dixmier's book [1]).

There is an another observation about the existence of smooth maximal abelian subalgebras. Let $M$ be a countably decomposable von Neumann algebra and $\varphi$ a faithful normal state. Let $\sigma_t^\varphi$ be the modular automorphism group associated to $\varphi$ (cf. [13, 14]). Define

$$M_\varphi = \{a \in M \mid \langle ax, \varphi \rangle = \langle xa, \varphi \rangle \text{ for all } x \in M\}.$$

[1] This answers to the question by R. Kadison about the existence of normal projections to maximal abelian subalgebras raised at the International Congress of Mathematicians in Nice, 1970. Takesaki [14] also gives the answer.

Then it can be shown

$$M_\varphi = \{a \mid \sigma_t^\varphi(a) = a\}$$

(cf. [4, 13]). With these things we may proceed.

PROPOSITION 4.4. *A maximal abelian subalgebra $A$ in $M$ is smooth if and only if there exists a faithful normal state $\varphi$ of $M$ such that $M_\varphi \supset A$.*

*Proof.* Suppose we get a normal projection $\pi$ to $A$. Let $\psi$ be a faithful normal state on $A$. Since $\pi$ is faithful, $\varphi = {}^t\pi(\psi)$ is a faithful normal state. Take $a \in A$ and $x \in M$, then

$$\begin{aligned}\langle ax, \varphi\rangle &= \langle \pi(ax), \psi\rangle = \langle a\pi(x), \psi\rangle \\ &= \langle \pi(x)a, \psi\rangle = \langle \pi(xa), \psi\rangle \\ &= \langle xa, \varphi\rangle.\end{aligned}$$

Hence $a \in M_\varphi$ and $A \subset M_\varphi$.

Conversely if $A \subset M_\varphi$, there exists a normal projection $\pi_1$ from $M_\varphi$ to $A$ because $\varphi \mid M_\varphi$ induces a faithful trace and $M_\varphi$ is finite. Furthermore there also exists a normal projection $\pi_2$ from $M$ to $M_\varphi$ (cf. [14]). Combining these two mappings we get a normal projection from $M$ to $A$.

It had been conjectured whether $M_\varphi$ always contains a maximal abelian subalgebra or not and it has been settled by Herman–Takesaki [4] in the negative sense showing a factor $M$ and a normal state $\varphi$ with $M_\varphi = (\lambda 1)$. However we ask whether there exists always a faithful normal state $\varphi$ on $M$ such that $M_\varphi$ contains a maximal abelian subalgebra.

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