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Vertex Operator Algebras and Associative Algebras

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Let V be a vertex operator algebra. We construct a sequence of associative algebras $A_n(V)$ $(n = 0, 1, 2, \ldots)$ such that $A_n(V)$ is a quotient of $A_{n+1}(V)$ and a pair of functors between the category of $A_n(V)$-modules which are not $A_{n-1}(V)$-modules and the category of admissible V-modules. These functors exhibit a bijection between the simple modules in each category. We also show that V is rational if and only if all $A_n(V)$ are finite-dimensional semisimple algebras.

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1. INTRODUCTION

For a vertex operator algebra V Zhu constructed an associative algebra $A(V)$ [Z] such that there is a one-to-one correspondence between irreducible admissible V-modules and irreducible $A(V)$-modules. In the case that V is rational the admissible V-module category and $A(V)$-module category are in fact equivalent. But if V is not rational, $A(V)$ does not carry enough information for the representation of V.

In this paper we construct a sequence of associative algebras $A_n(V)$ $(n = 0, 1, 2, \dots)$ such that $A_0(V) = A(V)$ and $A_n(V)$ is an epimorphic image of $A_{n+1}(V)$. As in [Z], we use $A_n(V)$ to study representation theory of V. Let $M = \oplus_{k \geq 0} M(k)$ be an admissible V-module as defined in [DLM] with $M(0) \neq 0$. Then each $M(k)$ for $k \leq n$ is an $A_n(V)$-module. In some sense, $A_n(V)$ takes care of the first $n + 1$ homogeneous subspaces of M while $A(V)$ concerns the top level $M(0)$. The results of the present paper are modeled on the results in [DLM] and the methods are also similar. However, the situation for constructing admissible V-modules from $A_n(V)$-modules turns out to be very complicated. As in [L2, DLM] we extensively use the Lie algebra

$$\hat{V} = V \otimes \mathbb{C}[t, t^{-1}] / \left(L(-1) \otimes 1 + 1 \otimes \frac{d}{dt} \right) \left(V \otimes \mathbb{C}[t, t^{-1}] \right)$$

to construct admissible V-modules from $A_n(V)$-modules.

It should be pointed out that the $\{A_n(V)\}$ in fact form an inverse system. So it is natural to consider the inverse limit $\lim\limits_{\leftarrow} A_n(V)$ and its representations. This problem will be addressed in a separate paper.

One of the important motivations for constructing $A_n(V)$ is to study induced modules from a subalgebra to V as initiated in [DL]. Induced module theory is very important in the representation theory of classical objects such as groups, rings, Lie algebras. The theory of $A_n(V)$ developed in this paper will definitely play a role in the study of induced modules for vertex operator algebras. In order to see this, we consider a subalgebra U of V and a U-submodule W of M which is an admissible V-module. In general, the top level of W is not necessarily a subspace of the top level of M. In other words, an $A(U)$-module can be a subspace of an $A_n(V)$-module for some $n > 0$. One can now see how the $A_n(V)$ enter the picture of studying the induced module for the pair (U, V) along this line.

This paper is organized as follows: In Section 2 we introduce the algebra $A_n(V)$ which is a quotient of V modulo a subspace $O_n(V)$ consisting of $u \circ_n v$ (see Section 2 for the definition) and $L(-1)u + L(0)u$ for $u, v \in V$. In the case $n = 0, (L(-1) + L(0))u$ can be expressed as $\omega \circ_0 u$. But in general it is not clear if one can write $(L(-1) + L(0))u$ as a linear combination of $v \circ_n w$'s. On the other hand, the weight zero component of the vertex operator $Y((L(-1)+L(0))u,z)$ is zero on any weak V-module. So we have to put $(L(-1)+L(0))V$ artificially in $O_n(V)$ for general n. We also show in this section how the identity map on V induces an epimorphism of algebras from $A_{n+1}(V)$ to $A_n(V)$. In Section 3, we construct a functor Ω_n from the category of weak V-modules to the category of $A_n(V)$-modules such that if $M = \oplus_{k\geq 0} M(k)$ is an admissible V-module then $\oplus_{k=0}^n M(k)$ with $M(0) \neq 0$ is contained in $\Omega_n(M)$ and each $M(k)$ for $k \leq n$ is an $A_n(V)$-submodule. In particular, if M is irreducible then $\oplus_{k=0}^n M(k) = \Omega_n(M)$ and each $M(k)$ is an irreducible $A_n(V)$-module.

Section 4 is the core of this paper. In this section we construct a functor L_n from the category of $A_n(V)$-modules which cannot factor through $A_{n-1}(V)$ to the category of admissible V-modules. For any such $A_n(V)$-module U we first construct a universal admissible V-module $\overline{M}_n(U)$ which is somehow a "generalized Verma module." The $L_n(V)$ is then a suitable quotient of $\overline{M}_n(U)$; the proof of this result is technically the most difficult part of this paper. We also show that $\Omega_n(L_n(U))/\Omega_{n-1}(L_n(U))$ is isomorphic to U as $A_n(V)$-modules. Moreover, V is rational if and only if the $A_n(V)$ are finite-dimensional semisimple algebras for all n. Section 5 deals with several combinatorial identities used in previous sections.

We assume that the reader is familiar with the basic knowledge on vertex operator algebras as presented in [B, FHL, FLM]. We also refer the reader to [DLM] for the definitions of weak modules, admissible modules, and (ordinary) modules.

2. THE ASSOCIATIVE ALGEBRA $A_n(V)$

Let $V = (V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra. We will construct an associative algebra $A_n(V)$ for any nonnegative integer n generalizing the Zhu's algebra $A(V)$ which is our $A_0(V)$.

Let $O_n(V)$ be the linear span of all $u \circ_n v$ and $L(-1)u + L(0)u$ where for homogeneous $u \in V$ and $v \in V$,

$$u \circ_n v = \mathrm{Res}_z Y(u,z) v \frac{(1+z)^{\mathrm{wt}\,u+n}}{z^{2n+2}}. \tag{2.1}$$

Define the linear space $A_n(V)$ to be the quotient $V/O_n(V)$.

We also define a second product $*_n$ on V for u and v as above:

$$u *_n v = \sum_{m=0}^{n} (-1)^m \binom{m+n}{n} \mathrm{Res}_z Y(u,z) \frac{(1+z)^{\mathrm{wt}\,u+n}}{z^{n+m+1}} v. \tag{2.2}$$

Extend linearly to obtain a bilinear product on V which coincides with that of Zhu [Z] if $n = 0$. We denote the product (2.2) by $*$ in this case. Note that (2.2) may be written in the form

$$u *_n v = \sum_{m=0}^{n} \sum_{i=0}^{\infty} (-1)^m \binom{m+n}{n} \binom{\mathrm{wt}\, u + n}{i} u_{i-m-n-1} v. \tag{2.3}$$

The following lemma generalizes Lemmas 2.1.2 and 2.1.3 of [Z].

LEMMA 2.1. (i) *Assume that* $u \in V$ *is homogeneous,* $v \in V$*, and* $m \geq k \geq 0$. *Then*

$$\mathrm{Res}_z Y(u, z) v \frac{(1+z)^{\mathrm{wt}\, u + n + k}}{z^{2n+2+m}} \in O_n(V).$$

(ii) *Assume that* v *is also homogeneous. Then*

$$u *_n v - \sum_{m=0}^{n} \binom{m+n}{n} (-1)^n \mathrm{Res}_z Y(v, z) u \frac{(1+z)^{\mathrm{wt}\, v + m - 1}}{z^{1+m+n}} \in O_n(V)$$

and

(iii) $u *_n v - v *_n u - \mathrm{Res}_z Y(u,z) v (1+z)^{\mathrm{wt}\, u - 1} \in O_n(V)$.

Proof. The proof of (i) is similar to that of Lemma 2.1.2 of [Z]. As in [Z] we use $L(-1)u + L(0)u \in O_n(V)$ to derive the formula

$$Y(u, z) v \equiv (1+z)^{-\mathrm{wt}\, u - \mathrm{wt}\, v} Y\left(v, \frac{-z}{1+z}\right) u \qquad \mathrm{mod}\ O_n(V).$$

Thus we have

$$\begin{aligned} u *_n v &= \sum_{m=0}^{n} (-1)^m \binom{n+m}{n} \mathrm{Res}_z Y(u, z) v \frac{(1+z)^{\mathrm{wt}\, u + n}}{z^{m+n+1}} \\ &\equiv \sum_{m=0}^{n} (-1)^m \binom{n+m}{n} \mathrm{Res}_z Y\left(v, \frac{-z}{1+z}\right) u \frac{(1+z)^{-\mathrm{wt}\, v + n}}{z^{m+n+1}} \\ &\qquad \mathrm{mod}\ O_n(V) \\ &= \sum_{m=0}^{n} (-1)^n \binom{n+m}{n} \mathrm{Res}_z Y(v, z) u \frac{(1+z)^{\mathrm{wt}\, v + m - 1}}{z^{m+n+1}} \end{aligned}$$

and (ii) is proved.

Using (ii) we have

$$\begin{aligned} u *_n v - v *_n u &\equiv \mathrm{Res}_z Y(u, z) v (1+z)^{\mathrm{wt}\, u - 1} \\ &\quad \times \left(\sum_{m=0}^{n} \binom{m+n}{n} \frac{(-1)^m (1+z)^{n+1} - (-1)^n (1+z)^m}{z^{n+m+1}} \right). \end{aligned}$$

By Proposition 5.2 in the Appendix we know that

$$\sum_{m=0}^{n}\binom{m+n}{n}\frac{(-1)^m(1+z)^{n+1}-(-1)^n(1+z)^m}{z^{n+m+1}}=1.$$

The proof is complete. ∎

LEMMA 2.2. *$O_n(V)$ is a 2 sided ideal of V under $*_n$.*

Proof. First we show that $(L(-1)u+L(0)u)*_n v\in O_n(V)$ for any homogeneous $u\in V$. From the definition we see that

$$\begin{aligned}(L(-1)u)*_n v &= \sum_{m=0}^{n}\binom{m+n}{n}(-1)^m\operatorname{Res}_z Y(L(-1)u,z)\frac{(1+z)^{\mathrm{wt}\,u+n+1}}{z^{n+m+1}}v\\
&= \sum_{m=0}^{n}\binom{m+n}{n}(-1)^m\operatorname{Res}_z\left(\frac{d}{dz}Y(u,z)\right)v\frac{(1+z)^{\mathrm{wt}\,u+n+1}}{z^{n+m+1}}\\
&= \sum_{m=0}^{n}\binom{m+n}{n}(-1)^{m+1}\operatorname{Res}_z Y(u,z)v\\
&\quad\times\left(\frac{(-n-m-1)(1+z)^{\mathrm{wt}\,u+n+1}}{z^{n+m+2}}+\frac{z(\mathrm{wt}\,u+n+1)(1+z)^{\mathrm{wt}\,u+n}}{z^{n+m+2}}\right).\end{aligned}$$

Thus

$$\begin{aligned}(L(-1)u+\mathrm{wt}\,uu)*_n v &= \sum_{m=0}^{n}\binom{m+n}{n}(-1)^m\operatorname{Res}_z Y(u,z)v(1+z)^{\mathrm{wt}\,u+n}\frac{mz+n+m+1}{z^{n+m+2}}.\end{aligned}$$

It is straightforward to show that

$$\begin{aligned}&\sum_{m=0}^{n}\binom{m+n}{n}(-1)^m\frac{mz+n+m+1}{z^{n+m+2}}\\
&\quad=\sum_{m=0}^{n}\binom{m+n}{n}(-1)^m\frac{mz}{z^{n+m+2}}+\sum_{m=0}^{n}\binom{m+n+1}{n}(-1)^m\frac{m+1}{z^{n+m+2}}\\
&\quad=(-1)^n\binom{2n+1}{n}\frac{2n+1}{z^{2n+2}}.\end{aligned}$$

It is clear now that $(L(-1)u+L(0)u)*_n v\in O_n(V)$.

Second, we show that $u *_n (L(-1)v + L(0)v) \in O_n(V)$. Using the result that $(L(-1)v + L(0)v) *_n v \in O_n(V)$ and Lemma 2.1(iii) we have

$$
\begin{aligned}
&u *_n (L(-1)v + L(0)v) \\
&\quad \equiv -\mathrm{Res}_z\big(Y(L(-1)v, z)u(1+z)^{\mathrm{wt}\, v} + Y(L(0)v, z)u(1+z)^{\mathrm{wt}\, v - 1}\big) \\
&\qquad\qquad \mod O_n(V) \\
&\quad = \mathrm{Res}_z\left(Y(v,z)u\frac{d}{dz}(1+z)^{\mathrm{wt}\, v} - Y(L(0), v, z)u(1+z)^{\mathrm{wt}\, v-1}\right) \\
&\quad = 0.
\end{aligned}
$$

Third, a similar argument as in [Z] using Lemma 2.1(i) shows that $u *_n (v \circ_n w) \in O_n(V)$ for $u, v, w \in V$.

Finally, use $u *_n (v \circ_n w) \in O_n(V)$ and Lemma 2.1(iii) to obtain

$$
\begin{aligned}
&(v \circ_n w) *_n u \\
&\quad \equiv -\mathrm{Res}_{z_1}\,\mathrm{Res}_{z_2} Y(u, z_1)Y(v, z_2)w \frac{(1+z_1)^{\mathrm{wt}\, u-1}(1+z_2)^{\mathrm{wt}\, v+n}}{z_2^{2n+2}} \\
&\qquad\qquad \mod O_n(V) \\
&\equiv -\mathrm{Res}_{z_2}\,\mathrm{Res}_{z_1-z_2} Y(Y(u, z_1 - z_2)v, z_2)w \frac{(1+z_1)^{\mathrm{wt}\, u-1}(1+z_2)^{\mathrm{wt}\, v+n}}{z_2^{2n+2}} \\
&= -\sum_{i\geq 0}\binom{\mathrm{wt}\, u - 1}{i}\mathrm{Res}_{z_2} Y(u_i v, z_2)w\frac{(1+z_2)^{\mathrm{wt}\, u + \mathrm{wt}\, v + n - 1 - i}}{z_2^{2n+2}}
\end{aligned}
$$

which belongs to $O_n(V)$ as $\mathrm{wt}\, u_i v = \mathrm{wt}\, u + \mathrm{wt}\, v - i - 1$. This completes the proof. ∎

Our first main result is the following.

THEOREM 2.3. (i) *The product* $*_n$ *induces the structure of an associative algebra on* $A_n(V)$ *with identity* $1 + O_n(V)$.

(ii) *The linear map*

$$\phi : v \mapsto e^{L(1)}(-1)^{L(0)}v$$

induces an anti-isomorphism $A_n(V) \to A_n(V)$.

(iii) $\omega + O_n(V)$ *is a central element of* $A_n(V)$.

Proof. For (i) we only need to prove that $A_n(V)$ is associative. Let $u, v, w \in V$ be homogeneous. Then

$$
\begin{aligned}
&(u *_n v) *_n w \\
&= \sum_{m_1=0}^{n} \sum_{i\geq 0} (-1)^{m_1} \binom{m_1+n}{n} \binom{\mathrm{wt}\, u+n}{i} (u_{-m_1-n-1+i}v) *_n w \\
&= \sum_{m_1, m_2=0}^{n} \sum_{i\geq 0} (-1)^{m_1+m_2} \binom{m_1+n}{n} \binom{m_2+n}{n} \binom{\mathrm{wt}\, u+n}{i} \\
&\quad \times \mathrm{Res}_{z_2} Y(u_{-m_1-n-1+i}v, z_2) w \frac{(1+z_2)^{\mathrm{wt}\, u+\mathrm{wt}\, v+2n+m1-i}}{z_2^{1+m_2+n}} \\
&= \sum_{m_1, m_2=0}^{n} (-1)^{m_1+m_2} \binom{m_1+n}{n} \binom{m_2+n}{n} \mathrm{Res}_{z_2} \mathrm{Res}_{z_1-z_2} \\
&\quad \times Y(Y(u, z_1-z_2)v, z_2) w \frac{(1+z_1)^{\mathrm{wt}\, u+n} (1+z_2)^{\mathrm{wt}\, v+n+m_1}}{(z_1-z_2)^{m_1+n+1} z_2^{1+m_2+n}} \\
&= \sum_{m_1, m_2=0}^{n} (-1)^{m_1+m_2} \binom{m_1+n}{n} \binom{m_2+n}{n} \\
&\quad \times \mathrm{Res}_{z_1} \mathrm{Res}_{z_2} Y(u, z_1) Y(v, z_2) w \frac{(1+z_1)^{\mathrm{wt}\, u+n} (1+z_2)^{\mathrm{wt}\, v+n+m_1}}{(z_1-z_2)^{m_1+n+1} z_2^{1+m_2+n}} \\
&\quad - \sum_{m_1, m_2=0}^{n} (-1)^{m_1+m_2} \binom{m_1+n}{n} \binom{m_2+n}{n} \\
&\quad \times \mathrm{Res}_{z_2} \mathrm{Res}_{z_1} Y(v, z_2) Y(u, z_1) w \frac{(1+z_1)^{\mathrm{wt}\, u+n} (1+z_2)^{\mathrm{wt}\, v+n+m_1}}{(z_1-z_2)^{m_1+n+1} z_2^{1+m_2+n}} \\
&= \sum_{m_1, m_2=0}^{n} \sum_{i\geq 0} (-1)^{m_1+m_2} \binom{m_1+n}{n} \binom{m_2+n}{n} \binom{-m_1-n-1}{i} (-1)^i \\
&\quad \times \mathrm{Res}_{z_1} \mathrm{Res}_{z_2} Y(u, z_1) Y(v, z_2) w \frac{(1+z_1)^{\mathrm{wt}\, u+n} (1+z_2)^{\mathrm{wt}\, v+n+m_1}}{z_1^{m_1+n+1+i} z_2^{1+m_2+n-i}} \\
&\quad - \sum_{m_1, m_2=0}^{n} \sum_{i\geq 0} (-1)^{m_2+n+1+i} \binom{m_1+n}{n} \binom{m_2+n}{n} \binom{-m_1-n-1}{i} \\
&\quad \times \mathrm{Res}_{z_2} \mathrm{Res}_{z_1} Y(v, z_2) Y(u, z_1) w \frac{(1+z_1)^{\mathrm{wt}\, u+n} (1+z_2)^{\mathrm{wt}\, v+n+m_1}}{z_1^{m_1+n+1-i} z_2^{2+m_1+m_2+2n+i}}.
\end{aligned}
$$

From Lemma 2.1 we know that

$$\operatorname{Res}_{z_2}\operatorname{Res}_{z_1}Y(v,z_2)Y(u,z_1)w\frac{(1+z_1)^{\mathrm{wt}\,u+n}(1+z_2)^{\mathrm{wt}\,v+n+m_1}}{z_1^{m_1+n+1-i}z_2^{2+m_1+m_2+2n+i}}$$

lies in $O_n(V)$. Also if $i > n - m_1$

$$\operatorname{Res}_{z_1}\operatorname{Res}_{z_2}Y(u,z_1)Y(v,z_2)w\frac{(1+z_1)^{\mathrm{wt}\,u+n}(1+z_2)^{\mathrm{wt}\,v+n+m_1}}{z_1^{m_1+n+1+i}z_2^{1+m_2+n-i}}$$

is in $O_n(V)$. Thus

$$(u*_n v)*_n w \equiv u*_n(v*_n w) + \sum_{m_1,m_2=0}^{n}(-1)^{m_1+m_2}\binom{m_1+n}{n}\binom{m_2+n}{n}$$
$$\times\operatorname{Res}_{z_1}\operatorname{Res}_{z_2}Y(u,z_1)Y(v,z_2)\frac{(1+z_1)^{\mathrm{wt}\,u+n}(1+z_2)^{\mathrm{wt}\,v+n}}{z_1^{m_1+n+1}z_2^{1+m_2+n}}$$
$$\times\left(\sum_{i=0}^{n-m_1}\sum_{j\geq 0}\binom{-m_1-n-1}{i}\binom{m_1}{j}(-1)^i\frac{z_2^{i+j}}{z_1^i}-1\right).$$

From Proposition 5.3 in the Appendix we know that

$$\sum_{m_1=0}^{n}(-1)^{m_1}\binom{m_1+n}{n}$$
$$\left(\sum_{i=0}^{n-m_1}\sum_{j\geq 0}\binom{-m_1-n-1}{i}\binom{m_1}{j}(-1)^i\frac{z_2^{i+j}}{z_1^{i+m_1}}-\frac{1}{z_1^{m_1}}\right)=0.$$

This implies that the product $*_n$ of $A_n(V)$ is associative.

The proof of (ii) is similar to that of (ii) of Theorem 2.4 of [DLM]. We refer the reader to [DLM] for detail.

Note that $1*_n u = u$ for any $u \in V$ and that

$$u*_n 1 - 1*_n u \equiv \operatorname{Res}_z Y(u,z)1(1+z)^{\mathrm{wt}\,u-1} = 0.$$

This shows that $1 + O_n(V)$ is the identity of $A_n(V)$. Again by Lemma 2.1(iii),

$$\omega *_n u - u *_n \omega = \operatorname{Res}_z Y(\omega,z)u(1+z) = L(-1)u + L(0)u \in O_n(V).$$

So (iii) is proved. ∎

PROPOSITION 2.4. *The identity map on V induces an onto algebra homomorphism from $A_n(V)$ to $A_{n-1}(V)$.*

Proof. First by Lemma 2.1(i), $O_n(V) \subset O_{n-1}(V)$. It remains to show that $u *_n v \equiv u *_{n-1} v \mod O_{n-1}(V)$. Let u be homogeneous. Then

$$\begin{aligned} u *_n v &= \sum_{m=0}^{n} \binom{m+n}{n} (-1)^m \operatorname{Res}_z Y(u,z) v \frac{(1+z)^{\mathrm{wt}\, u+n-1}}{z^{n+m}} \\ &\quad + \sum_{m=0}^{n} \binom{m+n}{n} (-1)^m \operatorname{Res}_z Y(u,z) v \frac{(1+z)^{\mathrm{wt}\, u+n-1}}{z^{n+m+1}} \\ &\equiv \sum_{m=0}^{n-1} \binom{m+n}{n} (-1)^m \operatorname{Res}_z Y(u,z) v \frac{(1+z)^{\mathrm{wt}\, u+n-1}}{z^{n+m}} \\ &\quad + \sum_{m=0}^{n-2} \binom{m+n}{n} (-1)^m \operatorname{Res}_z Y(u,z) \frac{(1+z)^{\mathrm{wt}\, u+n-1}}{z^{n+m+1}} \\ &\qquad \mod O_{n-1}(V) \\ &= \operatorname{Res}_z Y(u,z) v \frac{(1+z)^{\mathrm{wt}\, u+n-1}}{z^{n}} + \sum_{m=1}^{n-1} \operatorname{Res}_z Y(u,z) v \frac{(1+z)^{\mathrm{wt}\, u+n-1}}{z^{n+m}} \\ &\quad \cdot \left((-1)^m \binom{m+n}{n} + (-1)^{m+1} \binom{m+n-1}{n} \right) \\ &= u *_{n-1} v, \end{aligned}$$

as desired. ∎

From Proposition 2.4 we in fact have an inverse system $\{A_n(V)\}$. Denote by $I(V)$ the inverse limit $\varprojlim A_n(V)$. Then

$$I(V) = \left\{ a = (a_n + O_n(V)) \in \prod_{n=0}^{\infty} A_n(V) | a_n - a_{n-1} \in O_{n-1}(v) \right\}. \tag{2.4}$$

Define $i: V \to I(V)$ such that $i(v) = (v + O_n(V))$ for $v \in V$. Then $V/\ker i$ is linearly isomorphic to a subspace of $I(V)$. It is easy to see that $i(V)$ is not closed under the product. But one can introduce a suitable topology on $I(V)$ so that $i(V)$ is a dense subspace of $I(V)$ under the topology. An interesting problem is to determine the kernel of i. From the definition of $O_n(V)$ we see immediately that $(L(-1) + L(0))V$ is contained in the kernel. It will be proved in Section 3 that if $v \in O_n(V)$ then $a_{\mathrm{wt}\, a-1} = 0$ on

$\oplus_{k=0}^{n} M(n)$ for any admissible V-module $\oplus_{k=0}^{\infty} M(k)$. Thus $a \in \ker i$ if and only if $a_{\mathrm{wt}\, a-1} = 0$ on any admissible V-module. It is proved in [DLMM] that if V is a simple vertex operator algebra satisfying $V_k = 0$ for all $k < 0$, and $V_0 = \mathbb{C}1$ then the subspace of V consisting of vectors v whose component operators $v_{\mathrm{wt}\, v-1}$ are 0 on V is essentially $(L(0) + L(-1))V$. We suspect that if V is a rational vertex operator algebra then the kernel of i is exactly $(L(0) + L(-1))V$.

3. THE FUNCTOR Ω_n

Consider the quotient space

$$\hat{V} = \mathbb{C}[t, t^{-}] \otimes V / D\mathbb{C}[t, t^{-}] \otimes V, \tag{3.1}$$

where $D = \frac{d}{dt} \otimes 1 + 1 \otimes L(-1)$. Denote by $v(m)$ the image of $v \otimes t^m$ in $\hat{V}$ for $v \in V$ and $m \in \mathbb{Z}$. Then $\hat{V}$ is $\mathbb{Z}$-graded by defining the degree of $v(m)$ to be $\mathrm{wt}\, v - m - 1$ if v is homogeneous. Denote the homogeneous subspace of degree m by $\hat{V}(m)$. The space $\hat{V}$ is, in fact, a $\mathbb{Z}$-graded Lie algebra with bracket

$$[a(p), b(q)] = \sum_{i=0}^{\infty} \binom{p}{i} (a_i b)(p + q - i) \tag{3.2}$$

(see [L2, DLM]). In particular, $\hat{V}(0)$ is a Lie subalgebra. By Lemma 2.1(iii) we have

PROPOSITION 3.1. *Regarded $A_n(V)$ as a Lie algebra, the map $v(\mathrm{wt}\, v - 1) \mapsto v + O_n(V)$ is a well-defined onto Lie algebra homomorphism from $\hat{V}(0)$ to $A_n(V)$.*

By Lemmas 5.1 and 5.2 of [DLM], any weak V-module M is a module for $\hat{V}$ under the map $a(m) \mapsto a_m$ and a weak V-module which carries a $\mathbb{Z}_+$-grading is an admissible V-module if, and only if, M is a $\mathbb{Z}_+$-graded module for the graded Lie algebra $\hat{V}$.

For a module W for the Lie algebra $\hat{V}$ and a nonnegative m we let $\Omega_m(W)$ denote the space of "mth lowest weight vectors," that is,

$$\Omega_m(W) = \left\{ u \in W \mid \hat{V}(-k)u = 0 \text{ if } k > m \right\}. \tag{3.3}$$

Then $\Omega_m(W)$ is a module for the Lie algebra $\hat{V}(0)$.

THEOREM 3.2. *Suppose that M is a weak V-module. Then there is a representation of the associative algebra $A_n(V)$ on $\Omega_n(M)$ induced by the map $a \mapsto o(a) = a_{\mathrm{wt}\, a-1}$ for homogeneous $a \in V$.*

Proof. We need to show that $o(a) = 0$ for all $a \in O_n(V)$ and $o(u *_n v) = o(u)o(v)$ for $u, v \in V$. Using $Y(L(-1)u, z) = \frac{d}{dz}Y(u, z)$ we immediately see that $o(L(-1)u + L(0)u) = 0$. From the proof of Lemma 2.1 we know that $(L(-1)u + L(0)u) *_n v = (-1)^n\binom{2n}{n}(2n+1)u \circ_n v$. It suffices to show that $o(u *_n v) = o(u)o(v)$.

Let u, v be homogeneous and $0 \le k \le n$. Note that $v_{\mathrm{wt}\, v + p} = u_{\mathrm{wt}\, u + p} = 0$ on $\Omega_n(M)$ if $p \ge n$. We assert that the following identity holds on $\Omega_n(M)$,

$$\sum_{m=0}^{k} (-1)^m \binom{2n+m-k}{m} o\left(\mathrm{Res}_z Y(u, z) v \frac{(1+z)^{\mathrm{wt}\, u + n}}{z^{2n+1-k+m}}\right) = u_{\mathrm{wt}\, u - n + k - 1} v_{\mathrm{wt}\, v + n - k - 1} \tag{3.4}$$

which reduces to $o(u *_n v) = o(u)o(v)$ if $k = n$. The proof of (3.4) is a straightforward computation involving the Jacobi identity on modules in terms of residues.

On $\Omega_n(M)$ we have

$$\begin{aligned}
&\sum_{m=0}^{k} (-1)^m \binom{2n+m-k}{m} o\left(\mathrm{Res}_z Y(u, z) v \frac{(1+z)^{\mathrm{wt}\, u + n}}{z^{2n+1-k+m}}\right) \\
&= \sum_{m=0}^{k} \sum_{i \ge 0} (-1)^m \binom{2n+m-k}{m} \binom{\mathrm{wt}\, u + n}{i} o(u_{i-2n-1-m+k} v) \\
&= \sum_{m=0}^{k} \sum_{i \ge 0} (-1)^m \binom{2n+m-k}{m} \binom{\mathrm{wt}\, u + n}{i} \\
&\quad \times (u_{i-2n-1-m+k} v)_{\mathrm{wt}\, u + \mathrm{wt}\, v - i + 2n + m - 1 - k} \\
&= \sum_{m=0}^{n} \sum_{i \ge 0} (-1)^m \binom{2n+m-k}{m} \binom{\mathrm{wt}\, u + n}{i} \\
&\quad \times \mathrm{Res}_{z_2} \mathrm{Res}_{z_1 - z_2} Y(Y(u, z_1 - z_2) v, z_2) \\
&\quad \times (z_1 - z_2)^{i-2n-m-1+k} z_2^{\mathrm{wt}\, u + \mathrm{wt}\, v - i + 2n + m - 1 - k} \\
&= \sum_{m=0}^{k} (-1)^m \binom{2n+m-k}{m} \mathrm{Res}_{z_2} \mathrm{Res}_{z_1 - z_2} \\
&\quad \times Y(Y(u, z_1 - z_2) v, z_2) \frac{z_1^{\mathrm{wt}\, u + n} z_2^{\mathrm{wt}\, v + n + m - 1 - k}}{(z_1 - z_2)^{2n+m+1-k}} \\
&= \sum_{m=0}^{k} (-1)^m \binom{2n+m-k}{m} \mathrm{Res}_{z_1} \mathrm{Res}_{z_2}
\end{aligned}$$

$$\times Y(u,z_1)Y(v,z_2)\frac{z_1^{\mathrm{wt}\,u+n}z_2^{\mathrm{wt}\,v+n+m_{-1-k}}}{(z_1-z_2)^{2n+m+1-k}}$$

$$-\sum_{m=0}^{k}(-1)^m\binom{2n+m-k}{m}\mathrm{Res}_{z_2}\,\mathrm{Res}_{z_1}$$

$$\times Y(v,z_2)Y(u,z_1)\frac{z_1^{\mathrm{wt}\,u+n}z_2^{\mathrm{wt}\,v+n+m-1-k}}{(z_1-z_2)^{2n+m+1-k}}$$

$$=\sum_{m=0}^{k}\sum_{i=0}^{k-m}(-1)^{m+i}\binom{2n+m-k}{m}\binom{-m-2n-1+k}{i}$$

$$\times u_{\mathrm{wt}\,u-n-m-1+k-i}v_{\mathrm{wt}\,v+n+m-1-k+i}$$

$$=\sum_{m=0}^{k}\sum_{i=m}^{k}\binom{2n+m-k}{m}\binom{-m-2n-1+k}{i-m}(-1)^i$$

$$\times u_{\mathrm{wt}\,u-n+k-i-1}v_{\mathrm{wt}\,v+n-k-1+i}$$

$$=\sum_{i=0}^{k}\sum_{m=0}^{i}\binom{2n+m-k}{m}\binom{-m-2n-1+k}{i-m}(-1)^i$$

$$\times u_{\mathrm{wt}\,u-n+k-i-1}v_{\mathrm{wt}\,v+n-k-1+i}$$

$$=u_{\mathrm{wt}\,u-n+k-1}v_{\mathrm{wt}\,v+n-k-1}$$

$$+\sum_{i=1}^{k}\sum_{m=0}^{i}\binom{2n+m-k}{m}\binom{-m-2n-1+k}{i-m}(-1)^i$$

$$\times u_{\mathrm{wt}\,u-n+k-i-1}v_{\mathrm{wt}\,v+n-k-1+i}.$$

It is enough to show that for $i=1,\ldots,k$.

$$\sum_{m=0}^{i}\binom{2n+m-k}{m}\binom{-m-2n-1+k}{i-m}=0,$$

which follows from an easy calculation:

$$\sum_{m=0}^{i}\binom{2n+m-k}{m}\binom{-m-2n-1+k}{i-m}$$

$$=\sum_{m=0}^{i}(-1)^{i-m}\binom{2n+m-k}{m}\binom{2n+i-k}{i-m}$$

$$=\sum_{m=0}^{i}(-1)^{i-m}\binom{2n+i-k}{2n-k}\binom{i}{m}$$

$$=0.$$

This completes the proof. ∎

Remark 3.3. For homogeneous $u, v \in V$ and $j \in \mathbb{Z}$ we set $o_j(u) = u_{\mathrm{wt}\, u - 1 - j}$ and extend to all $u \in V$ by linearity. Then $o_0(u) = o(u)$. Using associativity of the vertex operators

$$(z_0 + z_2)^{\mathrm{wt}\, u + n} Y(u, z_0 + z_2) Y(v, z_2) = (z_2 + z_0)^{\mathrm{wt}\, u + n} Y\big(Y(u, z_0) v, z_2\big)$$

on $\Omega_n(M)$ we have that for $i \geq j$ with $i + j \geq 0$ these exists a unique $w_{u,v}^{i,j} \in V$ such that $o_i(u) o_j(v) = o_{i+j}(w_{u,v}^{i,j})$ on $\Omega_n(M)$. In fact one can write $w_{u,v}^{i,j}$ explicitly in terms of u and v. But for our later purpose it is enough to know the explicit expression of $w_{u,v}^{i,-i}$ $(i \geq 0)$ which is given by

$$w_{u,v}^{i,-i} = \sum_{m=0}^{n-i} (-1)^m \binom{n+m+i}{m} \mathrm{Res}_z Y(u, z) v \frac{(1+z)^{\mathrm{wt}\, u + n}}{z^{n+1+i+m}}$$

in the proof of Theorem 3.2.

It is clear that Ω_n is a covariant functor from the category of weak V-modules to the category of $A_n(V)$-modules. To be more precise, if f: $M \to N$ is a morphism in the first category we define $\Omega_n(f)$ to be the restriction of f to $\Omega_n(M)$. Then f induces a morphism of $\hat{V}$-modules $M \to N$ by Lemma 5.1 of [DLM]. Moreover $\Omega_n(f)$ maps $\Omega_n(M)$ to $\Omega_n(N)$. Now Theorem 3.2 implies that $\Omega_n(f)$ is a morphism of $A_n(V)$-modules.

Let M be such a module. As long as $M \neq 0$, then some $M(m) \neq 0$, and it is no loss to shift the grading so that in fact $M(0) \neq 0$. If $M = 0$, let $M(0) = 0$. With these conventions we prove

PROPOSITION 3.4. *Suppose that M is an admissible V-module. Then the following hold*

(i) $\Omega_n(M) \supset \oplus_{i=0}^{n} M(i)$. *If M is simple then* $\Omega_n(M) = \oplus_{i=0}^{n} M(i)$.

(ii) *Each $M(p)$ is an $\hat{V}(0)$-module and $M(p)$ and $M(q)$ are inequivalent if $p \neq q$ and both $M(p)$ and $M(q)$ are nonzero. If M is simple then each $M(p)$ is an irreducible $\hat{V}(0)$-module.*

(iii) *Assume that M is simple. Then each $M(i)$ for $i = 0, \ldots, n$ is a simple $A_n(V)$-module and $M(i)$ and $M(j)$ are inequivalent $A_n(V)$-modules.*

Proof. An easy argument shows that $\Omega_n(M)$ is a graded subspace of M. That is,

$$\Omega_n(M) = \bigoplus_{i \geq 0} \Omega_n(M) \cap M(i). \tag{3.5}$$

Set $\Omega_n(i) = \Omega_n(M) \cap M(i)$. It is clear that $M(i) \subset \Omega_n(M)$ if $i \leq n$. In order to prove (i) we must show that $\Omega_n(i) = 0$ if $i > n$.

By Proposition 2.4 of [DM] or Lemma 6.1.1 of [L2], $M = \mathrm{span}\{u_n w | u \in V,\ n \in \mathbb{Z}\}$ where w is any nonzero vector in M. If $\Omega_n(i) \neq 0$ for some $i > n$ we can take $0 \neq w \in \Omega_n(i)$. Since $u_{\mathrm{wt}\,u+p}w = 0$ for all $p \geq n$ we see that $M = \mathrm{span}\{u_{\mathrm{wt}\,u+p}w | u \in V, p \in \mathbb{Z}, p < n\}$. This implies that $M(0) = 0$, a contradiction.

It is clear that (iii) follows from (ii). For (ii), note that $M = \hat{V}w = \oplus_{p\in\mathbb{Z}}\hat{V}(p)w$. Thus if $0 \neq w \in M(i)$ then $\hat{V}(p)w = M(i+p)$. In particular, $\hat{V}(0)w = M(i)$, as required. It was pointed out in [Z] that $L(0)$ is semisimple on M and $M(k) = \{w \in M | L(0)w = (h+k)w\}$ for some fixed h. The inequivalence follows from the fact that $L(0)$ has different eigenvalues on $M(p)$ and $M(q)$. ∎

4. THE FUNCTOR L_n

We show in this section that there is a universal way to construct an admissible V-module from an $A_n(V)$-module which cannot factor through A_{n-1}. (If it can factor through $A_{n-1}(V)$ we can consider the same procedure for $A_{n-1}(V)$.) Moreover a certain quotient of the universal object is an admissible V-module $L_n(U)$ and L_n defines a functor which is a right inverse to the functor Ω_n/Ω_{n-1}, where Ω_n/Ω_{n-1} is the quotient functor $M \mapsto \Omega_n(M)/\Omega_{n-1}(M)$.

Fix an $A_n(V)$-module U which cannot factor through $A_{n-1}(V)$. Then it is a module for $A_n(V)_{Lie}$ in an obvious way. By Proposition 3.1 we can lift U to a module for the Lie algebra $\hat{V}(0)$, and then to one for $P_n = \oplus_{p>n}\hat{V}(-p) \oplus \hat{V}(0)$ by letting $\hat{V}(-p)$ act trivially. Define

$$M_n(U) = \mathrm{Ind}_{P_n}^{\hat{V}}(U) = U(\hat{V}) \otimes_{U(P_n)} U. \tag{4.1}$$

If we give U degree n, the $\mathbb{Z}$-gradation of $\hat{V}$ lifts to $M_n(U)$ which thus becomes a $\mathbb{Z}$-graded module for $\hat{V}$. It is easy to see that $M_n(U)(i) = U(\hat{V})_{i-n}U$.

We define for $v \in V$,

$$Y_{M_n(U)}(v, z) = \sum_{m\in\mathbb{Z}} v(m) z^{-m-1}. \tag{4.2}$$

As in [DLM], $Y_{M(U)}(v, z)$ satisfies all conditions of a weak V-module except the associativity which does not hold on $M_n(U)$ in general. We have to divide out by the desired relations.

Let W be the subspace of $M_n(U)$ spanned linearly by the coefficients of

$$(z_0+z_2)^{\mathrm{wt}\,a+n}Y(a, z_0+z_2)Y(b, z_2)u - (z_2+z_0)^{\mathrm{wt}\,a+n}Y(Y(a, z_0)b, z_2)u \tag{4.3}$$

for any homogeneous $a \in V, b \in V, u \in U$. Set

$$\overline{M}_n(U) = M_n(U)/U(\hat{V})W. \tag{4.4}$$

THEOREM 4.1. *The space* $\overline{M}_n(U) = \sum_{m \geq 0} \overline{M}_n(U)(m)$ *is an admissible* V*-module with* $\overline{M}_n(U)(0) \neq 0$, $\overline{M}_n(U)(n) = U$ *and with the following universal property: for any weak* V*-module* M *and any* $A_n(V)$*-morphism* $\phi: U \to \Omega_n(M)$, *there is a unique morphism* $\overline{\phi}: \overline{M}_n(U) \to M$ *of weak* V*-modules which extends* ϕ.

Proof. By Proposition 6.1 of [DLM], we know that $\overline{M}_n(U)$ is a $\mathbb{Z}$-graded weak V-module generated by $U + U(\hat{V})W$. By Proposition 2.4 of [DM] or Lemma 6.1.1 of [L2], $\overline{M}_n(U)$ is spanned by

$$\left\{a_n\left(U + U(\hat{V})W\right) | a \in V, n \in \mathbb{Z}\right\}.$$

Thus $\overline{M}_n(U)(m) = \hat{V}(m-n)(U + U(\hat{V})W)$ for all $m \in \mathbb{Z}$. In particular, $\overline{M}_n(U)(m) = 0$ if $m < 0$ and $\overline{M}_n(U)(n) = A_n(V)(U + U(\hat{V})W)$ which is a quotient module of U. A proof that $\overline{M}_n(U)(0) \neq 0$ and $\overline{M}_n(U)(n) = U$ will be given after Proposition 4.7. The universal property of $\overline{M}_n(U)$ follows from its construction. ∎

In the following we let $U^* = \mathrm{Hom}_{\mathbb{C}}(U, \mathbb{C})$ and let U_s be the subspace of $M_n(U)(n)$ spanned by "length" s vectors

$$o_{p_1}(a_1) \cdots o_{p_s}(a_s)U,$$

where $p_1 \geq \cdots \geq p_s$, $p_1 + \cdots p_s = 0$, $p_i \neq 0$, $p_s \geq -n$, and $a_i \in V$. Then by the PBW theorem $M_n(U)(n) = \sum_{s \geq 0} U_s$ with $U_0 = U$ and $U_s \cap U_t = 0$ if $s \neq t$. Recall Remark 3.3. We extend U^* to $M_n(U)(n)$ inductively so that

$$\langle u', o_{p_1}(a_1) \cdots o_{p_s}(a_s)u \rangle = \langle u', o_{p_1+p_2}\left(w^{p_1, p_2}_{a_1, a_2}\right) o_{p_3}(a_3) \cdots o_{p_s}(a_s)u \rangle, \tag{4.5}$$

where $o_j(a) = a(\mathrm{wt}\, a - 1 - j)$ for homogeneous $a \in V$. We further extend U^* to $M_n(U)$ by letting U^* annihilate $\oplus_{i \neq n} M(U)(i)$.

Set

$$J = \left\{v \in M_n(U) | \langle u', xv \rangle = 0 \text{ for all } u' \in U^*, \text{ all } x \in U(\hat{V})\right\}.$$

We can now state the second main result of this section.

THEOREM 4.2. *The space* $L_n(U) = M_n(U)/J$ *is an admissible* V*-module satisfying* $L_n(U)(0) \neq 0$ *and* $\Omega_n/\Omega_{n-1}(L_n(U)) \cong U$. *Moreover* L_n *defines a functor from the category of* $A_n(V)$*-modules which cannot factor through* $A_{n-1}(V)$ *to the category of admissible* V*-modules such that* $\Omega_n/\Omega_{n-1} \circ L_n$ *is naturally equivalent to the identity.*

The main point in the proof of the theorem is to show that $U(\hat{V})W \subset J$. The next three results are devoted to this goal.

PROPOSITION 4.3. *The following hold for all homogeneous* $a \in V, b \in V, u' \in U^*, u \in U, j \in \mathbb{Z}_+$,

$$\langle u', (z_0 + z_2)^{\mathrm{wt}\, a+n+j} Y_{M_n(U)}(a, z_0 + z_2) Y_{M_n(U)}(b, z_2) u\rangle$$
$$= \langle u', (z_2 + z_0)^{\mathrm{wt}\, a+n+j} Y_{M_n(U)}(Y(a, z_0) b, z_2) u\rangle. \tag{4.6}$$

In the following we simply write Y for $Y_{M_n(U)}$, which should cause no confusion. The following is the key lemma.

LEMMA 4.4. *For any* $i, j \in \mathbb{Z}_+$,

$$\mathrm{Res}_{z_0} z_0^{-1+i}(z_0 + z_2)^{\mathrm{wt}\, a+n+j}\langle u', Y(a, z_0 + z_2) Y(b, z_2) u\rangle$$
$$= \mathrm{Res}_{z_0} z_0^{-1+i}(z_2 + z_0)^{\mathrm{wt}\, a+n+j}\langle u', Y(Y(a, z_0) b, z_2) u\rangle.$$

Proof. Since $j \geq 0$ then $a(\mathrm{wt}\, a + n + j)$ lies in $\oplus_{p>n} \hat{V}(-p)$ and hence annihilates u. Then for all $i \in \mathbb{Z}_+$ we get

$$\mathrm{Res}_{z_1}(z_1 - z_2)^i z_1^{\mathrm{wt}\, a+n+j} Y(b, z_2) Y(a, z_1) u = 0. \tag{4.7}$$

Note that (3.2) is equivalent to

$$[Y(a, z_1), Y(b, z_2)] = \mathrm{Res}_{z_0} z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0) v, z_2). \tag{4.8}$$

Using (4.7) and (4.8) we obtain

$$\mathrm{Res}_{z_0} z_0^i (z_0 + z_2)^{\mathrm{wt}\, a+n+j} Y(a, z_0 + z_2) Y(b, z_2) u$$
$$= \mathrm{Res}_{z_1}(z_1 - z_2)^i z_1^{\mathrm{wt}\, a+n+j} Y(a, z_1) Y(b, z_2) u$$
$$= \mathrm{Res}_{z_1}(z_1 - z_2)^i z_1^{\mathrm{wt}\, a+n+j} Y(a, z_1) Y(b, z_2) u$$
$$- \mathrm{Res}_{z_1}(z_1 - z_2)^i z_1^{\mathrm{wt}\, a+n+j} Y(b, z_2) Y(a, z_1) u$$
$$= \mathrm{Res}_{z_1}(z_1 - z_2)^i z_1^{\mathrm{wt}\, a+n+j} [Y(a, z_1), Y(b, z_2)] u$$
$$= \mathrm{Res}_{z_0} \mathrm{Res}_{z_1}(z_1 - z_2)^i z_1^{\mathrm{wt}\, a+n+j} z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(a, z_0) b, z_2) u$$
$$= \mathrm{Res}_{z_0} \mathrm{Res}_{z_1} z_0^i z_1^{\mathrm{wt}\, a+n+j} z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) Y(Y(a, z_0) b, z_2) u$$
$$= \mathrm{Res}_{z_0} z_0^i (z_2 + z_0)^{\mathrm{wt}\, a+n+j} Y(Y(a, z_0) b, z_2) u. \tag{4.9}$$

Thus Lemma 4.4 holds if $i \geq 1$, and we may now assume $i = 0$.
Next us (4.9) to calculate that

$$\begin{aligned}
&\mathrm{Res}_{z_0} z_0^{-1}(z_0+z_2)^{\mathrm{wt}\,a+n+j}\langle u', Y(a,z_0+z_2)Y(b,z_2)u\rangle \\
&= \sum_{k=0}^{\infty}\binom{j}{k}\mathrm{Res}_{z_0} z_0^{k-1}z_2^{j-k}(z_0+z_2)^{\mathrm{wt}\,a+n}\langle u', Y(a,z_0+z_2)Y(b,z_2)u\rangle \\
&= \sum_{k=1}^{\infty}\binom{j}{k}\mathrm{Res}_{z_0} z_0^{k-1}z_2^{j-k}(z_2+z_0)^{\mathrm{wt}\,a+n}\langle u', Y\big(Y(a,z_0)b,z_2\big)u\rangle \\
&\quad + \mathrm{Res}_{z_0} z_0^{-1}z_2^{j}(z_2+z_0)^{\mathrm{wt}\,a+n}\langle u', Y(a,z_0+z_2)Y(b,z_2)u\rangle. \qquad (4.10)
\end{aligned}$$

It reduces to show that

$$\mathrm{Res}_{z_0} z_0^{-1}(z_2+z_0)^{\mathrm{wt}\,a+n}\langle u', Y(a,z_0+z_2)Y(b,z_2)u\rangle \qquad (4.11)$$

$$= \mathrm{Res}_{z_0} z_0^{-1}(z_2+z_0)^{\mathrm{wt}\,a+n}\langle u', Y(a,z_0)u,z_2)u\rangle. \qquad (4.12)$$

Since $\langle u', M_n(U)(m)\rangle = 0$ if $m \neq n$, we see that

$$\begin{aligned}
&\mathrm{Res}_{z_0} z_0^{-1}(z_2+z_0)^{\mathrm{wt}\,a+n} z_2^{\mathrm{wt}\,b-n}\langle u', Y\big(Y(a,z_0)b,z_2\big)u\rangle \\
&= \left\langle u', \sum_{k\in\mathbb{Z}_+}\binom{\mathrm{wt}\,a+n}{k}(a_{k-1}b)\big(\mathrm{wt}(a_{k-1}b)-1\big)u\right\rangle \\
&= \left\langle u', \sum_{k\in\mathbb{Z}_+}\binom{\mathrm{wt}\,a+n}{k}o(a_{k-1}b)u\right\rangle \\
&= \left\langle u', o\left(\mathrm{Res}_z Y(a,z)b\frac{(1+z)^{\mathrm{wt}\,a+n}}{z}\right)u\right\rangle.
\end{aligned}$$

On the other hand, note that $b(\mathrm{wt}\,b-1+p)u = 0$ if $p > n$. So

$$\begin{aligned}
&\mathrm{Res}_{z_0} z_0^{-1}(z_0+z_2)^{\mathrm{wt}\,a+n} z_2^{\mathrm{wt}\,b-n}\langle u', Y(a,z_0+z_2)Y(b,z_2)u\rangle \\
&= \left\langle u', \sum_{i\in\mathbb{Z}_+} a(\mathrm{wt}\,a-2-i+n)\sum_{m\geq -n} b(\mathrm{wt}\,b-1-m)z_2^{-n+i+m}u\right\rangle \qquad (4.13) \\
&= \left\langle u', \sum_{i=0}^{n} a(\mathrm{wt}\,a-1-i)b(\mathrm{wt}\,b-1+i)u\right\rangle \qquad (4.14) \\
&\quad + \left\langle u', \sum_{i=1}^{n} a(\mathrm{wt}\,a-1+i)b(\mathrm{wt}\,b-1-i)u\right\rangle. \qquad (4.15)
\end{aligned}$$

Note that the $A_n(V)$-module structure on U is equivalent to

$$o(a)o(b)u = a(\text{wt}\, a - 1)b(\text{wt}\, b - 1)u$$
$$= \sum_{m=0}^{n} (-1)^m \binom{m+n}{n} o\left(\text{Res}_z Y(a,z)b \frac{(1+z)^{\text{wt}\, a+n}}{z^{m+n+1}}\right)u.$$

By (4.5) with $s = 2$, $a_1 = a$, $a_2 = b$, $p_1 = k = -p_2$ $(k > 0)$ we see that

$$\langle u', o_k(a)o_{-k}(b)u\rangle$$
$$= \langle u', a(\text{wt}\, a - 1 - k)b(\text{wt}\, b - 1 + k)u\rangle$$
$$= \left\langle u', \sum_{m=0}^{n-k} (-1)^m \binom{m+n+k}{m} o\left(\text{Res}_z Y(a,z)b \frac{(1+z)^{\text{wt}\, a+n}}{z^{m+n+1+k}}\right)u\right\rangle. \tag{4.16}$$

Thus

$$\left\langle u', \sum_{k=0}^{n} a(\text{wt}\, a - 1 - k)b(\text{wt}\, b - 1 + k)u\right\rangle$$
$$= \left\langle u', \sum_{k=0}^{n}\sum_{m=0}^{n-k} (-1)^m \binom{m+n+k}{m} o\left(\text{Res}_z Y(a,z)b \frac{(1+z)^{\text{wt}\, a+n}}{z^{m+n+1+k}}\right)u\right\rangle.$$

Use Lie algebra bracket (3.2) to get

$$a(\text{wt}\, a - 1 + k)b(\text{wt}\, b - 1 - k)$$
$$= b(\text{wt}\, b - 1 - k)a(\text{wt}\, a - 1 + k)$$
$$+ \sum_{i\geq 0} \binom{\text{wt}\, a - 1 + k}{i} (a_i b)(\text{wt}\, a + \text{wt}\, b - 2 - i).$$

By (4.16),

$$\langle u', b(\text{wt}\, b - 1 - k)a(\text{wt}\, a - 1 + k)u\rangle$$
$$= \left\langle \sum_{m=0}^{n-k} (-1)^m \binom{m+n+k}{m} o\left(\text{Res}_z Y(b,z)a \frac{(1+z)^{\text{wt}\, b+n}}{z^{m+n+1+k}}\right)u\right\rangle.$$

A proof similar to that of Lemma 2.1(ii) shows that

$$\sum_{m=0}^{n-k} (-1)^m \binom{m+n+k}{m} \text{Res}_z Y(b,z)a \frac{(1+z)^{\text{wt}\, b+n}}{z^{m+n+1+k}}$$
$$- \sum_{m=0}^{n-k} \binom{m+n+k}{m} (-1)^{n+k} \text{Res}_z Y(a,z)b \frac{(1+z)^{\text{wt}\, a+m+k-1}}{z^{1+m+n+k}} \in O_n(V).$$

We now have

$$\begin{aligned}
&\left\langle u', \sum_{k=1}^{n} a(\operatorname{wt} a - 1 + k) b(\operatorname{wt} b - 1 - k) u \right\rangle \\
&\quad = \sum_{k=1}^{n} \sum_{m=0}^{n-k} \binom{m+n+k}{m} (-1)^{n+k} \\
&\qquad \times \left\langle u', o\left(\operatorname{Res}_z Y(a,z) b \frac{(1+z)^{\operatorname{wt} a + m + k - 1}}{z^{1+m+n+k}} \right) u \right\rangle \\
&\qquad + \sum_{k=1}^{n} \sum_{i \ge 0} \binom{\operatorname{wt} a - 1 + k}{i} \langle u', (a_i b)(\operatorname{wt} a + \operatorname{wt} b - 2 - i) u \rangle \\
&\quad = \sum_{k=1}^{n} \sum_{m=0}^{n-k} \binom{m+n+k}{m} (-1)^{n+k} \\
&\qquad \times \left\langle u', o\left(\operatorname{Res}_z Y(a,z) b \frac{(1+z)^{\operatorname{wt} a + m + k - 1}}{z^{1+m+n+k}} \right) u \right\rangle \\
&\qquad + \sum_{k=1}^{n} \langle u', o\left(\operatorname{Res}_z Y(a,z) b (1+z)^{\operatorname{wt} a - 1 + k} \right) u \rangle.
\end{aligned}$$

So it is enough to show the identity

$$\begin{aligned}
&\sum_{k=0}^{n} \sum_{m=0}^{n-k} (-1)^m \binom{m+n+k}{m} \frac{(1+z)^{\operatorname{wt} a + n}}{z^{m+n+1+k}} \\
&\qquad + \sum_{k=1}^{n} \sum_{m=0}^{n-k} \binom{m+n+k}{m} (-1)^{n+k} \frac{(1+z)^{\operatorname{wt} a + m + k - 1}}{z^{1+m+n+k}} \\
&\qquad + \sum_{k=1}^{n} (1+z)^{\operatorname{wt} a - 1 + k} \\
&\qquad = \frac{(1+z)^{\operatorname{wt} a + n}}{z},
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\sum_{k=0}^{n} \sum_{m=0}^{n-k} (-1)^m \binom{m+n+k}{m} \frac{(1+z)^{n}}{z^{m+n+k}} \\
&\qquad + \sum_{k=1}^{n} \sum_{m=0}^{n-k} \binom{m+n+k}{m} (-1)^{n+k} \frac{(1+z)^{m+k-1}}{z^{m+n+k}} \\
&\qquad = 1.
\end{aligned}$$

This identity is proved in Proposition 5.1 in the Appendix. ∎

Proposition 4.3 is a consequence of the next lemma.

LEMMA 4.5. *For all $m \in \mathbb{Z}$ we have*

$$\operatorname{Res}_{z_0} z_0^m (z_0 + z_2)^{\mathrm{wt}\, a + m + j} \langle u', Y(a, z_0 + z_2) Y(b, z_2) u\rangle$$
$$= \operatorname{Res}_{z_0} z_0^m (z_2 + z_0)^{\mathrm{wt}\, a + m + j} \langle u', Y(Y(a, z_0) b, z_2) u\rangle.$$

Proof. This is true for $m \geq -1$ by Lemma 4.4. Let us write $m = -k + i$ with $i \in \mathbb{Z}_+$ and proceed by induction k. Induction yields

$$\operatorname{Res}_{z_0} z_0^{-k} (z_0 + z_2)^{\mathrm{wt}\, a + m + j} \langle u', Y(L(-1) a, z_0 + z_2) Y(b, z_2) u\rangle$$
$$= \operatorname{Res}_{z_0} z_0^{-k} (z_2 + z_0)^{\mathrm{wt}\, a + m + j} \langle u', Y(Y(L(-1) a, z_0) b, z_2) u\rangle.$$

Using the residue property $\operatorname{Res}_z f'(z) g(z) + \operatorname{Res}_z f(z) g'(z) = 0$ and the $L(-1)$-derivation property $Y(L(-1)a, z) = \frac{d}{dz} Y(a, z)$ we have

$$\operatorname{Res}_{z_0} z_0^{-k} (z_0 + z_2)^{\mathrm{wt}\, a + 1 + m + j} \langle u', Y(L(-1) a, z_0 + z_2) Y(b, z_2) u\rangle$$
$$= -\operatorname{Res}_{z_0} \left(\frac{\partial}{\partial z_0} z_0^{-k} (z_0 + z_2)^{\mathrm{wt}\, a + 1 + m + j} \right) \langle u', Y(a, z_0 + z_2) Y(b, z_2) u\rangle$$
$$= \operatorname{Res}_{z_0} k z_0^{-k-1} (z_0 + z_2)^{\mathrm{wt}\, a + 1 + m + j} \langle u', Y(a, z_0 + z_2) Y(b, z_2) u\rangle$$
$$- \operatorname{Res}_{z_0} (\mathrm{wt}\, a + 1 + m + j) z_0^{-k} (z_0 + z_2)^{\mathrm{wt}\, a + m + j}$$
$$\times \langle u', Y(a, z_0 + z_2) Y(b, z_2) u\rangle$$
$$= \operatorname{Res}_{z_0} k z_0^{-k-1} z_2 (z_0 + z_2)^{\mathrm{wt}\, a + m + j} \langle u', Y(a, z_0 + z_2) Y(b, z_2) u\rangle$$
$$+ \operatorname{Res}_{z_0} k z_0^{-k} (z_0 + z_2)^{\mathrm{wt}\, a + m + j} \langle u', Y(a, z_0 + z_2) Y(b, z_2) u\rangle$$
$$- \operatorname{Res}_{z_0} (\mathrm{wt}\, a + 1 + m + j) z_0^{-k} (z_2 + z_0)^{\mathrm{wt}\, a + m + j}$$
$$\times \langle u', Y(Y(a, z_0) b, z_2) u\rangle$$
$$= \operatorname{Res}_{z_0} k z_0^{-k-1} z_2 (z_0 + z_2)^{\mathrm{wt}\, a + m + j} \langle u', Y(a, z_0 + z_2) Y(b, z_2) u\rangle$$
$$+ \operatorname{Res}_{z_0} k z_0^{-k} (z_2 + z_0)^{\mathrm{wt}\, a + m + j} \langle u', Y(Y(a, z_0) b, z_2) u\rangle$$
$$- \operatorname{Res}_{z_0} (\mathrm{wt}\, a + 1 + m + j) z_0^{-k} (z_2 + z_0)^{\mathrm{wt}\, a + m + j}$$
$$\times \langle u', Y(Y(a, z_0) b, z_2) u\rangle,$$

and

$$\begin{aligned}
&\mathrm{Res}_{z_0} z_0^{-k}(z_2+z_0)^{\mathrm{wt}\,a+1+m+j}\langle u', Y(Y(L(-1)a,z_0)b,z_2)u\rangle \\
&\quad = -\mathrm{Res}_{z_0}\left(\frac{\partial}{\partial z_0} z_0^{-k}(z_2+z_0)^{\mathrm{wt}\,a+1+m+j}\right)\langle u', Y(Y(a,z_0)b,z_2)u\rangle \\
&\quad = \mathrm{Res}_{z_0} k z_0^{-k-1}(z_2+z_0)^{\mathrm{wt}\,a+1+m+j}\langle u', Y(Y(a,z_0)b,z_2)\rangle u\rangle \\
&\qquad - \mathrm{Res}_{z_0}(\mathrm{wt}\,a+1+m+j) z_0^{-k}(z_2+z_0)^{\mathrm{wt}\,a+m+j} \\
&\qquad\quad \times \langle u', Y(Y(a,z_0)b,z_2)u\rangle \\
&\quad = \mathrm{Res}_{z_0} k z_2 z_0^{-k-1}(z_2+z_0)^{\mathrm{wt}\,a+m+j}\langle u', Y(Y(a,z_0)b,z_2)u\rangle \\
&\qquad + \mathrm{Res}_{z_0} k z_0^{-k}(z_2+z_0)^{\mathrm{wt}\,a+m+j}\langle u', Y(Y(a,z_0)b,z_2)u\rangle \\
&\qquad - \mathrm{Res}_{z_0}(\mathrm{wt}\,a+1+m+j) z_0^{-k}(z_2+z_0)^{\mathrm{wt}\,a+m+j} \\
&\qquad\quad \times \langle u', Y(Y(a,z_0)b,z_2)u\rangle .
\end{aligned}$$

This yields the identity

$$\begin{aligned}
&\mathrm{Res}_{z_0} z_0^{-k-1}(z_0+z_2)^{\mathrm{wt}\,a+m+j}\langle u', Y(a,z_0+z_2)Y(b,z_2)u\rangle \\
&\quad = \mathrm{Res}_{z_0} z_0^{-k-1}(z_2+z_0)^{\mathrm{wt}\,a+m+j}\langle u', Y(Y(a,z_0)b,z_2)u\rangle ,
\end{aligned}$$

and the lemma is proved. ∎

Let us now introduce an arbitrary $\mathbb{Z}$-graded $\hat{V}$-module $M = \oplus_{m\in\mathbb{Z}} M(m)$. As before we extend $M(n)^*$ to M by letting it annihilate $M(m)$ for $m \neq n$. The proof of Proposition of 6.1 of [DLM] with $\langle u', \cdot\,\rangle$ suitably inserted gives:

PROPOSITION 4.6. *Let U be a subspace of $M(n)$ and U' a subspace of $M(n)'$ such that*

(i) $M = U(\hat{V})U$.

(ii) *For $a \in V$ and $u \in U$ there is $k \in \mathbb{Z}$ such that*

$$\begin{aligned}
&\langle u', (z_0+z_2)^{k+n} Y(a,z_0+z_2)Y(b,z_2)u\rangle \\
&\quad = \langle u', (z_2+z_0)^{k+n} Y(Y(a,z_0)b,z_2)u\rangle \qquad (4.17)
\end{aligned}$$

for any $b \in V, u' \in U'$. Then in fact (4.17) *holds for any $u \in M$.*

PROPOSITION 4.7. *Let M be as in Proposition* 4.6. *Then for any $x \in U(\hat{V})$, $a \in V$, $u \in M$, there is an integer k such that*

$$\langle u', (z_0 + z_2)^{k+n} x \cdot Y(a, z_0 + z_2) Y(b, z_2) u \rangle$$
$$= \langle u', (z_2 + z_0)^{k+n} x \cdot Y(Y(a, z_0) b, z_2) u \rangle \qquad (4.18)$$

for any $b \in V$, $u' \in U'$.

Proof. Let L be the subspace of $U(\hat{V})$ consisting of those x for which (4.18) holds. Let $x \in L$, let c be any homogeneous element of V, and let $m \in \mathbb{Z}$. Then from (4.8) we have

$$\langle u', xc(m) Y(a, z_0 + z_2) Y(b, z_2) u \rangle (z_0 + z_2)^{k+n}$$
$$= \sum_{i=0}^{\infty} \binom{m}{i} (z_0 + z_2)^{k+n+m-i} \langle u', xY(c(i)a, z_0 + z_2) Y(b, z_2) u \rangle$$
$$+ \sum_{i=0}^{\infty} \binom{m}{i} z_2^{m-i} (z_0 + z_2)^{k+n} \langle u', xY(a, z_0 + z_2) Y(c(i)b, z_2) u \rangle$$
$$+ (z_0 + z_2)^{k+n} \langle u', xY(a, z_0 + z_2) Y(b, z_2) c(m) u \rangle. \qquad (4.19)$$

The same method that was used in the proof of Proposition 4.6 shows that $xc(m) \in L$. Since $U(\hat{V})$ is generated by all such $c(n)$'s, and since (4.18) holds for $x = 1$ by Proposition 4.6, we conclude that $L = U(\hat{V})$, as desired. ∎

We can now finish the proof of Theorems 4.1 and Theorem 4.2. We can take $M = \overline{M}_n(U)$ in Proposition 4.7, as we may since $\overline{M}_n(U)$ certainly satisfies the conditions placed on M prior to Proposition 4.6 and in Proposition 4.6. Then from the definition of W (4.3) and Propositions 4.3, 4.6, and 4.7 we conclude that $U(\hat{V})W \subset J$. It is clear that $L(U)$ is a quotient of $\overline{M}_n(U)$ and hence an admissible V-module. Note that $J \cap U = 0$. So $L(U)(n)$ contains U as an $A_n(V)$-submodule. This shows that $\overline{M}_n(U)(n) \cong U$ as $A_n(V)$-modules. If $\overline{M}_n(U)(0) = 0$ then U will be an $A_{n-1}(V)$-module, contradicting the assumption on U. This finishes the proof of Theorem 4.1. Theorem 4.2 is now obvious. ∎

At this point we have a pair of functors Ω_n, L_n defined on appropriate module categories. It is clear that $\Omega_n / \Omega_{n-1} \circ L_n$ is equivalent to the identity.

LEMMA 4.8. *Suppose that U is a simple $A_n(V)$-module. Then $L_n(U)$ is a simple admissible V-module.*

Proof. If $0 \neq W \subset L_n(U)$ is an admissible submodule then, by the definition of $L_n(U)$, we have $W(n) = W \cap L_n(U)(n) \neq 0$. As $W(n)$ is an

$A_n(V)$-submodule of $U = L_n(U)(n)$ by Theorem 3.2 then $U = W(n)$, whence $W \supset U(\hat{V})W(n) = U(\hat{V})U = L_n(U)$. ∎

THEOREM 4.9. *L_n and Ω_n/O_{n-1} are equivalences when restricted to the full subcategories of completely reducible $A_n(V)$-modules whose irreducible components cannot factor through $A_{n-1}(V)$ and completely reducible admissible V-modules, respectively. In particular, L_n and Ω_n/Ω_{n-1} induce mutually inverse bijections on the isomorphism classes of simple objects in the category of $A_n(V)$-modules which cannot factor through $A_{n-1}(V)$ and admissible V-modules, respectively.*

Proof. We have $\Omega_n/O_{n-1}(L(U)) \cong U$ for any $A_n(V)$-module by Theorem 4.2.

If M is a completely reducible admissible V-module we must show $L_n(\Omega_n/\Omega_{n-1}(M)) \cong M$. For this we may take M simple, whence $\Omega_n/\Omega_{n-1}(M)$ is simple by Proposition 3.4(ii) and then $L_n(\Omega_n/\Omega_{n-1}(M))$ is simple by Lemma 4.8. Since both M and $L_n(\Omega_n/\Omega_{n-1}(M))$ are simple quotients of the universal object $\overline{M}_n(\Omega_n/\Omega_{n-1}(M))$ then they are isomorphic by Theorems 4.1 and 4.2. ∎

The following theorem is a generalization of Theorem 8.1 of [DLM].

THEOREM 4.10. *Suppose that V is a rational vertex operator algebra. Then the following hold*:

(a) *$A_n(V)$ is a finite-dimensional, semisimple associative algebra.*

(b) *The functors L_n, Ω_n/O_{n-1} are mutually inverse categorical equivalences between the category of $A_n(V)$-modules whose irreducible components cannot factor through $A_{n-1}(V)$ and the category of admissible V-modules.*

(c) *The functors L_n, Ω_n/Ω_{n-1} induce mutually inverse categorical equivalences between the category of finite-dimensional $A_n(V)$-modules whose irreducible components cannot factor through $A_{n-1}(V)$ and the category of ordinary V-modules.*

Proof. Part (b) follows from Theorem 4.9 and (a). Since V is rational any irreducible admissible V-module is an ordinary module by Theorem 8.1 of [DLM]. Now (c) follows from (b). It remains to prove (i).

Let W be an $A_n(V)$-module. Set $U = W \oplus V(n)$. Then U is an $A_n(V)$-module which cannot factor through $A_{n-1}(V)$. Now $L_n(U)$ is admissible and hence a direct sum of irreducible ordinary V-modules. Thus $\Omega_n(L_n(U))/\Omega_{n-1}(L_n(U)) \simeq U$ is a direct sum of finite-dimensional irreducible $A_n(V)$-modules and so is W. ∎

It is believed that if $A(V) = A_0(V)$ is semisimple then V is rational. We cannot solve this problem completely in this paper. But we have some partial results which are applications of $A_n(V)$-theory.

THEOREM 4.11. *If all $A_n(V)$ are finite-dimensional semisimple algebras then V is rational.*

Proof. Since $A(V)$ is semisimple V has only finitely many irreducible admissible modules which are necessarily ordinary V-modules. For any $\lambda \in \mathbb{C}$ let $\mathscr{M}_\lambda$ be the set of irreducible admissible modules whose weights are congruent to λ module $\mathbb{Z}$. Then for each $W \in \mathscr{M}_\lambda$ we have $W = \oplus_{n \in \mathbb{Z}_+} W_{\lambda + n_W + n} = \oplus_{n \in \mathbb{Z}_+} W(n)$ where $n_W \in \mathbb{Z}$ and $W_{\lambda + n_W + n} = W(n)$. Since $L(-1)$: $W(n) \to W(n+1)$ is injective if n is large (see [L1]) there exists an $m_\lambda \in \mathbb{N}$ such that the weight space $W_{\lambda+m} \neq 0$ for any $W \in \mathscr{W}_\lambda$ and $m \geq m_\lambda$.

Consider any admissible module M whose weights are in $\lambda + Z$ and whose homogeneous subspace $M_{\lambda+m}$ with some $m \geq m_\lambda$ is 0. Let U be an irreducible $A(V)$-submodule of $M(0)$. Then $L_0(U) = L(U)$ is an irreducible V-module such that $L(U)(0) = U$ and $L(U)_{\lambda+m} = 0$. Thus $L(U) = 0$ and $U = 0$. This implies that $M = 0$.

Now take an admissible module $M = \oplus_{k \in \mathbb{Z}_+} M(k)$. Then $M(0)$ is a direct sum of simple $A(V)$-modules as $A(V)$ is semisimple. Let U be an $A(V)$-submodule of $M(0)$ isomorphic to $W(0) = W_{\lambda+n_W}$ for some $W \in \mathscr{M}_\lambda$. We assert that the submodule N of M generated by U is irreducible and necessarily isomorphic to W. First note that N has an irreducible quotient isomorphic to W. Take $n \in \mathbb{N}$ such that $n + n_W \geq m_\lambda$. Observe that $\overline{M}_n(W(n))/\bar{J} = L_n(W(n))$ is isomorphic to W where $\bar{J}$ is a maximal submodule of $\overline{M}_n(W(n))$ such that $\bar{J} \cap W(n) = 0$. Since $\bar{J}_{\lambda+n_W+n} = 0$ we see that $\bar{J} = 0$ and $\overline{M}_n(W(n)) = L_n(W(n)) \simeq W$. Write $N(n)$ as a direct sum of $W(n)$ and another $A_n(V)$-submodule $N(n)'$ of $N(n)$ as $A_n(V)$ is semisimple. Clearly the submodule of $N(n)$ generated by $W(n)$ is isomorphic to W. This shows that N must be isomorphic to W, as claimed.

It is obvious now that the submodule $U(\hat{V})M(0)$ generated by $M(0)$ is completely reducible. Using the semisimplicity of $A_1(V)$ we can decompose $M(1)$ into a direct sum of $A_1(V)$-modules $(U(\hat{V})M(0))(1) \oplus M(1)'$. The same argument shows that $U(\hat{V})M(1)'$ is a completely reducible submodule of M. Continuing in this way proves that M is completely reducible. ∎

Remark 4.12. From the proof of Theorem 4.11, we see, in fact, that we can weaken the assumption in Theorem 4.11. Namely we only need to assume that $A_n(V)$ is semisimple if n is large.

5. APPENDIX

In this appendix we prove several combinatorial identities which are used in the previous sections.

For $n \geq 0$ define

$$A_n(z) = \sum_{k=0}^{n} \sum_{m=0}^{n-k} (-1)^m \binom{m+n+k}{m} \frac{(1+z)^n}{z^{m+n+k}} + \sum_{k=1}^{n} \sum_{m=0}^{n-k} \binom{m+n+k}{m} (-1)^{n+k} \frac{(1+z)^{m+k-1}}{z^{m+n+k}}.$$

Using the well-known identity

$$\sum_{k=0}^{i} (-1)^k \binom{n}{k} = (-1)^i \binom{n-1}{i}$$

we can rewrite $A_n(z)$ as

$$\begin{aligned} A_n(z) &= \sum_{k=0}^{n} \sum_{m=0}^{k} (-1)^m \binom{n+k}{m} \frac{(1+z)^n}{z^{n+k}} + \sum_{k=1}^{n} \sum_{m=0}^{k-1} \binom{n+k}{m} (-1)^{n+k+m} \frac{(1+z)^{k-1}}{z^{n+k}} \\ &= \sum_{k=0}^{n} (-1)^k \binom{n+k-1}{k} \frac{(1+z)^n}{z^{n+k}} - (-1)^n \sum_{k=1}^{n} \binom{n+k-1}{k-1} \frac{(1+z)^{k-1}}{z^{n+k}}. \end{aligned}$$

PROPOSITION 5.1. $A_n(z) = 1$ *for all* $n \geq 0$.

Proof. Set

$$B_n(z) = \sum_{k=0}^{n} (-1)^k \binom{n+k-1}{k} \frac{(1+z)^n}{z^{n+k}}$$

$$C_n(z) = \sum_{k=1}^{n} \binom{n+k-1}{k-1} \frac{(1+z)^{k-1}}{z^{n+k}}.$$

Then

$$B_n(z) = \sum_{k=0}^{n-1} (-1)^k \left(\binom{n+k-2}{k} + \binom{n+k-2}{k-1} \right) \frac{(1+z)^n}{z^{n+k}}$$

$$+(-1)^n \binom{2n-1}{n} \frac{(1+z)^n}{z^{2n}}$$

$$= \frac{1+z}{z} B_{n-1}(z) + \sum_{k=0}^{n-2} (-1)^{k+1} \binom{n+k-1}{k} \frac{(1+z)^n}{z^{n+k+1}}$$

$$+(-1)^n \binom{2n-1}{n} \frac{(1+z)^n}{z^{2n}}$$

$$= \frac{1+z}{z} B_{n-1}(z) - \frac{1}{z} B_n(z) + (-1)^{n-1} \binom{2n-2}{n-1} \frac{(1+z)^n}{z^{2n}}$$

$$+(-1)^n \binom{2n-1}{n} \frac{(1+z)^{n+1}}{z^{2n+1}}.$$

Solving $B_n(z)$ gives

$$B_n(z) = B_{n-1}(z) + (-1)^{n-1} \frac{(1+z)^{n-1}}{z^{2n-1}} \binom{2(n-1)}{n-1}$$

$$+(-1)^n \frac{(1+z)^n}{z^{2n}} \binom{2n-1}{n}.$$

Similarly,

$$C_n(z) = \sum_{k=1}^{n-1} \left(\binom{n+k-2}{k-1} + \binom{n+k-2}{k-2} \right) \frac{(1+z)^{k-1}}{z^{n+k}}$$

$$+ \binom{2n-1}{n-1} \frac{(1+z)^{n-1}}{z^{2n}}$$

$$= \frac{1}{z} C_{n-1}(z) + \sum_{k=0}^{n-2} \binom{n+k-1}{k-1} \frac{(1+z)^k}{z^{n+k+1}}$$

$$+ \binom{2n-1}{n-1} \frac{(1+z)^{n-1}}{z^{2n}}$$

$$= \frac{1}{z} C_{n-1}(z) + \frac{1+z}{z} C_n(z) + \binom{2n-2}{n-1} \frac{(1+z)^{n-1}}{z^{2n}}$$

$$- \binom{2n-1}{n-1} \frac{(1+z)^n}{z^{2n+1}}.$$

Thus

$$(-1)^{n+1}C_n(z) = (-1)^n C_{n-1}(z) + (-1)^n \binom{2n-2}{n-1} \frac{(1+z)^{n-1}}{z^{2n-1}}$$
$$+(-1)^{n-1}\binom{2n-1}{n-1}\frac{(1+z)^n}{z^{2n}}.$$

Thus

$$A_n(z) = B_n(z) + (-1)^{n-1}C_n(z) = A_{n-1}.$$

Note that $A_0(z) = 1$ and the proposition follows. ∎

For $n \geq 0$ we define

$$F_n(z) = \sum_{m=0}^{n} \binom{m+n}{n} \frac{(-1)^m(1+z)^{n+1} - (-1)^n(1+z)^m}{z^{n+m+1}}.$$

PROPOSITION 5.2. *$F_n(z) = 1$ for all n.*

Proof. Set

$$D_n(z) = \sum_{m=0}^{n} \binom{m+n}{n} (-1)^m \frac{(1+z)^{n+1}}{z^{n+m+1}}$$

$$E_n(z) = \sum_{m=0}^{n} \binom{m+n}{n} \frac{(1+z)^m}{z^{n+m+1}}.$$

Then

$$D_n(z) = B_{n+1}(z) + (-1)^n \binom{2n+1}{n} \frac{(1+z)^{n+1}}{z^{2n+2}}$$
$$= B_n(z) + (-1)^n \frac{(1+z)^n}{z^{2n+1}} \binom{2n}{n}$$
$$+(-1)^{n+1} \frac{(1+z)^{n+1}}{z^{2n+2}} \binom{2n+1}{n+1}$$
$$+(-1)^n \binom{2n+1}{n} \frac{(1+z)^{n+1}}{z^{2n+2}}$$
$$= B_n(z) + (-1)^n \binom{2n}{n} \frac{(1+z)^n}{z^{2n+1}}$$

and

$$E_n(z) = C_n(z) + \binom{2n}{n} \frac{(1+z)^n}{z^{2n+1}}.$$

Thus

$$F_n(z) = D_n(z) + (-1)^{n+1}E_n(z) = A_n(z) = 1,$$

as required. ∎

For $n \geq 0$ define

$$a_n(w,z) = \sum_{m=0}^{n} (-1)^m \binom{m+n}{n} \times \left(\sum_{i=0}^{n-m} \sum_{j\geq 0} \binom{-m-n-1}{i} \binom{m}{j} (-1)^i \frac{w^{i+j}}{z^{i+m}} - \frac{1}{z^m} \right).$$

Note that if $p > 0, k > 0$ then $\binom{-p}{k} = (-1)^k \binom{p+k-1}{k}$. We can rewrite $a_n(w,z)$ as

$$a_n(w,z) = \sum_{m=0}^{n} (-1)^m \binom{m+n}{n} \left(\sum_{i=0}^{n-m} \sum_{j\geq 0} \binom{m+n+i}{i} \binom{m}{j} \frac{w^{i+j}}{z^{i+m}} - \frac{1}{z^m} \right).$$

PROPOSITION 5.3. *The* $a_n(w,z) = 0$ *for all* $n \geq 0$.

Proof. Regarding $a_n(w,z)$ as a polynomial in z^{-1}, the coefficient of z^{-p} in $a_n(w,z)$ $(0 \leq p \leq n)$ is equal to (setting $m + i = p$)

$$\sum_{m=0}^{p} (-1)^m \binom{m+n}{n} \sum_{j\geq 0} \binom{n+p}{p-m} \binom{m}{j} w^{p-m+j} - (-1)^p \binom{p+n}{n}$$
$$= w^p \sum_{m=0}^{p} (-1)^m \binom{m+n}{n} \binom{n+p}{p-m} (1 + 1/w)^m - (-1)^p \binom{p+n}{n}.$$

So the coefficient of $z^{-p}w^0$ in $a_n(w,z)$ equals 0.

If $0 \leq q \leq p-1$, the coefficient of $z^{-p}w^{p-q}$ in $a_n(w,z)$ is equal to

$$c_n(p,q) = \sum_{m=0}^{p} (-1)^m \binom{m+n}{n} \binom{n+p}{n+m} \binom{m}{q}$$

which is defined for any $n, p, q \geq 0$. So we must prove that $a_n(p,q) = 0$ for $1 \leq q+1 \leq p \leq n$. Recall $\binom{l}{k} = \binom{l-1}{k} + \binom{l-1}{k-1}$. Then $c_n(p,q)$ is equal to

$$\sum_{m=0}^{p} (-1)^m \binom{m+n}{n}\left(\binom{n+p-1}{n+m} + \binom{n+p-1}{n+m-1}\right)\binom{m}{q}$$

$$= (-1)^p \binom{p+n}{n}\binom{p}{q} + \sum_{m=0}^{p-1} (-1)^m \binom{m+n}{n}\binom{n+p-1}{n+m-1}\binom{m}{q}$$

$$+ c_n(p-1,q)$$

$$= \sum_{m=0}^{p-1} (-1)^m \left(\binom{m+n-1}{n-1} + \binom{m+n-1}{n}\right)\binom{n+p-1}{n+m-1}\binom{m}{q}$$

$$+ (-1)^p \binom{p+n}{n}\binom{p}{q} + c_n(p-1,q)$$

$$= c_{n-1}(p,q) + c_n(p-1,q) - (-1)^p \binom{p+n-1}{n-1}\binom{p}{q}$$

$$+ (-1)^p \binom{p+n}{n}\binom{p}{q}$$

$$- \sum_{m=0}^{p-1} (-1)^{m-1} \binom{m+n-1}{n}\binom{n+p-1}{n+m-1}$$

$$\times \left(\binom{m-1}{q} + \binom{m-1}{q-1}\right)$$

$$= c_{n-1}(p,q) + c_n(p-1,q) + (-1)^p \binom{p+n-1}{n}\binom{p}{q}$$

$$- \sum_{m-1=0}^{p-2} (-1)^{m-1} \binom{m+n-1}{n}\binom{n+p-1}{n+m-1}$$

$$\times \left(\binom{m-1}{q} + \binom{m-1}{q-1}\right)$$

$$= c_{n-1}(p,q) - c_n(p-1,q-1) + (-1)^p \binom{p+n-1}{n}\binom{p}{q}$$

$$+ (-1)^{p-1} \binom{p-1+n}{n}\left(\binom{p-1}{q} + \binom{p-1}{q-1}\right)$$

$$= c_{n-1}(p,q) - c_n(p-1,q-1).$$

That is,

$$c_n(p,q) = c_{n-1}(p,q) - c_n(p-1,q-1).$$

so by induction it is enough to show that $c_0(p,q) = 0$ and $c_n(p,0) = 0$ if $p > q$. But this is clear from the definition. ∎

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