An efficient method for option pricing with discrete
dividend payment

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Abstract

This paper deals with the construction of a numerical solution of the Black–Scholes equation modeling option pricing with a discrete dividend payment. This model is a partial differential equation with two variables: the underlying asset and the time to maturity, and involves the shifted Dirac delta function centered at the dividend payment date. This generalized function is suitable for approximation by means of sequences of ordinary functions. By applying a semidiscretization technique on the asset, a numerical solution is obtained and the independence of the considered sequence in a wide class of delta defining sequences is proved. From the study of the influence of the spatial step $h$, it follows that the difference between the numerical solution for $h$ and $h/2$ is $O(h^2)$ as $h \rightarrow 0$. The proposed method is useful for different discrete dividend types like a dividend of present value $D_0$, a constant yield dividend or an arbitrary underlying asset-dependent yield dividend payment. Several illustrative examples are included.

Keywords: Black–Scholes equation; Discrete dividend; Numerical solution; Semidiscretization

1. Introduction

Stocks frequently pay dividends, which has implications for the value of options on these stocks. The Black–Scholes model for pricing stock options, when there are dividend payments $D(S, t)$, is

$$
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (rS - D(S, t)) \frac{\partial V}{\partial S} - rV = 0, \quad 0 < S < \infty, \ 0 < t < T.
$$

(1)

If a discrete dividend payment with dividend date $t_d$ is considered, $D(S, t)$ takes the form

$$
D(S, t) = D_0 \delta(t - t_d), \quad 0 < t_d < T,
$$

(2)

where $\frac{D_0(S)}{S}$ is the dividend yield and $\delta(t - t_d)$ is the shifted Dirac delta function, (see [1, p. 140]). Recently, an explicit solution of (1) with a discrete dividend yield, independent of $S$, and a general payoff function $V(S, T) = f(S)$ has

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been produced, (see [2]). Although many references deal with the study of option pricing with dividend payments — see [3–6] and references therein — many questions remain unclear.

This paper deals with the construction of numerical solutions of a modified Black–Scholes equation of the type

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (rS - D_{\delta}(S)) \frac{\partial V}{\partial S} - rV = 0,$$

$$V(S, T) = f(S), \quad 0 < S < \infty, \quad 0 < t < T.$$  \tag{3}

Let us denote times $t^-_d$ and $t^+_d$ just before and just after the dividend payment respectively. In order to guarantee that $S(t^-_d)$ is not negative, a general realistic discrete dividend yield verifies

$$\int_0^{S(t^-_d)} \frac{dS}{D_{\delta}(S)},$$

is infinite for any positive value of $S(t^-_d)$, see [1, p. 142].

This paper is organized as follows. Section 2 deals with preliminary results about the solution of the Black–Scholes equations without dividend payment as well as the approximation of the generalized function $\delta(t - t_d)$ by means of ordinary functions sequence $g_n(t)$. Also, included is a set of previous results in the matrix calculus.

Section 3 provides the numerical solution of the approximate problem

$$\begin{aligned}
\frac{\partial V_n}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_n}{\partial S^2} &+ (rS - D_{\delta}(S) g_n(t)) \frac{\partial V_n}{\partial S} - rV_n = 0, \\
&0 < S < \infty, \quad 0 < t < T, \\
V_n(S, T) &= f(S), \quad 0 < S < \infty,
\end{aligned}$$  \tag{5}

by a semidiscretization technique.

The numerical solution obtained in $t^-_d$ taking limits as $n$ tends to infinity, is extended to the interval $[0, t_d)$ in Section 4. Then it is shown that the difference between the numerical solution for a spatial step $h$ and a step $\frac{h}{2}$ is $O(h^2)$ as $h \to 0$. Finally, in Section 5, some illustrative examples are included.

2. Preliminaries

For the sake of clarity in the presentation, we recall some notation and results about the solution of the Black–Scholes equation without dividend payments as well as the concept and properties of the Dirac delta generalized function.

For $\eta, \nu \in \mathbb{R}$, with $\eta < \nu$ we define the set $\mathcal{M}(\eta, \nu)$ as follows:

$$\mathcal{M}(\eta, \nu) = \left\{ f : (0, \infty) \to \mathbb{R} \left| \int_0^{\infty} x^{a-1} |f(x)| dx < \infty, \eta < \alpha < \nu \right. \right\}.$$  \tag{7}

If there exists $\eta < \nu$ such that $f \in \mathcal{M}(\eta, \nu)$, then the solution of the Black–Scholes equation

$$\begin{aligned}
\frac{\partial V_{BS}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_{BS}}{\partial S^2} + rS \frac{\partial V_{BS}}{\partial S} - rV_{BS} &= 0, \\
V_{BS}(S, T) &= f(S), \quad 0 < S < \infty, \quad 0 < t < T,
\end{aligned}$$  \tag{6}

is given by

$$V_{BS}(S, t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} f(e^{-l}) e^{-\left[ \frac{\ln(S\exp(T-t)) - \frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{2(T-t)}} \right]^2} dl.$$  \tag{7}

See [7] for further details.

We denote by $K$ the space of functions $\varphi : \mathbb{R} \to \mathbb{R}$ in $C^\infty(\mathbb{R})$ having a compact support. A generalized function $g$ is defined as a continuous linear functional on $K$, and we denote $g(\varphi) = (g, \varphi)$, (see [8, p. 11]). The space of all generalized functions on $K$ will be called $K'$. The Dirac delta function is defined as the generalized function which
assigns value ϕ(0) to each function ϕ(x) ∈ K, i.e., (δ, ϕ) = ϕ(0). Note that the shifted Dirac delta function δ(t − td) acts on K in the form (δ(t − td), ϕ(t)) = ϕ(td), see [8, p. 11–13].

A sequence of ordinary functions \( \{g_n(t)\} \) converges in \( K' \) to the generalized function \( g \) if for all \( ϕ \in K \) (see [8, p. 63]),

\[
(g, ϕ) = \lim_{n \to \infty} (g_n, ϕ) = \lim_{n \to \infty} \int_{-\infty}^{+\infty} g_n(t)ϕ(t)dt.
\]

**Definition 2.1.** A sequence of ordinary functions \( \{g_n(t)\} \) is said to be a very nice shifted delta-defining if for each \( n \geq 0 \), \( g_n(t) \) has support \( [td - \frac{1}{2n}, td + \frac{1}{2n}] \) and is a continuous non-negative function in its support that verifies

\[
\int_{td - \frac{1}{2n}}^{td + \frac{1}{2n}} g_n(t)dt = 1.
\]

Taking into account [8, p. 65], a very nice shifted delta-defining sequence converges in \( K' \) to the generalized function \( δ(t − td) \). Concrete examples of these sequences may be found in [8, p. 66].

Throughout this paper, \( \|y\|_2 \) denotes the usual Euclidean norm of a vector \( y \). If \( A \) is a matrix in \( \mathbb{C}^{p \times p} \), its two-norm, denoted by \( \|A\| \) is defined as

\[
\|A\| = \max \left\{ +\sqrt{\lambda} / \lambda \in \sigma (A^H A) \right\},
\]

where \( A^H \) denotes the transconjugate of \( A \).

If \( A \) is a matrix in \( \mathbb{C}^{p \times p} \), then

\[
\|e^{tA}\| \leq e^{t \mu(A)}, \quad t \geq 0,
\]

where \( \mu(A) \) is the logarithmic norm of \( A \), defined by

\[
\mu(A) = \max \left\{ \lambda / \lambda \in \sigma \left( A + A^H 2 \right) \right\}.
\]

If \( A \) and \( B \) are matrices in \( \mathbb{C}^{p \times p} \), it holds that

\[
\mu(A + B) \leq \mu(A) + \mu(B), \tag{11}
\]

\[
|\mu(A)| \leq \|A\|, \tag{12}
\]

\[
\mu(c A) = |c| \mu(\text{sgn}(c)A), \quad c \in \mathbb{R}, \tag{13}
\]

See [9, p. 110], [10] for details.

By [9, p. 112], the solution of the linear system

\[
X'(t) = P(t)X(t) + b(t), \quad X(0) = X_0 \in \mathbb{C}^m, \quad t \geq 0,
\]

satisfies

\[
\|X(t)\| \leq \|X_0\| e^{\int_0^t \mu(P(s))ds} + \int_0^t e^{\int_0^s \mu(P(z))dz} \|b(v)\|_2 dv, \tag{14}
\]

where \( P(t) \) and \( b(t) \) are continuous functions taking values in \( \mathbb{C}^{m \times m} \) and \( \mathbb{C}^m \) respectively.
3. Numerical solution of the approximate problem

Let us consider problem (5) where \( \{g_n(t)\} \) is an arbitrary very nice shifted delta defining sequence. Taking into account Definition 2.1, for \( t_d + \frac{1}{2n} < t < T \) we take

\[
\begin{align*}
\frac{\partial V_n}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_n}{\partial S^2} + rS \frac{\partial V_n}{\partial S} - rV_n &= 0, \\
V_n(S, T) &= f(S), \quad 0 < S < \infty, \quad t_d + \frac{1}{2n} < t < T, \\
\end{align*}
\]

that is the Black–Scholes equation (6). Hence, solution of (5) in the interval \( [t_d + \frac{1}{2n}, T] \) is given by

\[ V_n(S, t) = V_{BS}(S, t), \quad t_d + \frac{1}{2n} \leq t < T. \] (15)

Then, problem (5) in the interval \( t_d + \frac{1}{2n} \leq t < t_d + \frac{1}{2n} \) can be written by

\[
\begin{align*}
\frac{\partial V_n}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_n}{\partial S^2} + (rS - D_b(S)g_n(t)) \frac{\partial V_n}{\partial S} - rV_n &= 0, \\
V_n\left(S, t_d + \frac{1}{2n}\right) &= V_{BS}\left(S, t_d + \frac{1}{2n}\right), \quad 0 < S < \infty, \quad t_d - \frac{1}{2n} \leq t < t_d + \frac{1}{2n}. \\
\end{align*}
\] (16)

We propose a semidiscretization method, see [11, p. 111], [12], for solving (16). Let us consider an interval \([a, b]\) and \( h \), an increment of \( S \), \( 0 < a \leq S \leq b \), where \( b - a = hN \), \( S_j = a + jh \), \( j = 0, 1, \ldots, N \). We introduce the notation \( v_{nj}(t) \approx V_n(S_j, t) \) for \( j = 0, 1, \ldots, N \). Then we replace the partial derivatives by finite expressions of the form

\[
\begin{align*}
\frac{\partial V_n(S, t)}{\partial S} &\approx \frac{v_{nj+1}(t) - v_{nj-1}(t)}{2h}, \quad j = 1, \ldots, N-1, \\
\frac{\partial^2 V_n(S, t)}{\partial S^2} &\approx \frac{v_{nj+1}(t) - 2v_{nj}(t) + v_{nj-1}(t)}{h^2}, \quad j = 1, \ldots, N-1. \\
\end{align*}
\] (17) (18)

From (16)–(18), one gets

\[
\begin{align*}
\frac{dv_{nj}(t)}{dt} &= \alpha_j v_{nj-1}(t) + \beta_j v_{nj}(t) + \gamma_j v_{nj+1}(t) - \frac{d_j}{2h} (v_{nj-1}(t) - v_{nj+1}(t)), \\
& \quad j = 1, \ldots, N-1. \\
\end{align*}
\] (19)

where

\[
\begin{align*}
d_j &= D_b(S_j), \\
\alpha_j &= -\frac{1}{2} \sigma^2 S_j^2 + rS_j \frac{1}{2h}, \\
\beta_j &= \frac{\sigma^2 S_j^2}{h^2} + r, \\
\gamma_j &= -\frac{1}{2} \sigma^2 S_j^2 h^2 - rS_j \frac{1}{2h}. \\
\end{align*}
\] (20)

In order to link the boundary values \( v_{n0}(t) \) and \( v_{nN}(t) \) with the rest of the solution, let us assume a quadratic approximation given by interpolation Lagrange polynomial of second degree for obtain auxiliary values \( v_{n-1}(t) \) and \( v_{nN+1}(t) \).

Taking into account

\[
P(S) = \frac{(S - S_1)(S - S_2)}{(S_0 - S_1)(S_0 - S_2)} v_{n0}(t) + \frac{(S - S_0)(S - S_2)}{(S_1 - S_0)(S_1 - S_2)} v_{n1}(t) + \frac{(S - S_0)(S - S_1)}{(S_2 - S_0)(S_2 - S_1)} v_{n2}(t),
\] (21)
one gets for $S_{-1} = S_0 - h$

$$v_{n-1}(t) = 3v_{n0}(t) - 3v_{n1}(t) + v_{n2}(t).$$

and assuming Eq. (19) for $j = 0$, it follows that

$$\frac{dv_{n0}(t)}{dt} = (3\alpha_0 + \beta_0) v_{n0}(t) + (-3\alpha_0 + \gamma_0) v_{n1}(t) + \alpha_0 v_{n2}(t) - \frac{d_0 g_n(t)}{2h} (3v_{n0}(t) - 4v_{n1}(t) + v_{n2}(t)). \quad (22)$$

where $d_0$, $\alpha_0$, $\beta_0$ and $\gamma_0$ are done by (20) for $j = 0$.

Similarly, one gets for $v_{nN}(t)$ the equation

$$\frac{dv_{nN}(t)}{dt} = \gamma_N v_{nN-2}(t) + (\alpha_N - 3\gamma_N) v_{nN-1}(t) + (\beta_N + 3\gamma_N) v_{nN}(t) - \frac{d_N g_n(t)}{2h} (-v_{nN-2}(t) + 4v_{nN-1}(t) - 3v_{nN}(t)), \quad (23)$$

where $d_N$, $\alpha_N$, $\beta_N$ and $\gamma_N$ are given by (20) for $j = N$.

Let us denote

$$v_n(t) = \begin{bmatrix} v_{n0}(t) \\ v_{n1}(t) \\ \vdots \\ v_{nN}(t) \end{bmatrix} \in \mathbb{R}^{(N+1) \times 1}. \quad (24)$$

Taking into account Eqs. (19), (22) and (23), one gets

$$\frac{dv_n(t)}{dt} = (M - g_n(t)B) v_n(t), \quad t_d - \frac{1}{2n} \leq t < t_d + \frac{1}{2n}, \quad (25)$$

where

$$M = \begin{bmatrix} 3\alpha_0 + \beta_0 & -3\alpha_0 + \gamma_0 & \alpha_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \alpha_2 & \beta_2 & \gamma_2 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & \alpha_{N-1} & \beta_{N-1} & \gamma_{N-1} & \alpha_N - 3\gamma_N \\
0 & 0 & 0 & 0 & \cdots & \gamma_N & \alpha_N - 3\gamma_N & \beta_N + 3\gamma_N & \end{bmatrix}, \quad (26)$$

$$B = \frac{1}{2h} \begin{bmatrix} 3d_0 & -4d_0 & d_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
d_1 & 0 & -d_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & d_2 & 0 & -d_2 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & d_{N-1} & 0 & -d_{N-1} & \end{bmatrix}. \quad (27)$$
Note that $M$ and $B$ depend on $S_j$ and $h$, but not on $t$. The solution vector $v_n(t)$ is attained for the final condition

$$v_n(t_d + \frac{1}{2n}) = \begin{bmatrix} V_{BS}(S_0, t_d + \frac{1}{2n}) \\ \vdots \\ V_{BS}(S_N, t_d + \frac{1}{2n}) \end{bmatrix} \in \mathbb{R}^{(N+1)\times 1}. \quad (28)$$

Problem (25)–(28) has not got an explicit formula for its solution if $M$ and $B$ are arbitrary non-commuting matrices. However, we are interested in obtaining, if it is possible, not the solution of this problem, but the limit

$$v(t_d^-) = \lim_{n \to \infty} v_n(t_d - \frac{1}{2n}). \quad (29)$$

Let us consider, first, the problem

$$\begin{align*}
\frac{d\tilde{v}_n(t)}{dt} &= -g_n(t) B \tilde{v}_n(t), \quad t_d - \frac{1}{2n} \leq t < t_d + \frac{1}{2n}; \\
\tilde{v}_n(t_d + \frac{1}{2n}) &= v_n(t_d + \frac{1}{2n}).
\end{align*} \quad (30)$$

The solution of (30) is given by

$$\tilde{v}_n(t) = e^{-\int_{t_d + \frac{1}{2n}}^{t} g_n(s) ds B} v_n(t_d + \frac{1}{2n}). \quad (31)$$

Taking into account Definition 2.1, one gets

$$\tilde{v}_n(t_d - \frac{1}{2n}) = e^{B} v_n(t_d + \frac{1}{2n}) \quad (32)$$

and hence

$$\tilde{v}(t_d^-) = \lim_{n \to \infty} \tilde{v}_n(t_d - \frac{1}{2n}) = e^{B} v_{i_d^+}, \quad (33)$$

where

$$v(t_d^+) = \begin{bmatrix} V_{BS}(S_0, t_d) \\ \vdots \\ V_{BS}(S_N, t_d) \end{bmatrix}. \quad (34)$$

We will show that

$$v(t_d^-) = \tilde{v}(t_d^-). \quad (35)$$

In order to prove (35), let us denote the error vector $\varphi_n(t)$ such that

$$v_n(t) = \tilde{v}_n(t) + \varphi_n(t). \quad (36)$$

It is easy to see that $\varphi_n(t)$ verifies the differential equation

$$\begin{align*}
\frac{d\varphi_n(t)}{dt} &= (M - g_n(t) B) \varphi_n(t) + M \tilde{v}_n(t), \\
\varphi_n(t_d + \frac{1}{2n}) &= 0, \quad t_d - \frac{1}{2n} \leq t < t_d + \frac{1}{2n};
\end{align*} \quad (37)$$

Taking into account the substitution $\tau = t_d + \frac{1}{2n} - t$ and denoting

$$\Psi_n(\tau) = \varphi_n(t_d + \frac{1}{2n} - \tau),$$
\[ h_n(\tau) = g_n \left( t_d + \frac{1}{2n} - \tau \right), \]

\[ \tilde{w}_n(\tau) = \tilde{v}_n \left( t_d + \frac{1}{2n} - \tau \right). \]

Problem (37) can be rewritten as
\[
\begin{align*}
\frac{d}{d\tau} \psi_n(\tau) &= (-M + h_n(\tau)B) \psi_n(\tau) - M \tilde{w}_n(\tau), \\
\psi_n(0) &= 0, \quad 0 \leq \tau \leq \frac{1}{n}.
\end{align*}
\] (38)

From (14) and (38), it follows that
\[
\| \psi_n(\tau) \|_2 \leq \int_0^\tau e^\int_s^\tau \mu(-M + h_n(z)B) dz \| M \| \| \tilde{w}_n(s) \|_2 ds.
\] (39)

By (11) and (13) and taking into account Definition 2.1 and (37), one gets
\[
\left\| \psi_n \left( \frac{1}{n} \right) \right\|_2 \leq e^{\mu(-M) + \| \mu(B) \|} \int_0^\frac{1}{n} \left\| \tilde{w}_n(s) \right\|_2 ds.
\] (40)

From (12), (31), (37) and (40), it follows that
\[
\left\| \psi_n \left( \frac{1}{n} \right) \right\|_2 \leq k \| M \| e^{\frac{\| M \|}{n} + \| B \|} \int_0^\frac{1}{n} e^{\mu(B) ds} ds,
\]
where
\[ k = \max_{t_d \leq t \leq T} \{ \| V_{BS}(S_0, t), \ldots, V_{BS}(S_N, t) \| \}. \]

Finally, taking into account (9) and (13), one gets
\[
\left\| \psi_n \left( \frac{1}{n} \right) \right\|_2 \leq k \| M \| e^{\frac{\| M \|}{n} + \| B \|} \int_0^\frac{1}{n} e^{\mu(B)} ds \leq k \| M \| e^{\| M \| + 2\| B \|} \frac{1}{n}.
\]

We have shown that
\[
\lim_{n \to \infty} \left\| \psi_n \left( \frac{1}{n} \right) \right\|_2 = 0,
\]
and by (33), (36) and (37), one gets
\[
\lim_{n \to \infty} \left\| v_n \left( t_d - \frac{1}{2n} \right) - e^B v(t_d^+) \right\|_2 \leq \lim_{n \to \infty} \left\| v_n \left( t_d - \frac{1}{2n} \right) - \tilde{v}_n \left( t_d - \frac{1}{2n} \right) \right\|_2 + \lim_{n \to \infty} \left\| \tilde{v}_n \left( t_d - \frac{1}{2n} \right) - e^B v(t_d^+) \right\|_2 = 0.
\]

Then the following result has been established.

**Theorem 3.1.** Let \( \{g_n(t)\} \) be a very nice shifted delta defining sequence done by Definition 2.1. Then the sequence of solutions \( \left\{ v_n \left( t_d - \frac{1}{2n} \right) \right\} \) of the problems (25)–(28) converges to
\[
v(t_d^+) = e^B v(t_d^+),
\]
where \( B \) and \( v(t_d^+) \) are defined by (27) and (34) respectively.
4. Prolongation of the solution and stability

In order to construct the numerical solution of (3) and (4) in the interval \([0, t_d]\), the semidiscretization method proposed in Section 3 is also appropriated. Taking into account (41) and the differential system (25) for \(g_n(t) = 0\), one gets

\[
\begin{align*}
\frac{dv(t)}{dt} &= M v(t), \quad 0 \leq t < t_d, \\
v(t_d) &= e^B v(t_d^+),
\end{align*}
\]  

(42)

where \(M\) is defined by (26). Solving (42) it follows that

\[
v(t) = e^{-M(t_d-t)} e^B v(t_d^+), \quad 0 \leq t < t_d.
\]  

(43)

Since the exact solution of problem (3) and (4) is not known for the case of nonconstant dividend yield, a measure of the stability of the numerical solution provided here can be given by studying its variation as the step size for discretization \(h\) changes. The difference between the numerical solution applying a semidiscretization technique with step \(h\) and the solution with \(\frac{h}{2}\) is \(O(h^2)\) as \(h \to 0\), as we are going to show.

Let us consider two partitions of the interval \([a, b]\) in order to study the influence of \(h\) on the approximate problem (16)

\[P_1 = \{S_0 = a, S_1 = S_0 + h, \ldots, S_j = S_0 + jh, \ldots, S_N = S_0 + Nh = b\},\]

with \(N + 1\) nodes and an \(h\) step size and

\[P_2 = \left\{S_0 = a, S_{\frac{1}{2}} = S_0 + \frac{h}{2}, S_1 = S_0 + h, \ldots, S_j = S_0 + jh, \ldots, S_N = S_0 + Nh = b\right\},\]

with \(2N + 1\) nodes and a \(\frac{h}{2}\) step size. Let us write Eq. (19) in the form

\[
\frac{dv_{nj}(t)}{dt} = f_1(h, S_j, S_{j-1}, S_{j+1}, v_{nj-1}(t), v_{nj}(t), v_{nj+1}(t)) + \frac{d_j g_n(t)}{2h} \left(v_{nj+1}(t) - v_{nj-1}(t)\right),
\]  

(44)

where \(f_1\) involves the three first terms of the second member of (19) corresponding with the discretization defined by \(P_1\). Similarly, for an increment of \(S\) equals to \(\frac{h}{2}\) and a partition \(P_2\), the solution \(\varphi_{nj}(t)\) satisfies the equation

\[
\frac{d\varphi_{nj}(t)}{dt} = f_2(h, S_j, S_{j-\frac{1}{2}}, S_{j+\frac{1}{2}}, \varphi_{nj-\frac{1}{2}}(t), \varphi_{nj}(t), \varphi_{nj+\frac{1}{2}}(t)) + \frac{d_j g_n(t)}{h} \left(\varphi_{nj+\frac{1}{2}}(t) - \varphi_{nj-\frac{1}{2}}(t)\right).
\]  

(45)

The objective is to estimate the difference between both solutions \(v_{nj}(t)\) and \(\varphi_{nj}(t)\) when \(n\) tends to infinity, denoted by \(v_j(t)\) and \(\varphi_j(t)\) respectively.

Let us denote \(F = f_1 - f_2\), and taking into account (44) and (45) it follows that

\[
\frac{d}{dt} \left(v_{nj}(t) - \varphi_{nj}(t)\right) = F + \frac{d_j g_n(t)}{2h} \left(v_{nj+1}(t) - 2\varphi_{nj+\frac{1}{2}}(t) + 2\varphi_{nj-\frac{1}{2}}(t) - v_{nj-1}(t)\right).
\]  

(46)

By integrating in the interval \(\left[t_d - \frac{1}{2n}, t_d + \frac{1}{2n}\right]\) one gets

\[
\int_{t_d - \frac{1}{2n}}^{t_d + \frac{1}{2n}} \frac{d}{dt} \left(v_{nj}(t) - \varphi_{nj}(t)\right) dt = v_{nj} \left(t_d - \frac{1}{2n}\right) - \varphi_{nj} \left(t_d + \frac{1}{2n}\right),
\]

since \(v_{nj}(t_d + \frac{1}{2n}) = \varphi_{nj}(t_d + \frac{1}{2n}) = V_{BS}(S_j, t_d + \frac{1}{2n})\). On the other hand, let us take \(M(h) > 0\) and a positive integer \(n_0\) such that

\[|F| \leq M(h) \forall t \in \left[t_d - \frac{1}{2n}, t_d + \frac{1}{2n}\right] \forall n \geq n_0, \quad j = 0, \frac{1}{2}, 1, \ldots, N - \frac{1}{2}, N.\]
Hence, it follows that

$$\left| v_{nj} \left( t_d - \frac{1}{2n} \right) - \varphi_{nj} \left( t_d - \frac{1}{2n} \right) \right| \leq \frac{M(h)}{n} + \frac{|d_j|}{2h} \int_{t_d + \frac{1}{2n}}^{t_d - \frac{1}{2n}} g_n(t) \left( v_{nj+1}(t) - 2\varphi_{nj+\frac{1}{2}}(t) + 2\varphi_{nj-\frac{1}{2}}(t) - v_{nj-1}(t) \right) dt.$$  \hspace{1cm} (47)

In order to estimate the integral in (47), we consider the second order Taylor’s expansion of the solution \( V_n(S, t) \) with respect to the variable \( S \) about \( S = S_j \).

$$\begin{align*}
v_{nj+1} &= V_n + V_n' h_{nj} + \frac{h}{2} V_n'' h_{nj} + \frac{h^3}{6} V_n''' h V_{nj} (z_1), \quad S_j < z_1 < S_j + h, \\
v_{nj+1} &= V_n - h V_n' h_{nj} + \frac{h^2}{2} V_n'' h_{nj} - \frac{h^3}{6} V_n''' h V_{nj} (z_2), \quad S_j - h < z_2 < S_j, \\
\varphi_{nj+\frac{1}{2}} &= V_n + h V_n' h_{nj} + \frac{h^2}{2} V_n'' h_{nj} + \frac{1}{48} h^3 V_n''' h V_{nj} (z_3), \quad S_j < z_3 < S_j + \frac{h}{2}, \\
\varphi_{nj-\frac{1}{2}} &= V_n - h V_n' h_{nj} + \frac{h^2}{2} V_n'' h_{nj} - \frac{1}{48} h^3 V_n''' h V_{nj} (z_4), \quad S_j - \frac{h}{2} < z_4 < S_j,
\end{align*}$$  \hspace{1cm} (48)

where \( V_n = V(S_j, t) \) and \( G' \) denote the partial derivatives of \( G \) with respect to \( S \). Taking into account (48), one gets

$$\begin{align*}
v_{nj+1} - 2v_{nj+\frac{1}{2}} + 2v_{nj-\frac{1}{2}} - v_{nj-1} &= \frac{h^3}{3} \left( V_n''' (z_1) + V_n''' (z_2) - \frac{1}{4} V_n''' (z_3) - \frac{1}{4} V_n''' (z_4) \right) \\
&= \frac{h^3}{3} \left( V_n''' (z) - \frac{1}{4} V''' (y) \right),
\end{align*}$$  \hspace{1cm} (49)

for some \( z \in (S_j - h, S_j + h), y \in (S_j - \frac{h}{2}, S_j + \frac{h}{2}). \)

Let \( m > 0 \) be such that \( |V_n''' (S, t)| < m \) for all \( t \) in \( \left( t_d - \frac{1}{2n}, t_d + \frac{1}{2n} \right) \), \( n \geq n_0, S \in [a, b] \), for a given positive integer \( n_0 \). Then by (47) and (49), one gets

$$\left| v_{nj} \left( t_d - \frac{1}{2n} \right) - \varphi_{nj} \left( t_d - \frac{1}{2n} \right) \right| \leq \frac{M(h)}{n} + \frac{5}{24} |d_j| h^2 m,$$  \hspace{1cm} (50)

where one uses that \( \int_{t_d - \frac{1}{2n}}^{t_d + \frac{1}{2n}} g_n(t) dt = 1 \), and Definition 2.1 because the sequence of functions \( g_n(t) \) is very nice shifted delta-defining. Taking limits for (50) as \( n \to \infty \), one gets

$$\left| v_j(t_d) - \varphi_j(t_d) \right| \leq C_1 h^2, \quad C_1 = \frac{5}{24} |d_j| m.$$  \hspace{1cm} (51)

In order to study the influence of \( h \) for each \( t \) in the interval \( [0, t_d] \), we also consider the partitions \( P_1 \) and \( P_2 \) for the interval \([a, b]\) with \( h \) and \( \frac{h}{2} \) step sizes respectively. Let us consider

$$\frac{dv_j(t)}{dt} = \alpha_j v_{j-1}(t) + \beta_j v_j(t) + \gamma_j v_{j+1}(t),$$  \hspace{1cm} (52)

where

$$\alpha_j = -K_j \frac{1}{h^2} + q_j \frac{1}{h}, \quad \beta_j = 2K_j \frac{1}{h^2} + r, \quad \gamma_j = -K_j \frac{1}{h^2} - q_j \frac{1}{h},$$

and \( K_j = \frac{1}{2} \sigma^2 S_j^2, q_j = \frac{rS_j^2}{h^2}, \) for the partition \( P_1 \), and

$$\frac{dv_j(t)}{dt} = \alpha'_j v_{j-1}(t) + \beta'_j v_j(t) + \gamma'_j v_{j+1}(t),$$  \hspace{1cm} (53)
where

\[ \alpha_j' = -4K_j \frac{1}{h^2} + 2q_j \frac{1}{h}, \quad \beta_j' = 8K_j \frac{1}{h^2} + r, \quad \gamma_j' = -4K_j \frac{1}{h^2} - 2q_j \frac{1}{h}, \]

for the partition \( P_2 \).

From (52) and (53), it follows that

\[
\begin{aligned}
\frac{d}{dt} (v_j(t) - \varphi_j(t)) &= -K_j \frac{1}{h^2} \left( v_{j-1}(t) + v_{j+1}(t) - 2v_j(t) - 4\varphi_{j+\frac{1}{2}}(t) + 8\varphi_j(t) - 4\varphi_{j-\frac{1}{2}}(t) \right) \\
&+ q_j \frac{1}{h} \left( v_{j-1}(t) - v_{j+1}(t) - 2\varphi_{j-\frac{1}{2}}(t) + 2\varphi_{j+\frac{1}{2}}(t) \right) + r \left( v_j(t) - \varphi_j(t) \right)
\end{aligned}
\]  

(54)

By considering the third order Taylor’s expansion of \( V(S, t) \) about \( S = S_j \), one gets

\[
\begin{aligned}
v_{j+1} &= v_j + hV_j' + \frac{h^2}{2}V_j'' + \frac{h^3}{6}V_j''' + \frac{h^4}{24}V_j^{(4)}(z_1), \quad S_j < z_1 < S_j + h, \\
v_{j-1} &= v_j - hV_j' + \frac{h^2}{2}V_j'' - \frac{h^3}{6}V_j''' + \frac{h^4}{24}V_j^{(4)}(z_2), \quad S_j - h < z_2 < S_j, \\
\varphi_{j+\frac{1}{2}} &= \varphi_j + \frac{h}{2}V_j' + \frac{h^2}{8}V_j'' + \frac{h^3}{48}V_j''' + \frac{h^4}{384}V_j^{(4)}(z_3), \quad S_j < z_3 < S_j + \frac{h}{2}, \\
\varphi_{j-\frac{1}{2}} &= \varphi_j - \frac{h}{2}V_j' + \frac{h^2}{8}V_j'' - \frac{h^3}{48}V_j''' - \frac{h^4}{384}V_j^{(4)}(z_4), \quad S_j - \frac{h}{2} < z_4 < S_j.
\end{aligned}
\]  

(55)

Taking into account expansions (48) and (55) for the second and the first brackets respectively in the right member of (54), one gets

\[
\begin{aligned}
\frac{d}{dt} (v_j(t) - \varphi_j(t)) &= -K_j \frac{1}{h^2} \left( V^{(4)}(z) - \frac{1}{4}V^{(4)}(y) \right) \\
&- \frac{q_j}{3} \frac{1}{h^2} \left( V^{(3)}(z') - \frac{1}{4}V^{(3)}(y') \right) + r(v_j(t) - \varphi_j(t))
\end{aligned}
\]  

(56)

for some \( z, z' \in (S_j - h, S_j + h) \) and \( y, y' \in (S_j - \frac{h}{2}, S_j + \frac{h}{2}) \).

Taking into account the first order Taylor’s expansion of \( V(S_j, t) \) with respect to \( t \) about \( t_d \), one gets

\[
\begin{aligned}
v_j(t) &= v_j(t_d) - \Delta t \frac{\partial V}{\partial t}(S_j, t) \bigg|_{t=t_d}, \\
\varphi_j(t) &= \varphi_j(t_d) - \Delta t \frac{\partial V}{\partial t}(S_j, t) \bigg|_{t=t_d}, \\
\Delta t &= t_d - t > 0, \quad t < t_i < t_d, \quad i = 1, 2.
\end{aligned}
\]  

(57)

By integrating (56) over the interval \([t, t_d]\) and taking into account (57), one gets

\[
|v_j(t) - \varphi_j(t) - (v_j(t_d) - \varphi_j(t_d))| \leq C_2 h^2 \Delta t + r |v_j(t_d) - \varphi_j(t_d)| \Delta t + r C_3 (\Delta t)^2,
\]

where \( C_2 \) satisfies

\[
C_2 \geq \frac{5}{4} \max \left\{ \frac{1}{12} K_j |V^{(4)}(S, \tau)| + \frac{1}{3} q_j |V^{(3)}(S, \tau)| \right\}, \quad S \in [a, b], \tau \in [t, t_d].
\]

and \( C_3 \geq \max \left\{ \frac{\partial V}{\partial t}(S, \tau), S \in [a, b], \tau \in [t, t_d]. \right\} \). Finally taking into account (51), it holds that \(|v_j(t) - \varphi_j(t)| \leq C_4 h^2\), where \( C_4 = C_2 \Delta t + C_1 (1 + r \Delta t) + C_3 r (\Delta t)^2 \). Thus we have proved that

\[
\begin{aligned}
v_j(t) &= \varphi_j(t) + O(h^2) \\
0 &\leq t_d
\end{aligned}
\]  


5. Examples

The following example compares the exact solution of the valuation of a vanilla call option with a constant yield discrete dividend payment with the numerical solution constructed using the previous approach.

Example 5.1. Let us consider the valuation problem of a call option with a discrete dividend, modeled by (3) and (4), where

\[ D_δ(S) = A S, \]  
and

\[ f(S) = \max\{S - E, 0\}, \quad 0 < S < \infty. \]  

In this case, the discrete dividend payment has got a constant dividend yield \( A \). The solution of (3), (4), (58) and (59) is given by (see [2])

\[ V(S, t) = \begin{cases} 
  S e^{-A N(d'_1)} - E e^{-r(T-t)} N(d'_2), & 0 \leq t < t_d, \\
  S N(d_1) - E e^{-r(T-t)} N(d_2), & t_d < t < T,
\end{cases} \]  

where

\[ d_1 = \frac{1}{\sigma \sqrt{T-t}} \left[ \ln \frac{S}{E} + (T-t) \left( r + \frac{\sigma^2}{2} \right) \right], \]
\[ d_2 = \frac{1}{\sigma \sqrt{T-t}} \left[ \ln \frac{S}{E} + (T-t) \left( r - \frac{\sigma^2}{2} \right) \right], \]
\[ d'_i = d_i - \frac{A}{\sigma \sqrt{T-t}} \quad i = 1, 2 \]

and

\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}} d\xi, \]

is the cumulative probability function for a standardized normal variable.

Note that (60) can be written in the form

\[ V(S, t) = \begin{cases} 
  V_{BS}(S e^{-A}, t), & 0 \leq t < t_d, \\
  V_{BS}(S, t), & t_d < t < T.
\end{cases} \]

where \( V_{BS}(S, t) = SN(d_1) - E e^{-r(T-t)} N(d_2) \).

For \( A = 0.1, \sigma = 0.1, r = 0.08, T = 1, t_d = 0.5, E = 7, S_0 = 3, N = 34 \) and \( h = 0.5 \), one gets for the value at \( t = 0 \) the mean error

\[ \frac{\| v(0) - V(0) \|}{N + 1} = 0.000459, \]

where \( v(0) \) is obtained by using (43) and \( V(0) \) is the vector of the exact solutions given by (45) evaluating at grid points \( S_i, 0 \leq i \leq 34 \), and time \( t = 0 \). The blue line, denoted by dividend type C, in Fig. 1 is the numerical valuation of this call option at \( t = 0 \) for \( N = 100 \).

The following examples deal with the application of the numerical method to variable yield discrete dividend payment cases.

Example 5.2. Let us consider the valuation problem of a vanilla call option modeled by (3) and (4), where

\[ D_δ(S) = A S^2, \]
and

\[ f(S) = \max\{S - E, 0\}, \quad 0 < S < \infty. \]
Fig. 1. Call options with strike price $E$ and different dividend payments.

Fig. 2. Different spatial semidiscretizations with Dividend Type A.

For $A = 0.01$, $\sigma = 0.1$, $r = 0.08$, $T = 1$, $t_d = 0.5$, $E = 7$, $S_0 = 3$, $N = 100$ and $h = 0.5$, by (43) one gets the numerical solution $v(0)$ — see the green line in Fig. 1, denoted by dividend type $A$.

Let us denote the numerical valuation of the call option at $t = 0$ by $C(S, t = 0, h)$, when a step size $h$ is used. In order to estimate the stability of the method, we show the difference $C(S, t = 0, h/2) - C(S, t = 0, h)$, for various values of $h$ for the data of this example; see Fig. 2.

**Example 5.3.** Let us consider the valuation problem of a vanilla call option modeled by (3) and (4), where

$$D_3(S) = A S^3,$$

and

$$f(S) = \max\{S - E, 0\}, \quad 0 < S < \infty.$$  

For $A = 0.001$, $\sigma = 0.1$, $r = 0.08$, $T = 1$, $t_d = 0.5$, $E = 7$, $S_0 = 3$, $N = 100$ and $h = 0.5$, from (43) one gets the numerical solution $v(0)$; see the red line in Fig. 1, denoted by dividend type $B$.

The numerical methods used by agents to value derivatives are the binomial method and Monte Carlo simulations. In the following example, we compare the semidiscretization technique with the binomial method described in [5] for a particular case where the dividend is a constant proportion of $S$, see the Example 5.1.
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**Fig. 3.** $Y = C(S, t = 0, \text{semidiscretization}) - C(S, t = 0, \text{binomial})$ for different values of $A$.

**Fig. 4.** $F_1(S, 0) - C(S, t = 0, \text{binomial})$ and $F_2(S, 0) - C(S, t = 0, \text{binomial})$ for $A = 0.05$.

**Example 5.4.** Let us consider the valuation problem of a vanilla call option with constant discrete dividend yield $A$ and the parameters $\sigma = 0.1$, $r = 0.04$, $T = 1$, $t_d = 0.5$, $E = 7$, $S_0 = 3$, $h = 0.25$, $N = 200$. Several values of $A$ with $n = 100$ steps for the binomial method are considered.

Fig. 3 shows the difference between the solutions given by semidiscretization technique and the binomial method. Note that the difference increases as the yield $A$ and the underlying asset $S$ increase. Let us denote the exact solution of the problem (3)–(4)–(58)–(59) by the expression $F_1(S, t)$ for $0 \leq t < t_d$, i.e. $F_1(S, t) = V_{BS}(Se^{-A}, t)$, and consider $F_2(S, t) = V_{BS}(S(1 - A), t)$. Fig. 4 shows the difference between a binomial solution and $F_i(S, t = 0)$, $i = 1, 2$, and suggests that the binomial numerical solution approaches $F_2(S, t)$ instead of the exact solution $F_1(S, t)$. This fact would explain also Fig. 3.

**References**