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Spectral Theory for a Differential Operator: Characteristic Determinant and Green's Function

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For a two-point differential operator L in $L^2[a, b]$, it is shown that the Green's function has the representation $G(t, s; \lambda) = H(t, s; \lambda)/D(\lambda)$ for λ belonging to the resolvent set $\rho(L)$, where $D(\lambda)$ is the characteristic determinant and $H(t, s; \lambda)$ is an entire function in the λ variable admitting a power series expansion about any point $\lambda_0 \in \mathbb{C}$. This representation is given several applications: first, to calculate the coefficient operators in the Laurent series for the resolvent $R_\lambda(L)$ about each point λ_0 in the spectrum $\sigma(L)$, and second, to relate the algebraic multiplicity $\nu(\lambda_0)$ of an eigenvalue λ_0 to the ascent m_0 of the operator $\lambda_0 I - L$.

1. INTRODUCTION

In developing the spectral theory of a two-point differential operator L in the Hilbert space $L^2[a, b]$, two key ingredients are the characteristic determinant and the Green's function (see [2, Chap. 12; 8, Chaps. I and II; 3, Chap. XIX]). The characteristic determinant $D(\lambda)$ completely specifies the spectrum $\sigma(L)$ by means of its zeros, while the Green's function $G(t, s; \lambda)$ is the L^2-kernel of the resolvent $R_\lambda(L)$ in its representation as an integral operator on $L^2[a, b]$. Together with the resolvent they determine much of the local and global structure of L.

For example, for any point $\lambda_0 \in \sigma(L)$: (i) the ascent m_0 of the operator $L_{\lambda_0} = \lambda_0 I - L$ is equal to the order of the pole of $R_\lambda(L)$ at λ_0, (ii) the algebraic multiplicity $\nu(\lambda_0) = \dim \mathcal{N}([L_{\lambda_0}]^{m_0})$ is identical to the order n_0 of

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λ_0 as a zero of $D(\lambda)$, where $\mathcal{N}([L_{\lambda_0}]^{m_0})$ denotes the null space of the product operator $[L_{\lambda_0}]^{m_0}$, i.e., the generalized eigenspace of λ_0, and (iii) the projection P_0 from $L^2[a, b]$ onto the null space $\mathcal{N}([L_{\lambda_0}]^{m_0})$ along the range $\mathcal{R}([L_{\lambda_0}]^{m_0})$ is given by

$$P_0 = \frac{1}{2\pi i} \int_\Gamma R_\lambda(L)\, d\lambda, \tag{1.1}$$

where Γ is any circle about λ_0 lying in the resolvent set $\rho(L)$ with λ_0 the only point of $\sigma(L)$ inside Γ. With regards to the global properties of L, $D(\lambda)$ and $G(t, s; \lambda)$ can be used to derive decay rates for $R_\lambda(L)$ as $\lambda \to \infty$, and in many cases this establishes the denseness of the generalized eigenspaces in $L^2[a, b]$. Also, they provide the tools needed to study the uniform boundedness (or unboundedness) of the family of all finite sums of the projections onto the generalized eigenspaces.

The purpose of this paper is to represent the Green's function in the form

$$G(t, s; \lambda) = \frac{H(t, s; \lambda)}{D(\lambda)}, \qquad \lambda \in \rho(L), \tag{1.2}$$

where H is an entire function in the λ variable admitting a power series expansion about any point $\lambda_0 \in \mathbb{C}$. This power series is quite simple to construct when one uses operator theory and works in the Sobolev space $H^n[a, b]$. Several applications of the representation (1.2) are then given. First, for any point $\lambda_0 \in \sigma(L)$ we calculate the coefficient operators in the Laurent series for $R_\lambda(L)$ about the point λ_0, expressing each one as an integral operator with L^2-kernel formulated in terms of $D(\lambda)$ and $H(t, s; \lambda)$. Included among these results is a useful representation of the projection P_0, namely

$$P_0 x(t) = \frac{1}{(n_0 - 1)!} \int_a^b \frac{\partial^{n_0 - 1}}{\partial \lambda^{n_0 - 1}} \left[\frac{H(t, s; \lambda)}{h(\lambda)} \right]_{\lambda = \lambda_0} x(s)\, ds, \qquad a \leqslant t \leqslant b, \tag{1.3}$$

for all $x \in L^2[a, b]$, where $D(\lambda) = (\lambda - \lambda_0)^{n_0} h(\lambda)$. Second, for each $\lambda_0 \in \sigma(L)$ we establish a fundamental relationship between the ascent m_0 and the algebraic multiplicity $\nu(\lambda_0)$:

$$\nu(\lambda_0) = n_0 = m_0 + p_0, \tag{1.4}$$

where p_0 is the order of λ_0 as a zero of the function $H(t, s; \lambda)$.

In the next section we list the basic definitions and background material used in the paper. The power series expansions in $H^n[a, b]$ are discussed in Section 3, and then applied in Section 4 to the characteristic determinant $D(\lambda)$ and in Section 5 to the Green's function $G(t, s; \lambda)$, where the representation (1.2) is studied in detail (see (5.11)). Section 6 contains the characterization of the coefficient operators in the Laurent series for $R_\lambda(L)$ about any point $\lambda_0 \in \sigma(L)$, with (1.3) as a special case (see (6.9)). The relationship (1.4) is established in Section 7 (see (7.5)).

In a future paper [6] we will combine the results of this paper with those in [4] to give a comprehensive study of the spectral theory of all differential operators in $L^2[a, b]$ determined by $\tau = -(d/dt)^2$ and all possible pairs of independent boundary conditions. This new work will encompass all the regular and irregular boundary conditions, including both spectral and nonspectral operators and both simple and multiple eigenvalue cases. Equations (1.2), (1.3), and (1.4) will play a major role in this work.

2. Basic Definitions and Preliminaries

In the complex Hilbert space $L^2[a, b]$ let $(\ ,\ )$ and $\|\ \|$ denote the standard inner product and norm. Let $H^n[a, b]$ denote the Sobolev space consisting of all functions $u \in C^{n-1}[a, b]$ with $u^{(n-1)}$ absolutely continuous on $[a, b]$ and $u^{(n)} \in L^2[a, b]$. On $H^n[a, b]$ we introduce the H^n-norm

$$|u|_{H^n} = \sum_{i=0}^{n-1} \max_{a \leqslant t \leqslant b} |u^{(i)}(t)| + \|u^{(n)}\|,$$

under which $H^n[a, b]$ becomes a Banach space; this norm is equivalent to the standard Sobolev norm. Let

$$\tau = \sum_{i=0}^{n} a_i(t) \left(\frac{d}{dt}\right)^i$$

be an nth-order formal differential operator on $[a, b]$, where the coefficients $a_i \in H^i[a, b]$ for $i = 0, 1, \ldots, n$ and $a_n(t) \neq 0$ on $[a, b]$; let

$$B_i(u) = \sum_{j=0}^{n-1} \alpha_{ij} u^{(j)}(a) + \sum_{j=0}^{n-1} \beta_{ij} u^{(j)}(b), \qquad i = 1, \ldots, n,$$

be a set of n linearly independent boundary values on $H^n[a, b]$; let L be the two-point differential operator in $L^2[a, b]$ defined by

$$\mathscr{D}(L) = \{u \in H^n[a, b] \mid B_i(u) = 0,\ i = 1, \ldots, n\}, \qquad Lu = \tau u.$$

It is well known that L is a Fredholm operator of index 0 in $L^2[a, b]$ whose Fredholm set is all of $\mathbb{C}$.

Throughout this paper we assume that $\rho(L) \neq \varnothing$, and hence, $\sigma(L)$ is a countable subset of $\mathbb{C}$ having no finite limit points. The points of $\sigma(L)$ are all eigenvalues of L, and for each $\lambda_0 \in \sigma(L)$ the operator $L_{\lambda_0} = \lambda_0 I - L$ has a finite ascent m_0. Introducing the notation

$$L(\lambda_0) = (\lambda_0 I - L)^{m_0},$$

the null space $\mathcal{N}(L(\lambda_0))$ is just the generalized eigenspace corresponding to the eigenvalue $\lambda_0 \in \sigma(L)$, and its dimension $\nu(\lambda_0)$ is the algebraic multiplicity of λ_0.

As we study the characteristic determinant $D(\lambda)$ and the Green's function $G(t, s; \lambda)$ in the following sections, we will utilize two Banach spaces of operators. First, let $\mathscr{B}(L^2[a, b])$ denote the Banach algebra of all bounded linear operators on $L^2[a, b]$ with the uniform operator norm

$$\|R\| = \sup_{\|u\| = 1} \|Ru\|.$$

The standard results in the spectral theory of the differential operator L are usually stated in terms of the space $\mathscr{B}(L^2[a, b])$, e.g., the representation (1.1) of the projection P_0 and the Laurent expansion of $R_\lambda(L)$ about any point $\lambda_0 \in \sigma(L)$. Second, let X denote the Banach space of all bounded linear operators from $L^2[a, b]$ under the L^2-structure into $H^n[a, b]$ under the H^n-structure, with the uniform operator norm

$$\|S\|_{H^n} = \sup_{\|u\| = 1} |Su|_{H^n}.$$

The key to developing the fine structure of $D(\lambda)$ and $G(t, s; \lambda)$ is to work with operators in the space X instead of $\mathscr{B}(L^2[a, b])$.

3. Expansions in the Sobolev Space $H^n[a, b]$

Fix a point $\lambda_0 \in \mathbb{C}$, and suppose $c_0, c_1, ..., c_{n-1}$ are given complex numbers. For each $\lambda \in \mathbb{C}$ let $\psi_\lambda = \psi(\cdot; \lambda)$ be the unique function in $H^n[a, b]$ satisfying the initial value problem

$$\begin{gathered} (\lambda I - \tau)\, \psi_\lambda = 0, \\ \psi_\lambda^{(j)}(a) = c_j, \qquad j = 0, 1, ..., n-1, \end{gathered} \tag{3.1}$$

and then set $\psi = \psi_{\lambda_0}$. Working in the space $H^n[a, b]$, we want to expand the function ψ_λ in a power series about λ_0. These series expansions will play a central role in the sequel.

With this in mind we introduce the differential operator T in $L^2[a, b]$ defined by

$$\mathscr{D}(T) = \{u \in H^n[a, b] \mid u^{(i)}(a) = 0,\ i = 0, 1, ..., n-1\}, \qquad Tu = \tau u,$$

which has $\rho(T)=\mathbb{C}$ and $\sigma(T)=\varnothing$. Note that $\psi_\lambda-\psi\in\mathscr{D}(T)$, and

$$T_\lambda(\psi_\lambda-\psi)=-(\lambda I-\tau)\,\psi=(\lambda_0-\lambda)\,\psi,$$

so $\psi_\lambda-\psi=(\lambda_0-\lambda)\,R_\lambda(T)\,\psi$ and

$$\psi_\lambda=\psi+(\lambda_0-\lambda)\,R_\lambda(T)\,\psi \qquad \text{for all} \quad \lambda\in\mathbb{C}. \tag{3.2}$$

Setting $R=R_{\lambda_0}(T)\in\mathscr{B}(L^2[a,b])$, we have

$$R_\lambda(T)=\sum_{k=0}^{\infty}(\lambda_0-\lambda)^k\,R^{k+1} \qquad \text{in} \quad \mathscr{B}(L^2[a,b]) \tag{3.3}$$

for all $\lambda\in\mathbb{C}$ (see Lemma 3.3 in [5] or [11, p. 269]). Upon substituting (3.3) into (3.2), we obtain the expansion

$$\psi_\lambda=\sum_{k=0}^{\infty}(\lambda_0-\lambda)^k\,R^k\psi \qquad \text{in} \quad L^2[a,b] \tag{3.4}$$

for all $\lambda\in\mathbb{C}$.

Next, we show that (3.3) and (3.4) are also valid in X and $H^n[a,b]$, respectively, and that the convergence is uniform in the λ variable on compact subsets of $\mathbb{C}$. Indeed, take arbitrary constants δ_0 and δ_1 with $0<\delta_0<\delta_1$, and set $\theta=\delta_0/\delta_1$. Since R is a compact operator in $\mathscr{B}(L^2[a,b])$ and $\sigma(T)=\varnothing$, it follows that $\sigma(R)=\{0\}$ and the spectral radius $r(R)=0$, and hence, we can choose a positive integer k_0 such that $\|R^k\|^{1/k}\leqslant 1/\delta_1$ for all $k\geqslant k_0$. Clearly

$$\|(\lambda_0-\lambda)^k\,R^k\|\leqslant\frac{|\lambda_0-\lambda|^k}{\delta_1^k}\leqslant\theta^k$$

for all $|\lambda-\lambda_0|\leqslant\delta_0$ and for $k\geqslant k_0$. If we define $\gamma=\max_{0\leqslant k<k_0}\delta_1^k\|R^k\|\geqslant 1$, then we obtain the estimate

$$\|(\lambda_0-\lambda)^k\,R^k\|\leqslant\gamma\theta^k \tag{3.5}$$

for all $|\lambda-\lambda_0|\leqslant\delta_0$ and for $k=0,1,2,\ldots$.

The linear operators $R_\lambda(T)$ and R^{k+1}, $k=0,1,2,\ldots$, which appear in (3.3), are also elements of X, and from (3.5) it is immediate that

$$\|(\lambda_0-\lambda)^k\,R^{k+1}\|_{H^n}\leqslant\gamma\|R\|_{H^n}\theta^k \tag{3.6}$$

for all $|\lambda-\lambda_0|\leqslant\delta_0$ and for $k=0,1,2,\ldots$, which shows that the series on the right side of (3.3) converges in X to an operator $S_\lambda\in X$ for each $\lambda\in\mathbb{C}$. Applying this series to an arbitrary $u\in L^2[a,b]$ and using the fact that H^n-convergence implies L^2-convergence, we conclude that $R_\lambda(T)=S_\lambda$ for all $\lambda \in \mathbb{C}$, i.e., the expansion (3.3) is valid in X for all $\lambda \in \mathbb{C}$. In addition, (3.6) yields the estimates

$$\left\| R_\lambda(T) - \sum_{k=0}^{N} (\lambda_0 - \lambda)^k R^{k+1} \right\|_{H^n} \leqslant \frac{\gamma \|R\|_{H^n}}{1-\theta} \theta^{N+1} \tag{3.7}$$

and

$$\|R_\lambda(T)\|_{H^n} \leqslant \frac{\gamma \|R\|_{H^n}}{1-\theta} \tag{3.8}$$

for all $|\lambda - \lambda_0| \leqslant \delta_0$ and for $N = 0, 1, 2, \ldots$, thereby proving the convergence to be uniform in the λ variable on compact subsets of $\mathbb{C}$.

Finally, by the above we see that the expansion (3.4) is also valid in $H^n[a, b]$ for all $\lambda \in \mathbb{C}$, and from (3.2) and (3.7) we obtain the error estimate

$$\begin{aligned} \left| \psi_\lambda - \sum_{k=0}^{N} (\lambda_0 - \lambda)^k R^k \psi \right|_{H^n} &= |\lambda_0 - \lambda| \left| R_\lambda(T) \psi - \sum_{k=0}^{N-1} (\lambda_0 - \lambda)^k R^{k+1} \psi \right|_{H^n} \\ &\leqslant |\lambda_0 - \lambda| \, \|\psi\| \frac{\gamma \|R\|_{H^n}}{1-\theta} \theta^N \end{aligned} \tag{3.9}$$

for all $|\lambda - \lambda_0| \leqslant \delta_0$ and for $N = 0, 1, 2, \ldots$ (for $N = 0$ the estimate (3.7) is replaced by (3.8) in the argument). Also, since the boundary values B_i, $i = 1, \ldots, n$, are continuous linear functionals on $H^n[a, b]$ under the H^n-structure, we obtain the result

$$B_i(\psi_\lambda) = \sum_{k=0}^{\infty} (\lambda_0 - \lambda)^k B_i(R^k \psi), \qquad i = 1, \ldots, n, \tag{3.10}$$

for all $\lambda \in \mathbb{C}$, and hence, the functions $f_i(\lambda) = B_i(\psi_\lambda)$, $i = 1, \ldots, n$, are entire functions on $\mathbb{C}$.

Let us summarize these results as

THEOREM 3.1. *Let $\lambda_0 \in \mathbb{C}$, let $c_0, c_1, \ldots, c_{n-1}$ be complex numbers, and for each $\lambda \in \mathbb{C}$ let ψ_λ be the unique function in $H^n[a, b]$ satisfying the initial value problem $(\lambda I - \tau) \psi_\lambda = 0$, $\psi_\lambda^{(j)}(a) = c_j$ for $j = 0, 1, \ldots, n-1$. Then there exists a sequence of functions u_k, $k = 0, 1, 2, \ldots$, in $H^n[a, b]$ such that*

$$\psi_\lambda = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k u_k \qquad \text{in} \quad H^n[a, b] \tag{3.11}$$

for all $\lambda \in \mathbb{C}$, where the convergence is in the H^n-norm and is uniform in the λ variable on compact subsets of $\mathbb{C}$. Moreover, the functions $B_i(\psi_\lambda)$, $i = 1, \ldots, n$, are entire functions on $\mathbb{C}$.

Remark 3.2. A more classical result along the lines of Theorem 3.1 is given in [10, p. 699].

4. The Characteristic Determinant of L

Utilizing the power series expansions of Theorem 3.1, we next introduce the characteristic determinant of the differential operator L and use it to begin the study of the spectrum and the generalized eigenspaces of L. For each $\lambda \in \mathbb{C}$ and for $i = 1, ..., n$ let $\psi_{\lambda i} = \psi_i(\cdot\,; \lambda)$ be the unique function in $H^n[a, b]$ which satisfies the initial value problem

$$\begin{gathered}(\lambda I - \tau)\,\psi_{\lambda i} = 0,\\ \psi_{\lambda i}^{(j)}(a) = \delta_{i, j+1}, \qquad j = 0, 1, ..., n-1.\end{gathered} \tag{4.1}$$

Clearly the functions $\psi_{\lambda 1}, ..., \psi_{\lambda n}$ form a basis for the solution space of $(\lambda I - \tau)\,\psi = 0$ for each $\lambda \in \mathbb{C}$, and by Theorem 3.1 the functions

$$f_{ij}(\lambda) = B_i(\psi_{\lambda j}), \qquad i, j = 1, ..., n,$$

are entire functions on $\mathbb{C}$.

The characteristic determinant of L is the entire function $D(\lambda)$ defined by

$$D(\lambda) = \det[B_i(\psi_{\lambda j})] \qquad \text{for} \quad \lambda \in \mathbb{C}.$$

It is well known that $\sigma(L)$ is precisely the set of zeros of $D(\lambda)$, and hence, our underlying assumption that $\rho(L) \neq \varnothing$ is equivalent to $D(\lambda)$ being not identically 0 on $\mathbb{C}$. For future reference we list the basic properties of $D(\lambda)$ in the following two theorems (see [8, Part I, pp. 13–20]).

THEOREM 4.1. *Let L be an nth-order differential operator in $L^2[a, b]$ determined by a formal differential operator τ and by independent boundary values $B_1, ..., B_n$, and assume $\rho(L) \neq \varnothing$. Then the characteristic determinant $D(\lambda)$ of L is an entire function on $\mathbb{C}$ which is not identically* 0, *and the spectrum $\sigma(L)$ is the set of zeros of $D(\lambda)$.*

THEOREM 4.2. *Under the hypothesis of Theorem* 4.1, *let $\lambda_0 \in \sigma(L)$, let m_0 be the ascent of the operator L_{λ_0}, let $\nu(\lambda_0) = \dim \mathcal{N}(L(\lambda_0))$ be the algebraic multiplicity of λ_0, and let n_0 be the order of λ_0 as a zero of $D(\lambda)$. Then $1 \leqslant m_0 \leqslant \nu(\lambda_0)$ and*

$$\nu(\lambda_0) = n_0. \tag{4.2}$$

5. Power Series Expansions for the Green's Function of L_λ

For each $\lambda \in \rho(L)$ we now construct the Green's function $G(t, s; \lambda)$ corresponding to the differential operator $L_\lambda = \lambda I - L$. The construction proceeds along the lines developed in [7, pp. 142–144].

Step 1. For each $\lambda \in \mathbb{C}$ and for $i = 1, ..., n$ let $\psi_{\lambda i}$ be the unique function in $H^n[a, b]$ which satisfies the initial value problem (4.1). By Theorem 3.1 for $i = 1, ..., n$ there exists a sequence of functions u_{ik}, $k = 0, 1, 2, ...,$ in $H^n[a, b]$ such that

$$\psi_{\lambda i} = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k u_{ik} \qquad \text{in } H^n[a, b] \tag{5.1}$$

for all $\lambda \in \mathbb{C}$, where the convergence is in the H^n-norm and is uniform in the λ variable on compact subsets of $\mathbb{C}$.

Step 2. For each $\lambda \in \mathbb{C}$ let $v_{\lambda i}$, $i = 1, ..., n$, be the unique functions determined by the linear system

$$\begin{aligned} &\sum_{i=1}^{n} \psi_{\lambda i}^{(j)} v_{\lambda i} = 0, \qquad && j = 0, 1, ..., n-2, \\ &\sum_{i=1}^{n} \psi_{\lambda i}^{(n-1)} v_{\lambda i} = -a_n^{-1}. && \end{aligned} \tag{5.2}$$

We can use induction to show that the $v_{\lambda i}$ belong to $H^n[a, b]$, and in fact, for $k = 0, 1, ..., n-1$ the kth derivatives satisfy a linear system of the form

$$\begin{aligned} &\sum_{i=1}^{n} \psi_{\lambda i}^{(j)} v_{\lambda i}^{(k)} = 0, \qquad && j = 0, 1, ..., n-k-2, \\ &\sum_{i=1}^{n} \psi_{\lambda i}^{(j)} v_{\lambda i}^{(k)} = \omega_{kj}, \qquad && j = n-k-1, ..., n-1, \end{aligned} \tag{5.3}$$

where the functions ω_{kj} belong to $H^{n-k}[a, b]$ and are independent of λ (see Lemma III.3.6 in [7]). It follows from (5.3) that the $v_{\lambda i}$ satisfy the initial conditions

$$\begin{aligned} &v_{\lambda i}^{(j)}(a) = 0, \qquad && j = 0, 1, ..., n-i-1, \\ &v_{\lambda i}^{(j)}(a) = \omega_{j,i-1}(a), \qquad && j = n-i, ..., n-1, \end{aligned} \tag{5.4}$$

which are independent of λ. Also, the $v_{\lambda i}$ satisfy the differential equation $(\bar{\lambda} I - \tau^*)\, \overline{v_{\lambda i}} = 0$ or

$$(\lambda I - \overline{\tau^*})\, v_{\lambda i} = 0 \tag{5.5}$$

(see Corollary III.3.15 in [7] or [2, Problem 19 on p. 101]). Applying Theorem 3.1 to each of the functions $v_{\lambda i}$ for $i=1, ..., n$, there exists a sequence of functions v_{ik}, $k=0, 1, 2, ...$, in $H^n[a, b]$ such that

$$v_{\lambda i}=\sum_{k=0}^{\infty} (\lambda-\lambda_0)^k v_{ik} \qquad \text{in } H^n[a, b] \tag{5.6}$$

for all $\lambda \in \mathbb{C}$, where the convergence is in the H^n-norm and is uniform in the λ variable on compact subsets of $\mathbb{C}$.

Step 3. Let C_i, D_i be the boundary values defined by

$$C_i(u)=\sum_{j=0}^{n-1} \alpha_{ij} u^{(j)}(a), \qquad D_i(u)=\sum_{j=0}^{n-1} \beta_{ij} u^{(j)}(b),$$

for $i=1, ..., n$, so $B_i=C_i+D_i$ for $i=1, ..., n$. For each $\lambda \in \rho(L)$ let $\eta_{\lambda i}$, $i=1, ..., n$, be the unique functions determined by the linear system

$$\sum_{i=1}^{n} B_j(\psi_{\lambda i}) \eta_{\lambda i}=\sum_{i=1}^{n} C_j(\psi_{\lambda i}) v_{\lambda i}, \qquad j=1, ..., n. \tag{5.7}$$

By Cramer's rule we have

$$\eta_{\lambda i}=\frac{1}{D(\lambda)}\sum_{j=1}^{n} \gamma_{ij}(\lambda) v_{\lambda j}, \qquad i=1, ..., n, \tag{5.8}$$

where the $\gamma_{ij}(\lambda)$ are entire functions on $\mathbb{C}$, and this clearly shows that the $\eta_{\lambda i}$ belong to $H^n[a, b]$. In (5.8) the functions $\sum_{j=1}^{n} \gamma_{ij}(\lambda) v_{\lambda j}$, $i=1, ..., n$, exist for all $\lambda \in \mathbb{C}$, while the functions $\eta_{\lambda i}$, $i=1, ..., n$, exist only for $\lambda \in \rho(L)$.

Step 4. For each $\lambda \in \rho(L)$ let $\xi_{\lambda i}$, $i=1, ..., n$, be the functions in $H^n[a, b]$ defined by

$$\begin{aligned} \xi_{\lambda i} &= \eta_{\lambda i}-v_{\lambda i} \\ &= \frac{1}{D(\lambda)}\sum_{j=1}^{n} \tilde{\gamma}_{ij}(\lambda) v_{\lambda j}, \end{aligned} \tag{5.9}$$

where the $\tilde{\gamma}_{ij}(\lambda)=\gamma_{ij}(\lambda)-\delta_{ij}D(\lambda)$ are entire functions on $\mathbb{C}$. Observe that the functions $\sum_{j=1}^{n} \tilde{\gamma}_{ij}(\lambda) v_{\lambda j}$, $i=1, ..., n$, exist for all $\lambda \in \mathbb{C}$, while the functions $\xi_{\lambda i}$, $i=1, ..., n$, exist only for $\lambda \in \rho(L)$.

Step 5. For each $\lambda \in \rho(L)$ the Green's function for the differential operator $L_\lambda=\lambda I-L$ is given by

$$\begin{aligned}G(t, s; \lambda) &= \sum_{i=1}^{n} \psi_{\lambda i}(t)\, \xi_{\lambda i}(s)\\ &= \frac{1}{D(\lambda)} \sum_{i, j=1}^{n} \tilde{\gamma}_{ij}(\lambda)\, \psi_{\lambda i}(t)\, v_{\lambda j}(s) \qquad \text{for} \quad a \leqslant t < s \leqslant b,\\ G(t, s; \lambda) &= \sum_{i=1}^{n} \psi_{\lambda i}(t)\, v_{\lambda i}(s)\\ &= \frac{1}{D(\lambda)} \sum_{i, j=1}^{n} \gamma_{ij}(\lambda)\, \psi_{\lambda i}(t)\, v_{\lambda j}(s) \qquad \text{for} \quad a \leqslant s < t \leqslant b.\end{aligned} \tag{5.10}$$

To simplify the notation, for each $\lambda \in \mathbb{C}$ we set

$$M(t, s; \lambda) = \sum_{i, j=1}^{n} \tilde{\gamma}_{ij}(\lambda)\, \psi_{\lambda i}(t)\, v_{\lambda j}(s), \qquad t, s \in [a, b],$$

and

$$N(t, s; \lambda) = \sum_{i, j=1}^{n} \gamma_{ij}(\lambda)\, \psi_{\lambda i}(t)\, v_{\lambda j}(s), \qquad t, s \in [a, b],$$

and then define

$$H(t, s; \lambda) = \begin{cases} M(t, s; \lambda), & a \leqslant t < s \leqslant b, \\ N(t, s; \lambda), & a \leqslant s < t \leqslant b. \end{cases}$$

In terms of these functions

$$G(t, s; \lambda) = \frac{H(t, s; \lambda)}{D(\lambda)} \tag{5.11}$$

for all $\lambda \in \rho(L)$ and for all $t \neq s$ in $[a, b]$. Again we emphasize the fact that the functions M, N, and H exist for all $\lambda \in \mathbb{C}$, while the Green's function G exists only for $\lambda \in \rho(L)$.

To conclude this section, we employ the series (5.1) and (5.6) to expand the functions M, N, and H in power series about the point $\lambda_0 \in \mathbb{C}$. The lemmas which follow provide the theoretical foundations for the convergence analysis. They are proved in the same manner as the classical results, e.g., see [1, p. 39; 9, p. 74].

LEMMA 5.1. *Let $\lambda_0 \in \mathbb{C}$, let u_k, $k = 0, 1, 2, \ldots$, be a sequence of functions in $H^n[a, b]$, and for each $\lambda \in \mathbb{C}$ let ψ_λ be a function in $H^n[a, b]$ such that*

$$\psi_\lambda = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k u_k \qquad \text{in } H^n[a, b],$$

where the convergence is in the H^n-norm and is uniform in the λ variable on compact subsets of $\mathbb{C}$. If Ω is any compact subset of $\mathbb{C}$, then there exists a constant $\omega > 0$ such that

$$\sum_{k=0}^{\infty} |\lambda - \lambda_0|^k \, |u_k|_{H^n} \leqslant \omega \qquad \text{for all} \quad \lambda \in \Omega. \tag{5.12}$$

In addition, if $\gamma(\lambda) = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k a_k$ is a scalar-valued entire function on $\mathbb{C}$ and if $w_k \in H^n[a, b]$ is defined by $w_k = \sum_{l=0}^{k} a_l u_{k-l}$ for $k = 0, 1, 2, ...,$ then

$$\gamma(\lambda)\, \psi_\lambda = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k w_k \qquad \text{in } H^n[a, b] \tag{5.13}$$

for each $\lambda \in \mathbb{C}$, where the convergence is in the H^n-norm and is uniform in the λ variable on compact subsets of $\mathbb{C}$.

LEMMA 5.2. *Let $\lambda_0 \in \mathbb{C}$, let w_k, $k = 0, 1, 2, ...,$ and v_k, $k = 0, 1, 2, ...,$ be sequences of functions in $H^n[a, b]$, and for each $\lambda \in \mathbb{C}$ let ϕ_λ and v_λ be functions in $H^n[a, b]$ such that*

$$\phi_\lambda = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k w_k \qquad \text{and} \qquad v_\lambda = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k v_k \qquad \text{in } H^n[a, b]$$

for all $\lambda \in \mathbb{C}$, where the convergence is in the H^n-norm and is uniform in the λ variable on compact subsets of $\mathbb{C}$. Define

$$z_k(t, s) = \sum_{l=0}^{k} w_l(t)\, v_{k-l}(s) \in C([a, b] \times [a, b])$$

for $k = 0, 1, 2, ...$. Then

$$\phi_\lambda(t)\, v_\lambda(s) = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k z_k(t, s) \tag{5.14}$$

for all $t, s \in [a, b]$ and for each $\lambda \in \mathbb{C}$, where for each compact subset Ω of $\mathbb{C}$ the convergence is uniform in the t, s, λ variables on the set $[a, b] \times [a, b] \times \Omega$.

To expand the functions M and N in power series about $\lambda_0 \in \mathbb{C}$, first apply Lemma 5.1 to the products $\tilde{\gamma}_{ij}(\lambda)\, \psi_{\lambda i}$ and $\gamma_{ij}(\lambda)\, \psi_{\lambda i}$, expanding each one in a power series about λ_0, where the convergence is in the H^n-norm and is uniform in the λ variable on compact subsets of $\mathbb{C}$. Second, apply Lemma 5.2 to the products $[\tilde{\gamma}_{ij}(\lambda)\, \psi_{\lambda i}(t)]\, v_{\lambda j}(s)$ and $[\gamma_{ij}(\lambda)\, \psi_{\lambda i}(t)]\, v_{\lambda j}(s)$

and then sum; this produces functions M_k and N_k, $k=0, 1, 2, \ldots$, in $C([a, b] \times [a, b])$ such that

$$M(t, s; \lambda) = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k M_k(t, s) \tag{5.15}$$

and

$$N(t, s; \lambda) = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k N_k(t, s) \tag{5.16}$$

for all $t, s \in [a, b]$ and for each $\lambda \in \mathbb{C}$, where for each compact subset Ω of $\mathbb{C}$ the convergence is uniform in the t, s, λ variables on the set $[a, b] \times [a, b] \times \Omega$. From (5.15) and (5.16) it is immediate that M and N are continuous on $[a, b] \times [a, b] \times \mathbb{C}$, and for fixed $t, s \in [a, b]$ the functions $M(t, s; \cdot)$ and $N(t, s; \cdot)$ are entire functions on $\mathbb{C}$ with

$$\frac{\partial^k M}{\partial \lambda^k} (t, s; \lambda_0) = k!\, M_k(t, s) \tag{5.17}$$

and

$$\frac{\partial^k N}{\partial \lambda^k} (t, s; \lambda_0) = k!\, N_k(t, s) \tag{5.18}$$

for $k=0, 1, 2, \ldots$.

Finally, to expand the function H in a power series about λ_0, we introduce the slit square

$$\Lambda = \{(t, s) \in \mathbb{R}^2 \mid a \leqslant t, s \leqslant b, t \neq s\}$$

and the sequence of functions

$$H_k(t, s) = \begin{cases} M_k(t, s), & a \leqslant t < s \leqslant b, \\ N_k(t, s), & a \leqslant s < t \leqslant b, \end{cases}$$

$k=0, 1, 2, \ldots$, defined on Λ. From the above the following properties are apparent:

(a) The H_k are continuous on Λ and are bounded measurable functions on $[a, b] \times [a, b]$ for $k=0, 1, 2 \ldots$.

(b) H has the representation

$$H(t, s; \lambda) = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k H_k(t, s) \tag{5.19}$$

for all $(t, s) \in \Lambda$ and for each $\lambda \in \mathbb{C}$, where for each compact subset Ω of $\mathbb{C}$ the convergence is uniform in the t, s, λ variables on the set $\Lambda \times \Omega$.

(c) H is continuous on $\Lambda \times \mathbb{C}$, and for fixed $\lambda \in \mathbb{C}$ the function $H(\cdot,\cdot;\lambda)$ is a bounded measurable function on $[a, b] \times [a, b]$.

(d) For fixed $(t, s) \in \Lambda$ the function $H(t, s; \cdot)$ is an entire function on $\mathbb{C}$ with

$$\frac{\partial^k H}{\partial \lambda^k}(t, s; \lambda_0) = k!\, H_k(t, s) \qquad \text{for} \quad k = 0, 1, 2, \ldots. \tag{5.20}$$

Remark 5.3. Suppose the point $\lambda_0 \in \sigma(L)$, so

$$D(\lambda_0) = D'(\lambda_0) = \cdots = D^{(n_0-1)}(\lambda_0) = 0 \tag{5.21}$$

and $D^{(n_0)}(\lambda_0) \neq 0$ with $n_0 = \nu(\lambda_0)$ (see Theorem 4.2). For each $\lambda \in \mathbb{C}$ set

$$Q(t, s; \lambda) = \sum_{i=1}^{n} \psi_{\lambda i}(t)\, v_{\lambda i}(s), \qquad t, s \in [a, b].$$

Then

$$M(t, s; \lambda) = N(t, s; \lambda) - D(\lambda)\, Q(t, s; \lambda) \tag{5.22}$$

for all $t, s \in [a, b]$ and for each $\lambda \in \mathbb{C}$; the function Q can be expanded in a series about λ_0 in the same way as M and N; for fixed $t, s \in [a, b]$ the function $Q(t, s; \cdot)$ is an entire function on $\mathbb{C}$. If we fix $t, s \in [a, b]$ and proceed to calculate successive derivatives of (5.22) with respect to λ, then in view of (5.21) we obtain

$$\frac{\partial^k M}{\partial \lambda^k}(t, s; \lambda_0) = \frac{\partial^k N}{\partial \lambda^k}(t, s; \lambda_0) = \frac{\partial^k H}{\partial \lambda^k}(t, s; \lambda_0) \tag{5.23}$$

for $k = 0, 1, \ldots, n_0 - 1$, and hence,

$$M_k(t, s) = N_k(t, s) = H_k(t, s) \tag{5.24}$$

for all $t, s \in [a, b]$ and for $k = 0, 1, \ldots, n_0 - 1$ ($t \neq s$ in the formulas for $\partial^k H / \partial \lambda^k$ and H_k).

Remark 5.4. If we examine how the functions $M_k(t, s)$ and $N_k(t, s)$ are fashioned out of Lemmas 5.1 and 5.2, then we see that each one is of the general form

$$\sum_{l=1}^{m_k} w_{kl}(t)\, v_{kl}(s),$$

where the $w_{kl}, v_{kl} \in H^n[a, b]$, and hence, the $H_k(t, s)$ have a similar structure on each of the two triangles which make up Λ.

6. Integral Representations in the Laurent Series for $R_\lambda(L)$

Throughout this section we assume that λ_0 is a fixed point of the spectrum $\sigma(L)$, m_0 is the ascent of L_{λ_0}, and $\nu(\lambda_0)=n_0$ is the algebraic multiplicity of λ_0. Let P_0 be the projection of $L^2[a,b]$ onto the generalized eigenspace $\mathcal{N}(L(\lambda_0))$ along $\mathcal{R}(L(\lambda_0))$, and let

$$d=\inf\{|\lambda-\lambda_0| \mid \lambda\in\sigma(L)-\{\lambda_0\}\}>0,$$

so the punctured disk $0<|\lambda-\lambda_0|<d$ lies in $\rho(L)$. Suppose δ is any number satisfying $0<\delta<d$, and let Γ be the circle with center λ_0 and radius δ parametrized by $\lambda=\lambda(\theta)=\lambda_0+\delta e^{i\theta}$, $0\leqslant\theta\leqslant 2\pi$.

It is well known (see [11, pp. 328–331]) that the resolvent $R_\lambda(L)$ has the Laurent series

$$R_\lambda(L)=\sum_{k=0}^{\infty}(\lambda-\lambda_0)^k R_k+\sum_{k=1}^{m_0}\frac{S_k}{(\lambda-\lambda_0)^k} \tag{6.1}$$

in $\mathcal{B}(L^2[a,b])$ for all $0<|\lambda-\lambda_0|<d$, where the R_k, S_k are operators in $\mathcal{B}(L^2[a,b])$ given by

$$R_k=\frac{1}{2\pi i}\int_\Gamma\frac{R_\lambda(L)}{(\lambda-\lambda_0)^{k+1}}\,d\lambda,\qquad k=0,1,2,..., \tag{6.2}$$

$$S_k=\frac{1}{2\pi i}\int_\Gamma(\lambda-\lambda_0)^{k-1}R_\lambda(L)\,d\lambda,\qquad k=1,...,m_0, \tag{6.3}$$

$S_k\neq 0$ for $1\leqslant k\leqslant m_0$, and $S_1=P_0$ (cf. (1.1)). These coefficient operators can also be expressed in terms of the Green's function $G(t,s;\lambda)=H(t,s;\lambda)/D(\lambda)$ for L_λ. Indeed, suppose

$$D(\lambda)=(\lambda-\lambda_0)^{n_0}h(\lambda),\qquad \lambda\in\mathbb{C}, \tag{6.4}$$

where h is an entire function on $\mathbb{C}$ with $h(\lambda_0)\neq 0$. Then by (5.11)

$$G(t,s;\lambda)=\frac{H(t,s;\lambda)}{(\lambda-\lambda_0)^{n_0}h(\lambda)} \tag{6.5}$$

for all $\lambda\in\rho(L)$ and for all $t\neq s$ in $[a,b]$, and

$$R_\lambda(L)\,x(t)=\int_a^b\frac{H(t,s;\lambda)\,x(s)}{(\lambda-\lambda_0)^{n_0}h(\lambda)}\,ds,\qquad a\leqslant t\leqslant b, \tag{6.6}$$

for all $\lambda\in\rho(L)$ and for all $x\in L^2[a,b]$.

Now take arbitrary functions $x,y\in L^2[a,b]$. Using (6.2), (6.6), and

Fubini's theorem on the rectangle $a \leqslant t \leqslant b$, $a \leqslant s \leqslant b$, $0 \leqslant \theta \leqslant 2\pi$ in $\mathbb{R}^3$, we obtain

$$\begin{aligned}
(R_k x, y) &= \frac{1}{2\pi i} \int_\Gamma \frac{(R_\lambda(L)x, y)}{(\lambda - \lambda_0)^{k+1}}\, d\lambda \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_a^b \int_a^b \frac{H(t, s; \lambda(\theta))\, x(s)\, \overline{y(t)}}{[\delta e^{i\theta}]^{n_0+k}\, h(\lambda(\theta))}\, ds\, dt\, d\theta \\
&= \int_a^b \left\{ \int_a^b \left(\frac{1}{2\pi i} \int_0^{2\pi} \frac{H(t, s; \lambda(\theta))\, i\delta e^{i\theta}}{[\delta e^{i\theta}]^{n_0+k+1}\, h(\lambda(\theta))}\, d\theta \right) x(s)\, ds \right\} \overline{y(t)}\, dt \\
&= \int_a^b \left\{ \int_a^b \left(\frac{1}{(n_0+k)!} \frac{\partial^{n_0+k}}{\partial \lambda^{n_0+k}} \left[\frac{H(t, s; \lambda)}{h(\lambda)} \right]_{\lambda=\lambda_0} \right) x(s)\, ds \right\} \overline{y(t)}\, dt,
\end{aligned}$$

which implies that

$$R_k x(t) = \frac{1}{(n_0+k)!} \int_a^b \frac{\partial^{n_0+k}}{\partial \lambda^{n_0+k}} \left[\frac{H(t, s; \lambda)}{h(\lambda)} \right]_{\lambda=\lambda_0} x(s)\, ds, \qquad a \leqslant t \leqslant b, \tag{6.7}$$

for all $x \in L^2[a, b]$ and for $k = 0, 1, 2, \ldots$. A similar argument shows that

$$S_k x(t) = \frac{1}{(n_0-k)!} \int_a^b \frac{\partial^{n_0-k}}{\partial \lambda^{n_0-k}} \left[\frac{H(t, s; \lambda)}{h(\lambda)} \right]_{\lambda=\lambda_0} x(s)\, ds, \qquad a \leqslant t \leqslant b, \tag{6.8}$$

for all $x \in L^2[a, b]$ and for $k = 1, \ldots, m_0$.

Equations (6.7) and (6.8) express the coefficient operators in (6.1) as integral operators whose kernels are determined from the functions $H(t, s; \lambda)$ and $h(\lambda)$ which make up the Green's function $G(t, s; \lambda)$. These equations also give us a primitive relationship between the ascent m_0 and the algebraic multiplicity $v(\lambda_0) = n_0$, which we will refine in the next section. Setting $k = 1$ in Eq. (6.8), we obtain the important result

$$P_0 x(t) = \frac{1}{(n_0-1)!} \int_a^b \frac{\partial^{n_0-1}}{\partial \lambda^{n_0-1}} \left[\frac{H(t, s; \lambda)}{h(\lambda)} \right]_{\lambda=\lambda_0} x(s)\, ds, \qquad a \leqslant t \leqslant b, \tag{6.9}$$

for all $x \in L^2[a, b]$. Equation (6.9) will play a central role in a future paper [6].

Remark 6.1. Let us briefly examine the kernels appearing in (6.7)–(6.9). For fixed $(t, s) \in \Lambda$ we know that the function $H(t, s; \cdot)/h$ is analytic on the disk $|\lambda - \lambda_0| < d$, and taking successive derivatives with respect to λ, we obtain

$$\frac{\partial^k}{\partial \lambda^k} \left[\frac{H(t, s; \lambda)}{h(\lambda)} \right] = [h(\lambda)]^{-2^k} \sum_{l=0}^{k} h_{kl}(\lambda) \frac{\partial^l H}{\partial \lambda^l}(t, s; \lambda) \tag{6.10}$$

for all $|\lambda-\lambda_0|<d$ and for $k=0, 1, 2, \ldots$, where the $h_{kl}(\lambda)$ are entire functions on $\mathbb{C}$. If we set $\lambda=\lambda_0$ in (6.10) and substitute (5.20), then it becomes

$$\frac{\partial^k}{\partial\lambda^k}\left[\frac{H(t, s; \lambda)}{h(\lambda)}\right]_{\lambda=\lambda_0}=[h(\lambda_0)]^{-2^k}\sum_{l=0}^{k} l!\, h_{kl}(\lambda_0)\, H_l(t, s) \qquad (6.11)$$

for all $(t, s)\in \Lambda$ and for $k=0, 1, 2, \ldots$. Thus, these kernels are continuous on the slit square Λ and are bounded measurable functions on the square $[a, b]\times[a, b]$. For $k=0, 1, \ldots, n_0-1$ these formulas simplify even further:

$$\frac{\partial^k}{\partial\lambda^k}\left[\frac{H(t, s; \lambda)}{h(\lambda)}\right]_{\lambda=\lambda_0}=[h(\lambda_0)]^{-2^k}\sum_{l=0}^{k} l!\, h_{kl}(\lambda_0)\, M_l(t, s), \qquad (6.12)$$

which are continuous functions on $[a, b]\times[a, b]$ (see Remark 5.3).

Remark 6.2. In [5] another characterization of the coefficient operators R_k, S_k is given, namely

$$R_k=(-1)^k R_{\lambda_0}(T_0)^{k+1}(I-P_0), \qquad k=0, 1, 2, \ldots,$$
$$S_k=(-N_0)^{k-1} P_0, \qquad k=1, \ldots, m_0,$$

where $N_0=L_{\lambda_0}|\mathcal{N}(L(\lambda_0))$, $T_0=L|\mathcal{D}(L)\cap\mathcal{R}(L(\lambda_0))$, and the resolvent $R_{\lambda_0}(T_0)$ belongs to $\mathcal{B}(\mathcal{R}(L(\lambda_0)))$. Also, using the methods of Section 3, it can be shown that (6.1) is valid in the operator space X, the convergence being uniform in the λ variable on compact subsets of $0<|\lambda-\lambda_0|<d$.

7. The Ascent-Algebraic Multiplicity Relationship

Fix a point $\lambda_0\in\mathbb{C}$, and let us reconsider the power series expansion (5.19) for the function H about the point λ_0. Since the functions $H(\cdot, \cdot; \lambda)$, $\lambda\in\mathbb{C}$, and H_k, $k=0, 1, 2, \ldots$, are bounded measurable functions on $[a, b]\times[a, b]$, we can use them as the L^2-kernels of the integral operators $T(\lambda)$ and T_k defined by

$$T(\lambda)\, x(t)=\int_a^b H(t, s; \lambda)\, x(s)\, ds, \qquad a\leqslant t\leqslant b,$$

and

$$T_k x(t)=\int_a^b H_k(t, s)\, x(s)\, ds, \qquad a\leqslant t\leqslant b,$$

for $x\in L^2[a, b]$. Clearly $T(\lambda)\in\mathcal{B}(L^2[a, b])$ for all $\lambda\in\mathbb{C}$ and $T_k\in\mathcal{B}(L^2[a, b])$ for $k=0, 1, 2, \ldots$.

Take any $\varepsilon > 0$, and let Ω be any compact subset of $\mathbb{C}$. Then from (5.19) there exists a nonnegative integer N_0 such that

$$\left| H(t, s; \lambda) - \sum_{k=0}^{N} (\lambda - \lambda_0)^k H_k(t, s) \right| \leqslant \varepsilon \qquad \text{for all} \quad N \geqslant N_0,$$

for all $(t, s) \in \Lambda$ and for all $\lambda \in \Omega$, and hence,

$$\begin{aligned} \left\| T(\lambda) - \sum_{k=0}^{N} (\lambda - \lambda_0)^k T_k \right\|^2 &\leqslant \int_a^b \int_a^b \left| H(t, s; \lambda) - \sum_{k=0}^{N} (\lambda - \lambda_0)^k H_k(t, s) \right|^2 dt\, ds \\ &\leqslant \varepsilon^2 (b-a)^2 \end{aligned}$$

for all $N \geqslant N_0$ and for all $\lambda \in \Omega$. Thus, we obtain the power series expansion

$$T(\lambda) = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k T_k \qquad \text{in } \mathscr{B}(L^2[a, b]) \tag{7.1}$$

for all $\lambda \in \mathbb{C}$, the convergence being uniform in the λ variable on compact subsets of $\mathbb{C}$. Also, from (5.11) it is obvious that

$$T(\lambda) = D(\lambda)\, R_\lambda(L) \qquad \text{for all} \quad \lambda \in \rho(L). \tag{7.2}$$

Suppose the operator $T(\lambda)$ has a *zero at* λ_0 *of order* p_0, i.e., p_0 is a nonnegative integer with

$$T_0 = T_1 = \cdots = T_{p_0 - 1} = 0 \qquad \text{and} \qquad T_{p_0} \neq 0.$$

Note that $T(\lambda_0) = 0$ in case $p_0 > 0$, while $T(\lambda_0) \neq 0$ for $p_0 = 0$. Since the kernels of $T_0, T_1, ..., T_{p_0}$ are continuous on Λ, it follows that

$$H_k \equiv 0 \qquad \text{on } \Lambda \text{ for } k = 0, 1, ..., p_0 - 1, \tag{7.3}$$

and

$$H_{p_0} \not\equiv 0 \qquad \text{on } \Lambda. \tag{7.4}$$

In view of these remarks we say that the function $H(t, s; \lambda)$ has a *zero at* λ_0 *of order* p_0 if the operator $T(\lambda)$ has a zero at λ_0 of order p_0.

THEOREM 7.1. *The operator $T(\lambda)$ has a zero at λ_0 of order $p_0 \geqslant 0$ iff*

$$\frac{\partial^k H}{\partial \lambda^k}(t, s; \lambda_0) \equiv 0 \qquad \textit{on } \Lambda$$

for $k=0, 1, \ldots, p_0-1$, *and*

$$\frac{\partial^{p_0} H}{\partial \lambda^{p_0}}(t, s; \lambda_0) \not\equiv 0 \qquad \text{on } \Lambda.$$

In addition, $\lambda_0 \in \sigma(L)$ *if* $p_0 > 0$.

Proof. The main part of the theorem is a direct consequence of (7.3), (7.4), and (5.20). For the last part, if $p_0 > 0$ and $\lambda_0 \in \rho(L)$, then by (7.2) we would have $T(\lambda_0) = D(\lambda_0) R_{\lambda_0}(L) \neq 0$, a contradiction. Q.E.D.

There are many examples where $\lambda_0 \in \sigma(L)$ but $T(\lambda_0) \neq 0$, e.g., this is always the case when $v(\lambda_0) = 1$ (see the next theorem).

Our final theorem relates the ascent m_0 and the algebraic multiplicity $v(\lambda_0)$ at a point $\lambda_0 \in \sigma(L)$, the connection being made in terms of the order of λ_0 as a zero of $D(\lambda)$ and as a zero of $H(t, s; \lambda)$. This result will also be important in our future work [6].

THEOREM 7.2. *Let* $\lambda_0 \in \sigma(L)$, *and assume that* λ_0 *is a zero of order* $n_0 \geqslant 1$ *for* $D(\lambda)$ *and is a zero of order* $p_0 \geqslant 0$ *for* $H(t, s; \lambda)$. *Then the ascent* m_0 *of* L_{λ_0} *and the algebraic multiplicity* $v(\lambda_0)$ *of* λ_0 *are related by the equation*

$$v(\lambda_0) = n_0 = m_0 + p_0. \tag{7.5}$$

Proof. The equation $v(\lambda_0) = n_0$ is the conclusion of Theorem 4.2. For the second equation take $x, y \in L^2[a, b]$. Then by (7.1) we have

$$(T(\lambda)\, x, y) = \sum_{k=p_0}^{\infty} (\lambda-\lambda_0)^k\, (T_k x, y) \tag{$*$}$$

for all $\lambda \in \mathbb{C}$, while from (7.2), (6.4), and (6.1) we get

$$\begin{aligned}(T(\lambda)\, x, y) &= (\lambda-\lambda_0)^{n_0}\, h(\lambda)(R_\lambda(L)\, x, y) \\ &= (\lambda-\lambda_0)^{n_0-m_0}\, h(\lambda) \sum_{k=1}^{m_0} (\lambda-\lambda_0)^{m_0-k}\, (S_k x, y) \\ &\quad + (\lambda-\lambda_0)^{n_0}\, h(\lambda) \sum_{k=0}^{\infty} (\lambda-\lambda_0)^k\, (R_k x, y)\end{aligned} \tag{$**$}$$

for all $0 < |\lambda-\lambda_0| < d$. If $p_0 < n_0 - m_0$, then dividing $(*)$ and $(**)$ by $(\lambda-\lambda_0)^{p_0}$ and letting $\lambda \to \lambda_0$ yields

$$(T_{p_0} x, y) = \lim_{\lambda \to \lambda_0} \frac{(T(\lambda)\, x, y)}{(\lambda-\lambda_0)^{p_0}} = 0$$

for all $x, y \in L^2[a, b]$, contradicting the fact that $T_{p_0} \neq 0$. On the other hand, if $n_0 - m_0 < p_0$, then a similar argument shows that

$$h(\lambda_0)(S_{m_0}x, y) = \lim_{\lambda \to \lambda_0} \frac{(T(\lambda)\, x, y)}{(\lambda - \lambda_0)^{n_0 - m_0}} = 0$$

for all $x, y \in L^2[a, b]$ with $h(\lambda_0) \neq 0$, which contradicts the fact that $S_{m_0} \neq 0$. Therefore, $p_0 = n_0 - m_0$. Q.E.D.

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