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First and second kind paraorthogonal polynomials and their zeros

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Abstract

Given a probability measure μ with infinite support on the unit circle $\partial\mathbb{D} = \{z : |z| = 1\}$, we consider a sequence of paraorthogonal polynomials $h_n(z, \lambda)$ vanishing at $z = \lambda$ where $\lambda \in \partial\mathbb{D}$ is fixed. We prove that for any fixed $z_0 \notin \text{supp}(d\mu)$ distinct from λ, we can find an explicit $\rho > 0$ independent of n such that either h_n or h_{n+1} (or both) has no zero inside the disk $B(z_0, \rho)$, with the possible exception of λ.

Then we introduce paraorthogonal polynomials of the second kind, denoted $s_n(z, \lambda)$. We prove three results concerning s_n and h_n. First, we prove that zeros of s_n and h_n interlace. Second, for z_0 an isolated point in $\text{supp}(d\mu)$, we find an explicit radius $\tilde{\rho}$ such that either s_n or s_{n+1} (or both) have no zeros inside $B(z_0, \tilde{\rho})$. Finally, we prove that for such z_0 we can find an explicit radius such that either h_n or h_{n+1} (or both) has at most one zero inside the ball $B(z_0, \tilde{\rho})$.

Keywords: Paraorthogonal polynomials; Second kind paraorthogonal polynomials; Interlacing zeros; Zeros of consecutive paraorthogonal polynomials

1. Introduction

Suppose we are given a probability measure μ on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ with infinite support. We form the inner product $\langle , \rangle$ and the norm in $L^2(d\mu)$ as follows:

$$\langle f, g\rangle = \int_{\partial\mathbb{D}} f(z)\overline{g(z)}\, d\mu(z), \qquad \|f\| = \langle f, f\rangle^{1/2}. \tag{1.1}$$

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By the Gram–Schmidt process, we then obtain a sequence of monic orthogonal polynomials $(\Phi_n)_{n=1}^{\infty}$, the normalized sequence being $(\varphi_n)_{n=1}^{\infty}$, such that φ_n is an nth degree polynomial with the property:

$$\langle \varphi_m, \varphi_n \rangle = \delta_{mn}. \tag{1.2}$$

These orthogonal polynomials satisfy the Szegő recursion relation:

$$\Phi_n(z) = z\Phi_{n-1}(z) - \overline{\alpha_{n-1}}\Phi_{n-1}^*(z), \tag{1.3}$$

where $\Phi_m^*(z) = z^m\overline{\Phi_m(1/\bar{z})}$.

The family of α_n's are known as the Verblunsky coefficients. There are a few important properties of orthogonal polynomials and Verblunsky coefficients which are relevant to this paper:

$$|\alpha_n| < 1, \tag{1.4}$$

$$\|\Phi_n\| = (1 - |\alpha_{n-1}|^2)^{1/2}\|\Phi_{n-1}\| = \prod_{j=0}^{n-1}(1 - |\alpha_j|^2)^{1/2}, \tag{1.5}$$

$$|\Phi_n(z)| = |\Phi_n^*(z)| \quad \Leftrightarrow \quad z \in \partial\mathbb{D}, \tag{1.6}$$

$$\Phi_n(z) \text{ has all its zeros inside } \mathbb{D}, \tag{1.7}$$

$$\langle \Phi_n(z), z^k \rangle = 0 \quad \text{for } k = 0, \ldots, n-1, \tag{1.8}$$

$$\langle \Phi_n^*(z), z^k \rangle = 0 \quad \text{for } k = 1, \ldots, n. \tag{1.9}$$

Paraorthogonal polynomials were introduced at least as early as in [5]. An nth degree paraorthogonal polynomial is of the form (up to multiplication with a constant)

$$H_n(z, \beta_{n-1}) = z\Phi_{n-1}(z) - \overline{\beta_{n-1}}\Phi_{n-1}^*(z) \tag{1.10}$$

with $\beta_{n-1} \in \partial\mathbb{D}$; $\Phi_{n-1}^*(z) = z^{n-1}\overline{\Phi_{n-1}(1/\bar{z})}$.

Paraorthogonal polynomials have a lot in common with orthogonal polynomials on the real line $(p_n)_{n=0}^{\infty}$. For instance, a paraorthogonal polynomial has simple zeros on the unit circle while p_n has simple zeros on the real line. Besides, for a specific family of paraorthogonal polynomials $(h_n)_{n=0}^{\infty}$ that we shall consider, it has been proven in [1,4] that zeros of h_n and h_{n+1} strictly interlace, this interlacing property is also shared by p_n and p_{n+1}.

In this paper we shall prove three results concerning this specific family of paraorthogonal polynomials $(h_n)_{n=0}^{\infty}$, namely Theorems 5.1–5.3. These results are in parallel with those proven for orthogonal polynomials of the real line p_n.

Theorems 5.1 and 5.3 are analogues of the following results by Denisov–Simon [3]:

Theorem 1.1. *Let* $\delta = \operatorname{dist}(x_0, \operatorname{supp}(d\mu)) > 0$. *Suppose* a_{n+1} *is the recursion coefficient as given by* $xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x)$. *Let* $r_n = \delta^2/(\delta + \sqrt{2}a_{n+1})$. *Then either* p_n *or* p_{n+1} *(or both) has no zeros in* $(x_0 - r_n, x_0 + r_n)$.

Theorem 1.2. *Let* x_0 *be an isolated point of* $\operatorname{supp}(d\mu)$ *on the real line. Then there exists* $d_0 > 0$ *so that if* $\delta_n = d_0^2/(d_0 + \sqrt{2}a_{n+1})$, *then at least one of* p_n *and* p_{n+1} *has no zeros or one zero in* $(x_0 - \delta_n, x_0 + \delta_n)$.

Theorem 5.2, which proves that first and second kind paraorthogonal polynomials of the same degree have interlacing zeros, is an analogue of the following well-known fact about first and second kind orthogonal polynomials on the real line, p_n and q_n:

Theorem 1.3. *Zeros of p_n and q_n strictly interlace.*

For a more comprehensive introduction to orthogonal polynomials and paraorthogonal polynomials, the reader should refer to Refs. [6,7,9].

2. Properties of paraorthogonal polynomials

A major difference between orthogonal polynomials and paraorthogonal polynomials lies in the fact that $\alpha_n \in \mathbb{D}$ is determined uniquely by the measure, while $\beta_n \in \partial\mathbb{D}$ could be chosen arbitrarily on the unit circle. These differences give rise to the following properties of H_n which are not shared by Φ_n:

1. *Zeros on $\partial\mathbb{D}$*: Unlike orthogonal polynomials which have zeros strictly inside the unit disk, paraorthogonal polynomials have zeros in $\partial\mathbb{D}$. To see that it suffices to note that

$$\left|\frac{z\Phi_n(z)}{\Phi_n^*(z)}\right| = 1 \quad \Leftrightarrow \quad z \in \partial\mathbb{D}. \tag{2.1}$$

2. *Orthogonality*: An nth degree paraorthogonal polynomial is orthogonal to $\{z, z^2, \ldots, z^{n-1}\}$ because of the orthogonal properties of Φ_{n-1} and Φ_{n-1}^* as in (1.8) and (1.9). However, we note that H_n is never orthogonal to 1 or z^n because

$$\langle H_n, 1\rangle = (\overline{\alpha_{n-1}} - \overline{\beta_{n-1}})\|\Phi_{n-1}\|^2 \neq 0, \tag{2.2}$$

$$\langle H_n, z^n\rangle = \left(1 - \overline{\beta_{n-1}}\alpha_{n-1}\right)\|\Phi_{n-1}\|^2 \neq 0. \tag{2.3}$$

3. *Representation*: Suppose λ is a zero of $H_n(z, \beta_{n-1})$. We prove that H_n could be represented using the reproducing kernel $K_n(z,\lambda) = \sum_{j=0}^n \varphi_j(z)\overline{\varphi_j(\lambda)}$ and a constant C as follows:

$$H_n(z, \beta_{n-1}) = C(z-\lambda)\sum_{j=0}^{n-1}\varphi_j(z)\overline{\varphi_j(\lambda)} = C(z-\lambda)K_{n-1}(z,\lambda). \tag{2.4}$$

The argument is related to Szegő [9] when he proved the Christoffel–Darboux formula. It goes as follows: since λ is a zero of H_n, $H_n(z) = (z-\lambda)h(z)$ for some polynomial h of degree $n-1$. By the orthogonality of H_n against $\{z, \ldots, z^{n-1}\}$, $\langle zh, z^m\rangle = \langle \lambda h, z^m\rangle$ for $1 \leqslant m \leqslant n-1$, which implies that $\overline{\lambda}\langle h, z^{m-1}\rangle = \langle h, z^m\rangle$. Applying this formula recursively, we conclude that

$$\langle h, z^m\rangle = \overline{\lambda^m}\langle h, 1\rangle \quad \text{for } 0 < m \leqslant n-1. \tag{2.5}$$

When $m = 0$ the argument is trivial. If $\varphi_s(z) = \sum_{j=0}^s a_j z^j$, then for $0 \leqslant s \leqslant n-1$,

$$\langle h, \varphi_s\rangle = \langle h, 1\rangle \sum_{j=0}^{s}\overline{a_j\lambda^j} = \langle h, 1\rangle\overline{\varphi_s(\lambda)}. \tag{2.6}$$

If we express h using Fourier series,

$$h(z) = \sum_{j=0}^{n-1}\langle h, \varphi_j\rangle\varphi_j(z) = \sum_{j=0}^{n-1}\langle h, 1\rangle\overline{\varphi_j(\lambda)}\varphi_j(z) = \langle h, 1\rangle K_{n-1}(z,\lambda). \tag{2.7}$$

4. *Simple zeros*: Let λ and h be defined as above. By (2.7), $\langle h, 1\rangle = 0$ implies $h = 0$, hence $\langle h, 1\rangle \neq 0$. In addition, $\varphi_0 = 1$ implies $K_{n-1}(\lambda, \lambda) > 0$. Therefore $h(\lambda) = \langle h, 1\rangle K_{n-1}(\lambda, \lambda) \neq 0$. This shows that zeros of paraorthogonal polynomials are simple.

5. *Linear independence*: The argument for property (3) above also tells us that a paraorthogonal polynomial could vanish at one arbitrary point on the unit circle, and that particular zero fixes the remaining ones. Therefore, two paraorthogonal polynomials of the same degree are linearly independent if and only if all their zeros are distinct.

The reader could refer to [1,7] for more properties of paraorthogonal polynomials.

3. Equivalent definitions of h_n

Fix $\lambda \in \partial\mathbb{D}$. We define the family of paraorthogonal polynomials $(h_n(z, \lambda))_n$ as follows:

$$h_n(z, \lambda) := (1 - \overline{\lambda} z) K_{n-1}(z, \lambda). \tag{3.1}$$

We will soon see that there are three equivalent definitions of h_n by the Christoffel–Darboux formula. The formula says that for $\overline{y} z \neq 1$, the reproducing kernel $K_{n-1}(z, y)$ could be expressed in the following ways:

$$K_{n-1}(z, y) = \frac{\overline{\varphi_n^*(y)}\varphi_n^*(z) - \overline{\varphi_n(y)}\varphi_n(z)}{1 - \overline{y} z} \tag{3.2}$$

$$= \frac{\overline{\varphi_{n-1}^*(y)}\varphi_{n-1}^*(z) - \overline{y} z \overline{\varphi_{n-1}(y)}\varphi_{n-1}(z)}{1 - \overline{y} z}. \tag{3.3}$$

Hence, we have the following three equivalent definitions of $h_n(z, \lambda)$:

$$h_n(z) = (1 - \overline{\lambda} z) \sum_{j=0}^{n-1} \varphi_j(z)\overline{\varphi_j(\lambda)} \tag{3.4}$$

$$= \overline{\varphi_n^*(\lambda)}\varphi_n^*(z) - \overline{\varphi_n(\lambda)}\varphi_n(z) \tag{3.5}$$

$$= \overline{\varphi_{n-1}^*(\lambda)}\varphi_{n-1}^*(z) - z\overline{\lambda \varphi_{n-1}(\lambda)}\varphi_{n-1}(z). \tag{3.6}$$

By rewriting (3.6) in the form of (1.10),

$$h_n(z) = -\overline{\lambda \varphi_{n-1}(\lambda)} \left(z\varphi_{n-1}(z) - \lambda \frac{\overline{\varphi_{n-1}^*(\lambda)}}{\overline{\varphi_{n-1}(\lambda)}} \varphi_{n-1}^*(z) \right) \tag{3.7}$$

we see that the coefficient β_{n-1} of this particular family of paraorthogonal polynomials are

$$\beta_{n-1}(h_n) = \overline{\lambda} \frac{\varphi_{n-1}^*(\lambda)}{\varphi_{n-1}(\lambda)}. \tag{3.8}$$

4. Paraorthogonal polynomials of the second kind s_n

Paraorthogonal polynomials of the second kind arise from orthogonal polynomials of the second kind, namely $\psi_k(z)$, which are orthogonal polynomials associated to the measure v with Verblunsky coefficients

$$\alpha_n(dv) = -\alpha_n(d\mu). \tag{4.1}$$

The existence of the measure is guaranteed by Verblunsky's theorem which says that for any given sequence of complex numbers inside $\mathbb{D}$, there corresponds a measure on the unit circle with such as Verblunsky coefficients.

With the same λ as we used to define $h_n(z,\lambda)$, we define our *paraorthogonal polynomials of the second kind* s_n as follows:

$$s_n(z) = \overline{\varphi_{n-1}^*(\lambda)}\psi_{n-1}^*(z) + z\overline{\lambda\varphi_{n-1}(\lambda)}\psi_{n-1}(z). \tag{4.2}$$

If we rewrite (4.2) in the form of (3.7)

$$s_n(z) = \overline{\lambda\varphi_{n-1}(\lambda)}\left(z\psi_{n-1}(z) + \lambda\frac{\overline{\varphi_{n-1}^*(\lambda)}}{\overline{\varphi_{n-1}(\lambda)}}\psi_{n-1}^*(z)\right) \tag{4.3}$$

we see that the β_n coefficient of this family of paraorthogonal polynomials $(s_n)_n$ is given by

$$\beta_n(s_n) = -\beta_n(h_n). \tag{4.4}$$

As in the case of h_n, we shall see that there are three equivalent definitions of s_n by means of the *mixed Christoffel–Darboux formulae*, which state that

$$\overline{\varphi_{n-1}^*(y)}\psi_{n-1}^*(z) + z\overline{y}\overline{\varphi_{n-1}(y)}\psi_{n-1}(z) = \overline{\varphi_n^*(y)}\psi_n^*(z) + \overline{\varphi_n(y)}\psi_n(z), \tag{4.5}$$

$$\sum_{j=0}^{n-1}\overline{\varphi_j(y)}\psi_j(z) = \frac{2 - \overline{\varphi_n^*(y)}\psi_n^*(z) - \overline{\varphi_n(y)}\psi_n(z)}{1-\overline{y}z} \quad \text{for } y \neq z. \tag{4.6}$$

The reader should refer to [7, Chapter 3.2] for the proof.

By (4.5) and (4.6), $s_n(z,\lambda)$ has the following three equivalent definitions:

$$s_n(z) = \overline{\varphi_{n-1}^*(\lambda)}\psi_{n-1}^*(z) + z\overline{\lambda\varphi_{n-1}(\lambda)}\psi_{n-1}(z) \tag{4.7}$$

$$= \overline{\varphi_n^*(\lambda)}\psi_n^*(z) + \overline{\varphi_n(\lambda)}\psi_n(z) \tag{4.8}$$

$$= -(1-\overline{\lambda}z)\sum_{j=0}^{n-1}\overline{\varphi_j(\lambda)}\psi_j(z) + 2. \tag{4.9}$$

5. Results

We prove four results concerning h_n, h_{n+1}, s_n and s_{n+1}. Some related results will be discussed.

Theorem 5.1. *Suppose $z_0 \in \partial\mathbb{D}$ distinct from λ and $\delta = \operatorname{dist}(z_0, \operatorname{supp}(d\mu)) > 0$. Then in the open disk around z_0 with radius*

$$\rho = \frac{\delta^3}{8+\delta^2} \tag{5.1}$$

either h_n or h_{n+1} (or both) has no zero inside, with the possible exception of λ.

Furthermore, if $L = \operatorname{dist}(\lambda, \operatorname{supp}(d\mu)) > 0$, then the radius could be taken as

$$\rho' = \frac{\delta^2 L}{8+\delta L}. \tag{5.2}$$

Note that when $L > \delta$, $\rho' > \rho$, hence (5.2) improves (5.1).

There is a related conjecture concerning double limit points which was proposed in [4] and proven in [2]. The result says that the set of double limit points of h_n coincides with $\mathrm{supp}(d\mu)$, except at most the point λ. In other words, if $\mathrm{dist}(z_0, \mathrm{supp}(d\mu)) > 0$, then for any sequence of integers I, there exists a subsequence $I' \subset I$ and $\varepsilon_I > 0$ such that for $n \in I'$, either h_n or h_{n+1} (or both) has no zero in the open disk $B(z_0, \varepsilon_I)$.

However, Theorem 5.1 is clearly stronger because we found an explicit radius ρ for which the double zero result holds (5.1) and the result does not depend on n.

Theorem 5.2. *The zeros of h_n and s_n strictly interlace, that is, between any two zeros of h_n (or s_n), there is one and only one zero of s_n (or h_n, respectively) in between.*

At the same time that this result was proven, Simon [8] demonstrated another way of proving the result using the theory of rank one perturbations of unitary operators. He made the observation that the CMV matrix associated to s_n is just the original one with the signs of α_j and β_{n-1} reversed, and it is unitarily equivalent to one where the signs are not reversed but the first column has opposite sign.

The main tools of the proof are the two real-valued functions σ_n and η_n which we will define in (7.3) and (7.4). They were used in Ref. [1] to prove that zeros of h_n and h_{n+1} interlace, but the method employed in our proof is different.

The remaining two results are:

Lemma 5.1. *Suppose z_0 is an isolated point in* $\mathrm{supp}(d\mu)$. *Then*

$$\tilde{\delta} = \mathrm{dist}(z_0, \mathrm{supp}(d\nu)) > 0 \tag{5.3}$$

and in the ball around z_0 with radius

$$\tilde{\rho} = \frac{\tilde{\delta}^2 |z_0 - \lambda|}{8 + |z_0 - \lambda|\tilde{\delta}} \tag{5.4}$$

either s_n or s_{n+1} (or both) has no zeros inside.

Theorem 5.3. *Suppose z_0 is an isolated point of* $\mathrm{supp}(d\mu)$ *and $\tilde{\delta}$ is as defined in* (5.3). *Then in the open disk around z_0 with radius*

$$\tilde{\rho} = \frac{\tilde{\delta}^2 |z_0 - \lambda|}{8 + |z_0 - \lambda|\tilde{\delta}} \tag{5.5}$$

either h_m or h_{n+1} (or both) has at most one zero inside.

6. Proof of Theorem 5.1

Before we start the proof, we refer to a theorem about zeros of h_n in a gap of the measure:

Theorem 6.1 (*Cantero et al. [1, Corollary 2], Golinskii [4, Theorem 2], Simon [8, Theorem 2.3]*)**.** *Let an arc $\Gamma = (\alpha, \beta)$ on $\partial\mathbb{D}$ be a gap in* $\mathrm{supp}(d\mu)$, *that is,* $\mathrm{supp}(d\mu) \cap \Gamma = \emptyset$ *and α goes to β counterclockwise. Then for each n, the paraorthogonal polynomial h_n has at most one zero in $\overline{\Gamma} = [\alpha, \beta]$.*

If λ is in a gap Γ, since λ is zero of all h_n, by Theorem 6.1 there are no other zeros of h_n or h_{n+1} in Γ. In other words, if z_0 and λ are in the same gap, in a radius $\delta = \mathrm{dist}(z_0, \mathrm{supp}(d\mu))$ around z_0 there could be no zeros other than λ. Since $\delta > \rho$, Theorem 5.1 holds. Hence if λ is in a gap, it suffices to look at the case when z_0 that sits in gaps other than Γ. In such a situation, $|z_0 - \lambda| \geqslant \mathrm{dist}(z_0, \mathrm{supp}(d\mu))$.

However, if λ is not in a gap, that is, λ is in the support of a measure, then clearly $|z_0 - \lambda| \geqslant \mathrm{dist}(z_0, \mathrm{supp}(d\mu))$.

Without loss of generality, we may assume that $|z_0 - \lambda| \geqslant \delta$ in this section.

We shall divide the proof into two lemmas:

Lemma 6.1.

$$\left| \frac{h_i(z_0)}{K_{n-1}(z_0, z_0)^{1/2}} \right| \geqslant \frac{1}{4} |\varphi_n(\lambda)| \delta^2 \tag{6.1}$$

where

$$i = \begin{cases} n & \text{if } |h_{n+1}(z_0)| \leqslant |h_n(z_0)|, \\ n+1 & \text{if } |h_n(z_0)| \leqslant |h_{n+1}(z_0)|. \end{cases}$$

Proof. Suppose $|h_{n+1}(z_0)| \leqslant |h_n(z_0)|$.

First, we give a bound for the $L^2(\mu)$ norm of $\|(z_0 - \cdot)K_{n-1}(z_0, \cdot)\|$.

By the parallelogram equality and the fact that $|\varphi_n^*(z_0)| = |\varphi_n(z_0)|$,

$$\begin{aligned} \|(z_0 - \cdot)K_{n-1}(z_0, \cdot)\|^2 &= \|\overline{\varphi_n^*(\cdot)}\varphi_n^*(z_0) - \overline{\varphi_n(\cdot)}\varphi_n(z_0)\|^2 \\ &\leqslant 2|\varphi_n^*(z_0)|^2 + 2|\varphi_n(z_0)|^2 = 4 \left| \frac{h_{n+1}(z_0) - h_n(z_0)}{(z_0 - \lambda)\overline{\varphi_n(\lambda)}} \right|^2 \\ &\leqslant \frac{4|h_{n+1}(z_0)|^2 + 4|h_n(z_0)|^2 + 8|h_{n+1}(z_0)h_n(z_0)|}{|\varphi_n(\lambda)|^2 |z_0 - \lambda|^2} \leqslant \frac{16|h_n(z_0)|^2}{|\varphi_n(\lambda)|^2|z_0 - \lambda|^2}. \quad \square \end{aligned} \tag{6.2}$$

Remark. Note that $h_{n+1}(z_0) - h_n(z_0) = (1 - \bar{\lambda} z_0)\overline{\varphi_n(\lambda)}\varphi_n(z_0)$, so it is impossible that both $h_{n+1}(z_0)$ and $h_n(z_0)$ are zero because φ has zeros inside the unit circle.

On the other hand, we observe that

$$\|K_{n-1}(z_0, \cdot)\| = \left(\int_{\partial \mathbb{D}} K_{n-1}(z_0, y)\overline{K_{n-1}(z_0, y)}\, d\mu(y) \right)^{1/2} = K_{n-1}(z_0, z_0)^{1/2}. \tag{6.3}$$

Hence

$$\|(z_0 - \cdot)K_{n-1}(z_0, \cdot)\|^2 \geqslant \mathrm{dist}(z_0, \mathrm{supp}(d\mu))^2 K_{n-1}(z_0, z_0). \tag{6.4}$$

As a result,

$$\mathrm{dist}(z_0, \mathrm{supp}(d\mu))^2 K_{n-1}(z_0, z_0) \leqslant \frac{16|h_n(z_0)|^2}{|\varphi_n(\lambda)|^2 |z_0 - \lambda|^2}. \tag{6.5}$$

This proves the case when $|h_{n+1}(z_0)| \leqslant |h_n(z_0)|$.

Now suppose $|h_{n+1}(z_0)| \leqslant |h_n(z_0)|$. The proof could be carried out in a similar manner, only that after (6.2) all appearances of h_n will be replaced by h_{n+1}. $\square$

Lemma 6.2. *Suppose τ is a zero of h_n which is distinct from λ. Let $T = \operatorname{dist}(\tau, \operatorname{supp}(d\mu))$, then*

$$|z_0 - \tau| \geqslant \frac{|h_n(z_0)|}{K_{n-1}(z_0, z_0)^{1/2}\|h_n\|} T. \tag{6.6}$$

Proof. Since τ is a zero of h_n, $g(z) = \frac{h_n(z)}{(z-\tau)}$ is a polynomial of degree $n-1$, so we can express it as

$$\frac{h_n(z)}{(z-\tau)} = \int_{\partial\mathbb{D}} K_{n-1}(z, y) g(y)\, d\mu(y). \tag{6.7}$$

By the Schwarz inequality,

$$\left|\frac{h_n(z_0)}{(z_0-\tau)}\right| \leqslant \|K_{n-1}(z_0, \cdot)\| \|g\| = K_{n-1}(z_0, z_0)^{1/2} \|g\|. \tag{6.8}$$

Also note that $\|g\| = \left\|\frac{h_n(z)}{(z-\tau)}\right\| \leqslant \frac{\|h_n\|}{T}$. Therefore,

$$|z_0 - \tau| \geqslant \frac{|h_n(z_0)|}{K_{n-1}(z_0, z_0)^{1/2}\|h_n\|} T. \qquad \square \tag{6.9}$$

Proof of Theorem 5.1. Notice that either one of the following must be true:

$$|h_{n+1}(z_0)| \leqslant |h_n(z_0)|, \tag{6.10}$$

$$|h_n(z_0)| \leqslant |h_{n+1}(z_0)|. \tag{6.11}$$

We observe that

$$\|h_n\| = \|\overline{\varphi_n^*(\lambda)}\varphi_n^*(y) - \overline{\varphi_n(\lambda)}\varphi_n(y)\|_{L^2(d\mu(y))} \leqslant 2|\varphi_n(\lambda)|. \tag{6.12}$$

If (6.10) is true, combining this with Lemmas 6.1 and 6.2, we obtain that

$$|z_0 - \tau| \geqslant \left(\frac{\delta^2|\varphi_n(\lambda)|}{4} \frac{1}{2|\varphi_n(\lambda)|}\right) T = \frac{\delta^2 T}{8}. \tag{6.13}$$

Finally, by the triangle inequality,

$$T = \operatorname{dist}(\tau, \operatorname{supp}(d\mu)) \geqslant \operatorname{dist}(z_0, \operatorname{supp}(d\mu)) - |z_0 - \tau| = \delta - |z_0 - \tau|. \tag{6.14}$$

This gives

$$|z_0 - \tau| \geqslant \frac{\delta^2(\delta - |z_0 - \tau|)}{8} \tag{6.15}$$

and the result follows.

On the other hand, if (6.11) is true, then instead of (6.12) we use the definition of h_{n+1} in (3.6) which will give the same bound of $\|h_{n+1}\|$ as in (6.12). Hence the same argument applies to h_{n+1}.

Now consider the special case where $L = \operatorname{dist}(\lambda, \operatorname{supp}(d\mu)) > 0$. Without loss of generality, suppose (6.10) is true. Since τ and λ are distinct zeros of h_n, we could apply a similar argument as in Lemma 6.2 to $\frac{h_n(z)}{(z-\tau)(z-\lambda)}$ and obtain the following:

$$|z_0 - \tau||z_0 - \lambda| \geqslant \frac{|h_n(z_0)|}{K_{n-2}(z_0, z_0)^{1/2}\|h_n\|} T L. \tag{6.16}$$

Since $K_{n-2}(z_0, z_0)^{1/2} \leqslant K_{n-1}(z_0, z_0)^{1/2}$, the desired inequality follows. Now we combine (6.16) with Lemma 6.1. The $|z_0 - \lambda|$ term cancels on both sides and it gives us

$$|z_0 - \tau| \geqslant \frac{\delta L T}{8}. \tag{6.17}$$

Again, we use the triangle inequality on T and the result follows. Clearly, if (6.11) is true, we could still apply the same argument to h_{n+1}. □

7. Proof of Theorem 5.2

Proof. According to the definitions of φ_n^* and ψ_n^*,

$$s_n(z) = \overline{\lambda^n} z^n \varphi_n(\lambda)\overline{\psi_n(z)} + \overline{\varphi_n(\lambda)}\psi_n(z), \tag{7.1}$$

$$h_n(z) = \overline{\lambda^n} z^n \varphi_n(\lambda)\overline{\varphi_n(z)} - \overline{\varphi_n(\lambda)}\varphi_n(z). \tag{7.2}$$

If we define for $z \in \partial\mathbb{D}$

$$\sigma_n(z) := \frac{s_n(z)}{(\bar{\lambda} z)^{n/2}}, \tag{7.3}$$

$$\eta_n(z) := \frac{h_n(z)}{i(\bar{\lambda} z)^{n/2}} \tag{7.4}$$

with $\mathrm{Arg}((\bar{\lambda}\mathrm{z})^{1/2}) \in [0, \pi)$, then σ_n and η_n are real-valued C^∞ functions and they have the same zeros as s_n and h_n, respectively.

To prove the interlacing condition of Theorem 5.2, it suffices to prove the following:

$$\frac{d\eta_n(e^{i\theta})}{d\theta}\sigma_n(e^{i\theta}) < 0 \quad \text{at every zero } e^{i\theta} \text{ of } \eta_n(z). \tag{7.5}$$

We shall prove condition (7.5) for $n+1$.

Suppose ζ is a zero of h_{n+1}. By (2.4), h_{n+1} could be expressed by the reproducing kernel. Hence η_{n+1} can be represented as

$$\eta_{n+1}(z) = \frac{1}{i(\bar{\lambda} z)^{(n+1)/2}} \frac{\overline{-\lambda\varphi_n(\lambda)}}{\overline{\varphi_n(\zeta)}}(z-\zeta)\sum_{j=0}^{n} \varphi_j(z)\overline{\varphi_j(\zeta)}. \tag{7.6}$$

The constant $\frac{\overline{-\lambda\varphi_n(\lambda)}}{\overline{\varphi_n(\zeta)}}$ is obtained by comparing the leading coefficients of the right-hand side of (7.6) and that of h_{n+1} when expressed in terms of (3.6).

As a result, the derivative of η_{n+1} at ζ is

$$\begin{aligned}\frac{d\eta_{n+1}}{dz}(\zeta) &= \lim_{z\to\zeta}\frac{\eta_{n+1}(z)-\eta_{n+1}(\zeta)}{z-\zeta} = \lim_{z\to\zeta}\frac{\eta_{n+1}(z)}{z-\zeta} \\ &= \frac{\overline{-\lambda\varphi_n(\lambda)}}{i\overline{\varphi_n(\zeta)}}\left(\frac{\lambda}{\zeta}\right)^{\frac{n+1}{2}} K_n(\zeta,\zeta).\end{aligned} \tag{7.7}$$

Let $\zeta = e^{i\theta}$ and $z = e^{i\omega}$. By the chain rule,

$$\frac{d\eta_{n+1}}{d\omega}(\theta) = i\zeta\frac{d\eta_{n+1}}{dz}(\zeta) = -\frac{\overline{\varphi_n(\lambda)}}{\overline{\varphi_n(\zeta)}}\left(\frac{\lambda}{\zeta}\right)^{\frac{n-1}{2}} K_n(\zeta,\zeta). \tag{7.8}$$

Now we go back to $\frac{d\eta_n(e^{i\theta})}{d\theta}\sigma_n(e^{i\theta})$ and compute

$$\begin{aligned}
&\frac{d\eta_{n+1}(e^{i\theta})}{d\theta}\sigma_{n+1}(e^{i\theta}) \\
&= -\frac{\overline{\varphi_n(\lambda)}}{\overline{\varphi_n(\zeta)}}\left(\frac{\lambda}{\zeta}\right)^n K_n(\zeta,\zeta)\left(\overline{\varphi_n^*(\lambda)}\psi_n^*(\zeta)+\bar{\lambda}\zeta\overline{\varphi_n(\lambda)}\psi_n(\zeta)\right) \\
&= -\left(\frac{\lambda}{\zeta}\right)^n K_n(\zeta,\zeta)\left(|\varphi_n(\lambda)|^2\left(\frac{\zeta}{\lambda}\right)^n\frac{\overline{\psi_n(\zeta)}}{\overline{\varphi_n(\zeta)}}+\bar{\lambda}\zeta\frac{\overline{\varphi_n(\lambda)}}{\overline{\varphi_n(\zeta)}}\overline{\varphi_n(\lambda)}\psi_n(\zeta)\right).
\end{aligned} \tag{7.9}$$

Recall that $\eta_{n+1}(\zeta)=0$, which implies that

$$\frac{\overline{\varphi_n(\lambda)}}{\overline{\varphi_n(\zeta)}}=\frac{\varphi_n(\lambda)}{\varphi_n(\zeta)}\left(\frac{\zeta}{\lambda}\right)^{n-1}. \tag{7.10}$$

We then apply this onto the second part of the summand in (7.9):

$$\begin{aligned}
(7.9) &= -\left(\frac{\lambda}{\zeta}\right)^n K_n(\zeta,\zeta)\left(|\varphi_n(\lambda)|^2\left(\frac{\zeta}{\lambda}\right)^n\frac{\overline{\psi_n(\zeta)}}{\overline{\varphi_n(\zeta)}}+\left(\frac{\zeta}{\lambda}\right)^n\frac{\varphi_n(\lambda)}{\varphi_n(\zeta)}\overline{\varphi_n(\lambda)}\psi_n(\zeta)\right) \\
&= -K_n(\zeta,\zeta)|\varphi_n(\lambda)|^2\left(\frac{\overline{\psi_n(\zeta)}}{\overline{\varphi_n(\zeta)}}+\frac{\psi_n(\zeta)}{\varphi_n(\zeta)}\right) \\
&= -K_n(\zeta,\zeta)\left|\frac{\varphi_n(\lambda)}{\varphi_n(\zeta)}\right|^2\left(\overline{\psi_n(\zeta)}\varphi_n(\zeta)+\overline{\varphi_n(\zeta)}\psi_n(\zeta)\right).
\end{aligned} \tag{7.11}$$

Now we use a formula that relates φ_n and ψ_n (see [7, Chapter 3.2]):

$$\overline{\psi_n(z)}\varphi_n(z)+\overline{\varphi_n(z)}\psi_n(z)=2 \quad \text{in } \partial\mathbb{D}. \tag{7.12}$$

We apply (7.12) to (7.11). This gives us the result that at any zero ζ of η_{n+1}:

$$\frac{d\eta_{n+1}(e^{i\theta})}{d\theta}\sigma_{n+1}(e^{i\theta})=(7.9)=-2K_n(\zeta,\zeta)\left|\frac{\varphi_n(\lambda)}{\varphi_n(\zeta)}\right|^2<0. \tag{7.13}$$

The interlacing theorem is proven. □

8. Proof of Lemma 5.1

We prove Lemma 5.1 by stating several lemmas which are similar to those in the proof of Theorem 5.1.

Lemma 8.1. *Suppose* $\tilde{\delta}=\operatorname{dist}(z_0,\operatorname{supp}(d\nu))>0$ *and* $\tilde{K}_n(x,y)=\sum_{j=0}^n\psi_j(x)\overline{\psi_j(y)}$ *is the reproducing kernel with respect to the measure* ν. *Then*

$$\left|\frac{s_i(z_0)}{\tilde{K}_{n-1}(z_0,z_0)^{1/2}}\right|\geqslant\frac{1}{4}|\varphi_n(\lambda)||z_0-\lambda|\tilde{\delta}, \tag{8.1}$$

where

$$i=\begin{cases} n & \text{if } |s_{n+1}(z_0)|\leqslant|s_n(z_0)|,\\ n+1 & \text{if } |s_n(z_0)|\leqslant|s_{n+1}(z_0)|.\end{cases}$$

Proof. The proof is essentially the same as the one of Lemma 6.1, except for a few differences. The L^2 norm here refers to the one taken with respect to v and h_n is replaced by s_n.

It is also worth noting that by the definition of s_n in (4.9),

$$s_{n+1}(z) - s_n(z) = -(1-\bar{\lambda}z)\overline{\varphi_n(\lambda)}\psi_n(z) \neq 0 \quad \text{on } \partial\mathbb{D}. \tag{8.2}$$

As a result,

$$|\psi_n(z_0)| = \left| \frac{s_{n+1}(z_0) - s_n(z_0)}{(z_0-\lambda)\varphi_n(\lambda)} \right| \tag{8.3}$$

which allows us to proceed in the same way as in the proof of Lemma 6.1. □

Lemma 8.2. *Suppose* $\tilde{\tau}$ *is a zero of* s_n. *Let* $\widetilde{T} = \operatorname{dist}(\tilde{\tau}, \operatorname{supp}(dv))$, *then*

$$|z_0 - \tilde{\tau}| \geqslant \frac{|s_n(z_0)|}{\tilde{K}_{n-1}(z_0, z_0)^{1/2} \|s_n\|_{L^2(dv)}} \widetilde{T}. \tag{8.4}$$

The proof of this lemma is omitted because it resembles that of Lemma 6.2.

Finally, we state the following lemma relating the support of μ and v:

Lemma 8.3. *Suppose* z_0 *is an isolated point in the support of* μ. *Then*

$$\tilde{\delta} = \operatorname{dist}(z_0, \operatorname{supp}(dv)) > 0. \tag{8.5}$$

The reader could refer to [7, Chapter 3.2, p. 225] for the proof.

Next, we are going to finish the proof of Lemma 5.1.

Proof. Suppose z_0 is an isolated point in the support of $d\mu$ which is distinct from λ. By Lemma 8.3, $\operatorname{dist}(z_0, \operatorname{supp}(dv)) > 0$.

Either $|s_n(z_0)| \geqslant |s_{n+1}(z_0)|$ or $|s_n(z_0)| \leqslant |s_{n+1}(z_0)|$ is true. Without loss of generality, we assume that $|s_n(z_0)| \geqslant |s_{n+1}(z_0)|$ and use Lemma 8.1.

Furthermore, we observe that

$$\|s_n\| \leqslant 2|\varphi_n(\lambda)| \|\psi_n\|_{L^2(dv)} = 2|\varphi_n(\lambda)|. \tag{8.6}$$

Then we combine these results to get

$$|z_0 - \tilde{\tau}| \geqslant \frac{|z_0 - \lambda| \tilde{\delta} \widetilde{T}}{8}. \tag{8.7}$$

Finally, we apply the triangle inequality to $\widetilde{T}$:

$$\widetilde{T} = \operatorname{dist}(\tilde{\tau}, \operatorname{supp}(dv)) \geqslant \operatorname{dist}(z_0, \operatorname{supp}(dv)) - |z_0 - \tilde{\tau}| = \tilde{\delta} - |z_0 - \tilde{\tau}|. \tag{8.8}$$

This gives us the following inequality which finishes the proof:

$$|z_0 - \tilde{\tau}| \geqslant \frac{\tilde{\delta}^2 |z_0 - \lambda|}{8 + |z_0 - \lambda| \tilde{\delta}}. \qquad \square \tag{8.9}$$

9. Proof of Theorem 5.3

Proof. By Lemma 5.1, inside the ball $B(z_0, \tilde{\rho})$ either s_n or s_{n+1} (or both) has no zero inside, with $\tilde{\rho}$ given by (8.9). Without loss of generality, we assume that s_n does not have zeros inside. By Theorem 5.2 the zeros of h_n and s_n interlace, therefore h_n cannot have more than two zeros inside $B(z_0, \tilde{\rho})$. □

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