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$(1/\alpha)$-Self Similar α-Stable Processes with Stationary Increments*

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In this note we settle a question posed by Kasahara, Maejima, and Vervaat. We show that the α-stable Lévy motion is the only $(1/\alpha)$-self-similar α-stable process with stationary increments if $0<\alpha<1$. We also introduce new classes of $(1/\alpha)$-self-similar α-stable processes with stationary increments for $1<\alpha<2$. © 1990 Academic Press, Inc.

1. INTRODUCTION

A stochastic process $\{X(t), t \geqslant 0\}$ is called *α-stable*, $0<\alpha \leqslant 2$, if its finite-dimensional distributions are α-stable, and it is called *H-self-similar*, $H>0$, if for every $c>0$, $\{X(ct), t \geqslant 0\} \stackrel{d}{=} \{c^H X(t), t \geqslant 0\}$ in the sense of equality of the finite-dimensional distributions. The class of α-stable H-self-similar processes with stationary increments (H-sssi processes) has been extensively studied in recent years. (Kasahara, Maejima, and Vervaat [4], Cambanis and Maejima [1], Samorodnitsky and Taqqu [9], Takenaka [10]. An extensive list of references can be found in Taqqu [11], and Maejima [7]). It is known in particular that the self-similarity parameter H can never exceed $\max(1, 1/\alpha)$ [6]. Much of the research in this area has been concentrated on constructing examples of α-stable H-sssi processes with (α, H) in the feasible region. One major problem is to show that two such stochastic

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processes are really different, i.e., that they do not satisfy $\{X_1(t), t \geqslant 0\} \stackrel{d}{=} \{cX_2(t), t \geqslant 0\}$ for some constant c.

The first goal of this note is to solve the problem posed by Kasahara, Maejima, and Vervaat [4], namely, to show that the only α-stable $(1/\alpha)$-sssi process with $0<\alpha<1$ is the α-stable Lévy motion. This is done in Section 2.

The second goal, achieved in Section 3, is to obtain new classes of $(1/\alpha)$-sssi processes with $1<\alpha<2$. This is done by considering classes of α-stable H-sssi processes, $0<H<1$, related to multiparameter processes described in Takenaka [10]. We use a new technique developed by Samorodnitsky and Taqqu [8] to show that these classes are disjoint. The technique is based on the properties of the conditional distributions of α-stable processes.

2. α-STABLE $(1/\alpha)$-SSSI PROCESSES WITH $0<\alpha<1$

It is easy to see that strictly α-stable Lévy motions (i.e., processes with stationary independent increments having a strictly α-stable distribution) are $(1/\alpha)$-sssi processes. Are there any others? In the Gaussian case $\alpha=2$, the answer is easily seen to be negative. The answer is positive when $1<\alpha<2$ (see [4] and Section 3 for more details). The answer is positive for $\alpha=1$ as well, because if $X(1)$ has a 1-stable law then the linear function with random slope $X(t)=tX(1)$, $t \geqslant 0$, is 1-sssi [4]. The problem has been open in the case $0<\alpha<1$. We settle it through the following result.

THEOREM 2.1. *The only non-degenerate α-stable $(1/\alpha)$-sssi processes with $0<\alpha<1$ are the strictly α-stable Lévy motions.*

Proof. Let $\{X(t), t \geqslant 0\}$ be a non-degenerate (i.e., $X(1) \neq 0$ a.s.) α-stable $(1/\alpha)$-sssi process with $0<\alpha<1$. It follows from Theorem A of [4] that $\{X(t), t \geqslant 0\}$ must be strictly α-stable. Let σ_t denote the scaling parameter of the α-stable random variable $X(t)$. Then $\sigma_t = t^{1/\alpha}\sigma_1$ by $(1/\alpha)$-self-similarity. Fix arbitrary $0 \leqslant s_1 < s_2 \leqslant t_1 < t_2$. The random variables $X(s_1)$, $X(s_2)$, $X(t_1)$, and $X(t_2)$ are jointly strictly α-stable, and thus there are functions $f_{s_1}, f_{s_2}, f_{t_1}$, and f_{t_2} in $L^\alpha([0,1])$ such that

$$(X(s_1), X(s_2), X(t_1), X(t_2))$$
$$\stackrel{d}{=} \left(\int_0^1 f_{s_1}(x)\, M(dx), \int_0^1 f_{s_2}(x)\, M(dx), \int_0^1 f_{t_1}(x)\, M(dx), \int_0^1 f_{t_2}(x)\, M(dx)\right),$$

where M is an independently scattered α-stable measure on $([0,1], \mathscr{B})$ with Lebesgue control measure and skewness intensity $\beta \equiv 1$ [3]. We have

$$\begin{aligned} t_2 \sigma_1^\alpha = \sigma_{t_2}^\alpha &= \int_0^1 |f_{t_2}(x)|^\alpha \, dx \\ &\leqslant \int_0^1 |f_{s_1}(x)|^\alpha \, dx + \int_0^1 |f_{s_2}(x) - f_{s_1}(x)|^\alpha \, dx \\ &\quad + \int_0^1 |f_{t_1}(x) - f_{s_2}(x)|^\alpha \, dx + \int_0^1 |f_{t_2}(x) - f_{t_1}(x)|^\alpha \, dx \qquad (2.1) \\ &= \sigma_{s_1}^\alpha + \sigma_{s_2 - s_1}^\alpha + \sigma_{t_1 - s_2}^\alpha + \sigma_{t_2 - t_1}^\alpha \\ &= s_1 \sigma_1^\alpha + (s_2 - s_1)\, \sigma_1^\alpha + (t_1 - s_2)\, \sigma_1^\alpha + (t_2 - t_1)\, \sigma_1^\alpha = t_2 \sigma_1^\alpha. \end{aligned}$$

Here we have used the stationarity of the increments of $\{X(t),\ t \geqslant 0\}$. Thus the inequality in (2.1) is, actually, an equality, implying

$$(f_{s_2}(x) - f_{s_1}(x))(f_{t_2}(x) - f_{t_1}(x)) = 0 \qquad \text{a.e.}$$

It follows from Theorem 2.3 of [3] that $X(s_2) - X(s_1)$ and $X(t_2) - X(t_1)$ are independent for any $0 \leqslant s_1 < s_2 \leqslant t_1 \leqslant t_2$, and since for jointly stable random variables pairwise independence is equivalent to total independence, we conclude that $\{X(t),\ t \geqslant 0\}$ has independent increments. That is, $\{X(t),\ t \geqslant 0\}$ is a strictly α-stable Lévy motion. ∎

3. New Classes of α-stable H-sssi Processes

Let $n \geqslant 2$, $0 < \alpha < 2$, and let M be an independently scattered α-stable random measure on $(\mathbf{R}^n, \mathscr{B}^n)$ with (n-dimensional) Lebesgue control measure and constant skewness intensity β. In the case $\alpha = 1$, we assume $\beta \equiv 0$. Let $\|\cdot\|$ be the Euclidean norm on $\mathbf{R}^n$ (any other norm will do as well). For a fixed $H \in (0, 1)$, set

$$X_{n,\alpha,H}(t) = \int_{\mathbf{R}^n} \left(\|\mathbf{x} - t\mathbf{1}\|^{H - (n/\alpha)} - \|\mathbf{x}\|^{H - (n/\alpha)} \right) M(d\mathbf{x}), \quad t \geqslant 0. \qquad (3.1)$$

Here $\mathbf{x} = (x_1, ..., x_n)$, and $\mathbf{1} = (1, ..., 1) \in \mathbf{R}^n$. It is easy to check that the integrand in (3.1) is in $L^\alpha(\mathbf{R}^n)$, and thus $\{X(t), t \geqslant 0\}$ is a well-defined strictly α-stable process. It is a matter of simple algebra to check that $\{X(t),\ t \geqslant 0\}$ is an H-sssi process. The process (3.1) is a natural extension of an α-stable fractional Lévy motion [5]. It is related to the processes introduced by Takenaka [10, Theorem 2].

Our goal is to prove that the processes $\{X_{n,\alpha,H}(t),\ t \geqslant 0\}$ and $\{X_{m,\alpha,H}(t),\ t \geqslant 0\}$ are different if $m \neq n$ in the sense that there is no constant c such that $\{X_{n,\alpha,H}(t),\ t \geqslant 0\} \overset{d}{=} \{cX_{m,\alpha,H}(t),\ t \geqslant 0\}$. They therefore form new families of α-stable H-sssi processes.

THEOREM 3.1. *For any $m,\ n \geqslant 2,\ m \neq n$, any $0 < \alpha < 2,\ 0 < H < 1$, the processes $\{X_{n,\alpha,H}(t),\ t \geqslant 0\}$ and $\{X_{m,\alpha,H}(t),\ t \geqslant 0\}$ are different.*

Proof. The idea of the proof is to show that the two-dimensional distributions of the two processes have different properties. Formally, suppose that there is a c such that $\{X_{n,\alpha,H}(t),\ t \geqslant 0\} \overset{d}{=} \{cX_{m,\alpha,H}(t),\ t \geqslant 0\}$. Letting $\{X^{(i)}_{n,\alpha,H}(t),\ t \geqslant 0\}$ and $\{X^{(i)}_{m,\alpha,H}(t),\ t \geqslant 0\}$, $i = 1, 2$, be independent copies of $\{X_{n,\alpha,H}(t),\ t \geqslant 0\}$ and $\{X_{m,\alpha,H}(t),\ t \geqslant 0\}$, respectively, and setting $Y_{n,\alpha,H}(t) = 2^{-1/\alpha}(X^{(1)}_{n,\alpha,H}(t) - X^{(2)}_{n,\alpha,H}(t)),\ t \geqslant 0,\ Y_{m,\alpha,H}(t) = 2^{-1/\alpha}(X^{(1)}_{m,\alpha,H}(t) - X^{(2)}_{m,\alpha,H}(t)),\ t \geqslant 0$, we conclude that $\{Y_{n,\alpha,H}(t),\ t \geqslant 0\}$ and $\{Y_{m,\alpha,H}(t),\ t \geqslant 0\}$ are $S\alpha S$ H-sssi processes having a representation (3.1), where now M is a symmetric α-stable (SαS) random measure with Lebesgue control measure; that is, its skewness intensity β is identically zero. Moreover, $\{Y_{n,\alpha,H}(t),\ t \geqslant 0\} \overset{d}{=} \{cY_{m,\alpha,H}(t),\ t \geqslant 0\}$. In particular,

$$(Y_{n,\alpha,H}(1),\ Y_{n,\alpha,H}(2)) \overset{d}{=} (cY_{m,\alpha,H}(1),\ cY_{m,\alpha,H}(2)). \tag{3.2}$$

We shall use

LEMMA 3.1. *Let (X_1, X_2) be a $S\alpha S$ random vector with two integral representations:*

$$(X_1, X_2) \overset{d}{=} \left(\int_{E_i} f_1^{(i)}(x)\, M_i(dx),\ \int_{E_i} f_2^{(i)}(x)\, M_i(dx) \right),\ i = 1, 2,$$

where M_1 and M_2 are $S\alpha S$ random measures on $(E_1, \mathscr{E}_1)$ and $(E_2, \mathscr{E}_2)$, respectively, whose corresponding control measures are m_1 and m_2, and $f_j^{(i)} \in L^\alpha(m_i)$, $j = 1, 2$, $i = 1, 2$. Then for every $\nu > 0$,

$$\int_{E_1^+} \frac{|f_2^{(1)}(x)|^{\alpha+\nu}}{|f_1^{(1)}(x)|^{\nu}}\, m_1(dx) < \infty \tag{3.3}$$

if and only if

$$\int_{E_2^+} \frac{|f_2^{(2)}(x)|^{\alpha+\nu}}{|f_1^{(2)}(x)|^{\nu}}\, m_2(dx) < \infty, \tag{3.4}$$

where $E_i^+ = \{x \in E_i : f_1^{(i)}(x)^2 + f_2^{(i)}(x)^2 \neq 0\}$, $i = 1, 2$.

Proof. Both (3.3) and (3.4) are equivalent to $\int_{S_2} (\Gamma(ds)/|s_1|^\nu) < \infty$, where S_2 is the unit circle and Γ is the spectral measure of (X_1, X_2). (See Samorodnitsky and Taqqu [8].) ∎

Applying (3.3) to (3.2), we obtain

$$\int_{\mathbf{R}^n} \frac{\left|\left(\sum_{i=1}^n (x_i-2)^2\right)^{(H/2)-(n/2\alpha)} - \left(\sum_{i=1}^n x_i^2\right)^{(H/2)-(n/2\alpha)}\right|^{\alpha+\nu}}{\left|\left(\sum_{i=1}^n (x_i-1)^2\right)^{(H/2)-(n/2\alpha)} - \left(\sum_{i=1}^n x_i^2\right)^{(H/2)-(n/2\alpha)}\right|^{\nu}} dx_1 \cdots dx_n < \infty \tag{3.5}$$

if and only if

$$\int_{\mathbf{R}^m} \frac{\left|\left(\sum_{i=1}^m (x_i-2)^2\right)^{(H/2)-(m/2\alpha)} - \left(\sum_{i=1}^m x_i^2\right)^{(H/2)-(m/2\alpha)}\right|^{\alpha+\nu}}{\left|\left(\sum_{i=1}^m (x_i-1)^2\right)^{(H/2)-(m/2\alpha)} - \left(\sum_{i=1}^m x_i^2\right)^{(H/2)-(m/2\alpha)}\right|^{\nu}} dx_1 \cdots dx_m < \infty.$$

It is now a matter of algebra to check that the left-hand side of (3.5) is finite if and only if

$$0 < \nu < \frac{\alpha H}{2/\alpha - H} \wedge 1 \qquad \text{if} \quad n = 2$$

and

$$0 < \nu < \frac{\alpha H}{n/\alpha - H} \qquad \text{if} \quad n \geqslant 3.$$

Since $m \neq n$, this contradicts (3.2), and thus completes the proof of the theorem. ∎

Remarks. 1. The relations $0 < \alpha < 2$ and $0 < H < 1$ imply $\alpha H/((n/\alpha) - H) < 1$ if $n \geqslant 3$.

2. Let M be an independently scattered SαS random measure with Lebesgue control measure. The log-fractional α-stable motion, $1 < \alpha < 2$, is the process $\int_{-\infty}^{+\infty} (\ln|t-x| - \ln|x|)\, M(dx)$, $t \geqslant 0$, discovered by Kasahara, Maejima, and Vervaat [4]. It is $(1/\alpha)$-sssi. Cambanis and Maejima [1] show that the linear combinations

$$\Delta_{a,b,\alpha}(t) = a \int_0^t M(dx) + b \int_{-\infty}^{+\infty} (\ln|t-x| - \ln|x|)\, M(dx), \qquad t \geqslant 0, \tag{3.6}$$

of the Lévy-stable motion and the log-fractional α-stable motion, define essentially different processes parametrized by $-\infty < a$, $b < \infty$, $|a| + |b| > 0$. These are "moving-average"-type processes, as are the processes (3.1). It is easy to check that the processes (3.6) satisfy (3.3) for any $\nu > 0$ if $b \neq 0$ and they satisfy it only for $\nu = 0$ if $b = 0$. Therefore, the classes of processes (3.1) with $H = 1/\alpha$ and (3.6) are different.

3. The supremum of $\nu > 0$ for which the integrals in (3.3) are finite is related to the existence of conditional moments of the type $E(|X_2|^p \mid X_1)$ (Samorodnitsky and Taqqu [8]). Therefore, the argument used in the

proof of Theorem 3.1 shows that the dependence structure of the processes $\{X_{n,\alpha,H}(t),\ t \geqslant 0\}$ for different n's is very different. For example, it follows from Theorems 3.1 and 4.1 of Samorodnitsky and Taqqu [8] that if $1 < \alpha < 2$, then $E(X_{n,\alpha,H}(t)^2 \mid X_{n,\alpha,H}(s)) < \infty$ a.s. for any $0 < s < t$ if $n \leqslant 2H/(2/\alpha - 1)$, and it follows from Theorem 1 of Cambanis and Wu [2] that the conditional second moment above is a.s. infinite if $n > 2H/(2/\alpha - 1)$.

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