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Revisiting Hua–Marcus–Bellman–Ando inequalities on contractive matrices

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Abstract

Loo-Keng Hua showed some elegant matrix and determinant inequalities via a matrix identity and proved the positive semidefiniteness of a matrix involving the determinants of contractive matrices through group representation theory. His study was followed by M. Marcus, R. Bellman and T. Ando. The purpose of current paper is to revisit the Hua's original work and the results of Marcus, Bellman and Ando with our comments, and to present analogs and extensions to their results.

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1. Introduction

In studying the theory of functions of several variables, Loo-Keng Hua (1910–1985) discovered a matrix identity and showed some elegant determinantal inequalities on contractive matrices in 1955 [7]. The results were soon reviewed by Shiing-shen Chern (1911–2004) [4]. Similar types of matrices were also used in Hua's research on automorphic functions of a matrix variable [5, p. 476] and on harmonic functions [5, pp. 1045–1088]. Both Hua and Chern were distinguished mathematicians and influential figures in modern Chinese history. Motivated by Hua's work, Marcus [8], Bellman [2] and Ando [1] continued the research on the topic and showed more results on contractive matrices. The purpose of this paper is to revisit these inequalities with our remarks and comments and to present some related inequalities. Our results are analogous or complement to Hua's; we will also show some results on contractive matrices that may be compared respectively to Marcus', Bellman's, and Ando's. To be precise, we show that the matrix $((I - u_i^* u_j)^{-1})$ and the matrix $((\mathrm{tr}(I - A_i^* A_j))^{-1})$ are positive semidefinite, where u_i are complex row vectors having norm less than 1, and A_i are strictly contractive matrices.

Let H be a square complex matrix of finite dimension. As usual, we write $H \geqslant 0$ if H is positive semidefinite and $H > 0$ if H is positive definite. For Hermitian matrices A and B of the same size, we write $A \geqslant B$ if $A - B \geqslant 0$. For a complex matrix A, we denote by A^* the conjugate transpose of A.

An $m \times n$ complex matrix A is said to be strictly *contractive* if $I_n - A^*A > 0$, where I_n, or simply I, is the identity matrix. Equivalently, A is a strict contraction if the largest singular value of A is less than 1. From now on, by writing $A \in \mathcal{SC}_{m\times n}$, we mean that A is an $m \times n$ strictly contractive matrix. If $I - A^*A \geqslant 0$, we say that A is a *contraction* and denote it by $A \in \mathcal{C}_{m\times n}$.

We begin by citing Hua's main results in [7] with our remarks.

Theorem 1 (Theorems 1 and 2 in [7]). *Let $A, B \in \mathcal{SC}_{n\times n}$. Then*

$$\begin{aligned}(I - B^*B) &+ (A - B)^*(I - AA^*)^{-1}(A - B)\\ &= (I - B^*A)(I - A^*A)^{-1}(I - A^*B).\end{aligned} \tag{1}$$

The matrix identity (1) *implies the determinantal inequalities*

$$\det(I - A^*A)\det(I - B^*B) + |\det(A - B)|^2 \leqslant |\det(I - A^*B)|^2 \tag{2}$$

and

$$\det(I - A^*A)\det(I - B^*B) \leqslant |\det(I - A^*B)|^2. \tag{3}$$

Equality in (2) *or* (3) *holds if and only if $A = B$.*

Since for positive semidefinite matrices M and S, $\det(M + S) \geqslant \det(M) + \det(S)$ (Lemma 1 in [7]), we see that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Actually the main goal of the first part of the Hua's paper was to show the determinantal inequality (3), which obviously follows from (2). In order to show (2), Hua proved the matrix identity (1) in the proof of his Theorem 1. In addition, the identity (1) yields

$$I - B^*B \leqslant (I - B^*A)(I - A^*A)^{-1}(I - A^*B), \tag{4}$$

which also gives (3). Notice that (3) is equivalent to

$$\begin{pmatrix}\det(I - A^*A)^{-1} & \det(I - A^*B)^{-1}\\ \det(I - B^*A)^{-1} & \det(I - B^*B)^{-1}\end{pmatrix} \geqslant 0. \tag{5}$$

Since (1) is the key to all his inequalities (2)–(5), we have singled it out.

Note that the contractiveness of B is not needed in the Hua's proof of (1). The original proof of (1) by Hua, though neat, is purely algebraic and rather technical. In a recent paper [9], two new proofs for Hua's matrix identity and inequality by using Schur complements and a generalization of Sylvester's law of inertia are given. In next section, we shall generalize the Hua's matrix identity, through which we will present a determinantal (upper bound) inequality that is complement to the Hua's determinantal (lower bound) inequality (2).

The second part of the Hua's paper was devoted to the positive semidefiniteness of a square matrix involving the determinants of contractive matrices.

Theorem 2 (Theorem 3 in [7]). *Let $A_1, \ldots, A_m \in \mathcal{SC}_{n\times n}$. Then for any integer $k \geqslant n-1$,*

$$\begin{pmatrix} \frac{1}{\det(I-A_1^*A_1)^k} & \frac{1}{\det(I-A_1^*A_2)^k} & \cdots & \frac{1}{\det(I-A_1^*A_m)^k} \\ \frac{1}{\det(I-A_2^*A_1)^k} & \frac{1}{\det(I-A_2^*A_2)^k} & \cdots & \frac{1}{\det(I-A_2^*A_m)^k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\det(I-A_m^*A_1)^k} & \frac{1}{\det(I-A_m^*A_2)^k} & \cdots & \frac{1}{\det(I-A_m^*A_m)^k} \end{pmatrix} \geqslant 0. \tag{6}$$

Apparently this is the main result of [7]. The proof of (6) by Hua is quite lengthy and the theory of group representation along with the complex variable functions is heavily used. Using integral representation of determinants, Bellman [2, Theorem 4] proved that (6) holds for all positive integers k. Notice that if A and B are positive semidefinite matrices of the same size, so are the Hadamard (also known as Schur or entrywise) product $A \circ B$ and the Hadamard power $A^{[k]} = (a_{ij}^k)$ for every positive integer k (see our remark below). Thus Bellman's result is equivalent to saying

$$\begin{pmatrix} \frac{1}{\det(I-A_1^*A_1)} & \frac{1}{\det(I-A_1^*A_2)} & \cdots & \frac{1}{\det(I-A_1^*A_m)} \\ \frac{1}{\det(I-A_2^*A_1)} & \frac{1}{\det(I-A_2^*A_2)} & \cdots & \frac{1}{\det(I-A_2^*A_m)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\det(I-A_m^*A_1)} & \frac{1}{\det(I-A_m^*A_2)} & \cdots & \frac{1}{\det(I-A_m^*A_m)} \end{pmatrix} \geqslant 0. \tag{7}$$

We shall present an analog to (7) with determinant replaced by trace.

In his Theorem 4 in [7], Hua stated that if $H = (h_{ij}) > 0$, then for any positive integer r, $H^{[r]} = (h_{ij}^r) > 0$. We point out that this follows immediately from a 1911 result of I. Schur which states that if $A, B > 0$ then $A \circ B > 0$ (see, e.g., [6] or [10]). Hua continued to assert in his Theorem 5 that if $H = (H_{ij}) > 0$ is a partitioned matrix, where all H_{ij} are square, then $(\det(H_{ij})) > 0$. In view of this, together with (5) and (7), it is natural and attempting to ask whether the matrix

$$H_m = (H_{ij}), \quad \text{where } H_{ij} = (I - A_i^*A_j)^{-1},\ i, j = 1, 2, \ldots, m$$

is positive semidefinite. If so, then Bellman's result (7), thus Hua's (6), would follow at once. Ando [1] considered the question with i, j switched for A's in H_{ij} and answered it in the negative for three strictly contractive matrices. We shall discuss H_m in Section 3.

2. Generalizing Hua's matrix identity

The Hua determinant inequality (2) provides a lower bound for $|\det(I - A^*B)|^2$, while the Hua matrix identity (1) is pivotal to obtaining the lower bound. In this section we generalize

Hua's matrix identity to arbitrary matrices without contractiveness, from which not only will Hua's matrix identity follow but also an (known) upper bound for $|\det(I-A^*B)|^2$ is immediate; the upper bound is as strong as the lower bound in the Hua's determinantal inequality.

Let $G=\begin{pmatrix} C & D \\ E & F \end{pmatrix}$ be a partitioned matrix. If C is square and nonsingular, then $F-EC^{-1}D$ is called the Schur complement of C in G, denoted by G/C. A celebrated theorem on the inverse of a partitioned matrix, known as the Banachiewicz inversion formula (see, e.g., [12, pp. 10–14]), is that if G and C are both invertible, then the inverse of G takes the form

$$G^{-1}=\begin{pmatrix} \times & \times \\ \times & (G/C)^{-1} \end{pmatrix},$$

where $\times$ denotes irrelevant entries in our following discussions.

Now we are in the position to present a matrix identity that generalizes Hua's matrix identity and determinantal inequalities that are analogous to Hua's.

Theorem 3. *Let X and Y be $m\times n$ matrices, and Z and W be $n\times m$ matrices. If $I+WY$ is nonsingular (equivalently, WY has no eigenvalues -1), then*

$$(I+XZ)-(X+Y)(I+WY)^{-1}(W+Z)=(I-XW)(I+YW)^{-1}(I-YZ). \tag{8}$$

Proof. Let I represent identity matrices of appropriate sizes and let

$$M=\begin{pmatrix} I & W \\ X & I \end{pmatrix}\begin{pmatrix} I & Z \\ Y & I \end{pmatrix}=\begin{pmatrix} I+WY & W+Z \\ X+Y & I+XZ \end{pmatrix}.$$

If $I-XW$ and $I-YZ$ are both nonsingular, then M is nonsingular, and

$$\begin{aligned} M^{-1}&=\begin{pmatrix} I & Z \\ Y & I \end{pmatrix}^{-1}\begin{pmatrix} I & W \\ X & I \end{pmatrix}^{-1} \\ &=\begin{pmatrix} \times & \times \\ -(I-YZ)^{-1}Y & (I-YZ)^{-1} \end{pmatrix}\begin{pmatrix} \times & -W(I-XW)^{-1} \\ \times & (I-XW)^{-1} \end{pmatrix} \\ &=\begin{pmatrix} \times & \times \\ \times & (I-YZ)^{-1}(I+YW)(I-XW)^{-1} \end{pmatrix}. \end{aligned}$$

On the other hand, since $I+WY$ is nonsingular, we have

$$M^{-1}=\begin{pmatrix} \times & \times \\ \times & [M/(I+WY)]^{-1} \end{pmatrix},$$

where

$$M/(I+WY)=(I+XZ)-(X+Y)(I+WY)^{-1}(W+Z).$$

Equating the lower right corners in the two expressions of M^{-1} yields (8).

If $I-XW$ (and/or similarly $I-YZ$) is singular, we may replace X with ϵX in our discussions, where ϵ is a positive number such that $\det(I-\epsilon XW)\neq 0$. A continuity argument by letting $\epsilon\to 1$ shows that (8) still holds. □

Corollary 1. *Let X, Y and Z be any three $m\times n$ complex matrices. Then*

$$(I+XZ^*)-(X+Y)(I+Y^*Y)^{-1}(Y+Z)^*=(I-XY^*)(I+YY^*)^{-1}(I-YZ^*).$$

Proof. Note that $I+Y^*Y$ is never singular. Putting $W=Y^*$ and replacing Z with Z^* in (8) results in the desired identity. □

When Y is a strictly contractive matrix, then $I - Y^*Y$ and $I - YY^*$ are nonsingular. Setting $W = -Y^*$ and $Z = -X^*$ in (8) reveals

$$(I - XX^*) + (X + Y)(I - Y^*Y)^{-1}(X^* + Y^*) = (I + XY^*)(I - YY^*)^{-1}(I + YX^*),$$

which gives immediately the Hua matrix identity (1) by replacing Y with $-Y$.

Corollary 2. *Let A and B be any two $m \times n$ complex matrices. Then*

$$\begin{aligned} I + A^*A = {} & (A + B)^*(I + BB^*)^{-1}(A + B) \\ & + (I - A^*B)(I + B^*B)^{-1}(I - A^*B)^*. \end{aligned} \tag{9}$$

Consequently, when $m = n$,

$$|\det(I - A^*B)|^2 \leqslant \det(I + A^*A)\det(I + B^*B) - |\det(A + B)|^2. \tag{10}$$

We note that (9) has appeared in the literature, see, e.g., [11, p. 184], and it can also be proved in a similar manner as for (8) by taking

$$N = \begin{pmatrix} I & A \\ B^* & I \end{pmatrix}^* \begin{pmatrix} I & A \\ B^* & I \end{pmatrix} = \begin{pmatrix} I + BB^* & A + B \\ A^* + B^* & I + A^*A \end{pmatrix}.$$

Combining (10) with Hua's determinantal inequalities, when A and B are square,

$$\begin{aligned} & \det(I - A^*A)\det(I - B^*B) + |\det(A - B)|^2 \\ & \quad \leqslant |\det(I - A^*B)|^2 \\ & \quad \leqslant \det(I + A^*A)\det(I + B^*B) - |\det(A + B)|^2. \end{aligned} \tag{11}$$

The first inequality in (11) is Hua's, it is valid only when A or B is (strictly) contractive, while the second inequality holds for all square matrices A and B. When A and B are $m \times n$ matrices, the weaker determinantal inequalities hold:

$$\begin{aligned} & \det(I - A^*A)\det(I - B^*B) \\ & \quad \leqslant |\det(I - A^*B)|^2 \\ & \quad \leqslant \det(I + A^*A)\det(I + B^*B). \end{aligned} \tag{12}$$

Note that the inequalities (11) generalize the scalar identity

$$(1 - |\alpha|^2)(1 - |\beta|^2) + |\alpha - \beta|^2 = |1 - \bar{\alpha}\beta|^2 = (1 + |\alpha|^2)(1 + |\beta|^2) - |\alpha + \beta|^2,$$

which is just the $m = n = 1$ case of (11).

For equality cases in (11) and (12), we assume $m = n > 1$. As Hua showed, the first equality holds in (12) if and only if the first equality holds in (11) if and only if $A = B$. We remark that the second equality in (11) holds if and only if $A = -B$ or $A^*B = I$, while the second equality in (12) holds if and only if $A = -B$. For this, write $I + A^*A = M + S$, where

$$M = (A + B)^*(I + BB^*)^{-1}(A + B)$$

and

$$S = (I - A^*B)(I + B^*B)^{-1}(I - A^*B)^*.$$

Since $M + S = I + A^*A$ is positive definite, equality occurs in (11) if and only if $M = 0$ or $S = 0$, that is, if and only if $A = -B$ or $A^*B = I$.

If the second equality in (12) holds, then the second equality in (11) holds, so $A = -B$ or $A^*B = I$. On the other hand, $I + A^*A$ and $I + B^*B$ are always positive definite. Thus $A^*B \neq I$. Therefore $A = -B$.

3. Marcus and Ando's results and new results

Motivated by Hua's work, Marcus [8] showed an inequality for a general inner product vector space and extended the Hua's determinantal inequality (3) to a family of eigenvalue inequalities.

Lemma 1 (Lemma in [8]). *If u and v are complex vectors and $\|u + v\| < 2$, then*

$$|1 - \langle u, v\rangle|^2 \geqslant (1 - \|u\|^2)(1 - \|v\|^2). \tag{13}$$

Marcus proved (13) by means of the Grassmann exterior product of vectors. We shall generalize this inequality to multiple complex vectors $u_1, u_2, \ldots, u_m$.

Theorem 4 (Theorem in [8]). *Let $A, B \in \mathcal{C}_{n\times n}$ and let λ_i, α_i and β_i be respectively the eigenvalues of $I - A^*B$, A^*A and B^*B so indexed that*

$$|\lambda_i| \geqslant |\lambda_{i+1}|, \quad \alpha_i \geqslant \alpha_{i+1}, \quad \beta_i \geqslant \beta_{i+1}, \quad i = 1, 2, \ldots, n-1.$$

Then for each integer k, $1 \leqslant k \leqslant n$,

$$\prod_{j=1}^{k} |\lambda_{n-j+1}|^2 \geqslant \prod_{j=1}^{k} (1 - \alpha_j)(1 - \beta_j). \tag{14}$$

The set of eigenvalue inequalities (14) gives a generalization of (3) which is the case where $k = n$. To prove (14), Marcus first used the above lemma to derive the inequality

$$|\langle (I - A^*B)x, x\rangle|^2 \geqslant \langle (I - A^*A)x, x\rangle \, \langle (I - B^*B)x, x\rangle, \tag{15}$$

then employed a theorem of Schur's to $I - A^*B$ and a theorem of Fan's to both $I - A^*A$ and $I - B^*B$ to convert (15) in vectors to (14) in terms of eigenvalues. What follows is a more general version of Theorem 4 with an elementary proof.

Theorem 5. *Let X, Y and Z be n-square complex matrices having eigenvalues x_j, y_j and z_j, respectively, and so indexed that $|s_j| \geqslant |s_{j+1}|$, $j = 1, \ldots, n$, where s_j is x_j, y_j or z_j. If $Y > 0$, $Z \geqslant 0$ and $X^*Y^{-1}X \geqslant Z$, then*

$$\prod_{j=1}^{k} |x_{n-j+1}|^2 \geqslant \prod_{j=1}^{k} y_{n-j+1} z_{n-j+1}.$$

Proof. Without loss of generality, we may assume that X is in the upper-triangular form with $x_n, x_{n-1}, \ldots, x_1$ on the main diagonal; otherwise we can replace X by $X = U^*DU$ in the discussion, where U is unitary and D is upper-triangular. Notice that for the upper-triangular X, $X\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X(k) & 0 \\ 0 & 0 \end{pmatrix}$, where $X(k)$ denotes the k-square leading principal submatrix of X. In addition, $\det X(k) = \prod_{j=1}^{k} x_{n-j+1}$. To extract the k-square leading principal submatrices from $X^*Y^{-1}X \geqslant Z$, we pre- and post-multiply both sides with $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$ to get

$$\begin{pmatrix} X^*(k) & 0 \\ 0 & 0 \end{pmatrix} Y^{-1} \begin{pmatrix} X(k) & 0 \\ 0 & 0 \end{pmatrix} \geqslant \begin{pmatrix} Z(k) & 0 \\ 0 & 0 \end{pmatrix}.$$

This yields

$$X^*(k)Y^{-1}(k)X(k) \geqslant Z(k).$$

By taking the determinants of both sides, we have

$$\prod_{j=1}^{k} |x_{n-j+1}|^2 \det(Y^{-1}(k)) \geqslant \det(Z(k)).$$

Notice that, by the eigenvalue interlacing theorem (see, e.g., [11, p. 222]),

$$\det(Z(k)) = \prod_{j=1}^{k} \lambda_j(Z(k)) \geqslant \prod_{j=1}^{k} \lambda_{n-j+1}(Z) = \prod_{j=1}^{k} z_{n-j+1}$$

and

$$\det(Y^{-1}(k)) = \prod_{j=1}^{k} \lambda_j(Y^{-1}(k)) \leqslant \prod_{j=1}^{k} \lambda_j(Y^{-1}) = \prod_{j=1}^{k} \lambda_{n-j+1}^{-1}(Y) = \left(\prod_{j=1}^{k} y_{n-j+1}\right)^{-1}.$$

The desired inequalities follow at once. □

An application of Theorem 5 to (4) reveals the inequalities (14). We point out that under the condition of Theorem 5, some inequalities for unitarily invariant norms (see, e.g., [12, p. 105]) can also be obtained.

Ando continued the research on contractive matrices and generalized (14) to an inequality in matrix form.

Theorem 6 (In Abstract of [1]). *If $A, B \in \mathcal{C}_{n\times n}$, then*

$$\mathcal{H}(I - A^*B) \geqslant 2^{-1}[(I - A^*A) + (I - B^*B)], \tag{16}$$

where $\mathcal{H}(X) = \frac{1}{2}(X^* + X)$ denotes the Hermitian part of the square matrix X.

As Ando observed, (16) is equivalent to the obvious inequality $(A - B)^*(A - B) \geqslant 0$ and it implies the Marcus' inequality (15). Inequality (16) is key to the results in [1] on contractions with Pick functions. Ando continued his study with multiple contractive matrices and considered the matrix with (i, j)-block $(I - A_j^* A_i)^{-1}$, where $A_1, \ldots, A_m$ are strictly contractive matrices. He showed that when $m = 3$ such a partitioned matrix is not positive semidefinite in general. However it is true for commuting normal matrices.

Theorem 7 (Theorem 4 in [1]). *Let $A_i \in \mathcal{SC}_{n\times n}$, $i = 1, 2, \ldots, m$. If all A_i are normal and commute with each other, then $\widetilde{H}_m = ((I - A_j^* A_i)^{-1}) \geqslant 0$.*

Ando's proof of this theorem is by induction and the Schur complement. Here we give another proof. Since all A_i are normal and communicate with each other, there exists a unitary matrix U

such that U^*A_iU are all diagonal. Thus, without loss of generality, we may assume that all A_i are diagonal. Let $A_i = \mathrm{diag}(a_{i1}, a_{i2}, \ldots, a_{in})$. Then

$$(I - A_j^*A_i)^{-1} = \mathrm{diag}((1 - \bar{a}_{j1}a_{i1})^{-1}, (1 - \bar{a}_{j2}a_{i2})^{-1}, \ldots, (1 - \bar{a}_{jn}a_{in})^{-1}).$$

By simultaneous permutation of rows and columns on $\widetilde{H}_m$, we get the block-diagonal matrix with n-square matrices $((1 - \bar{a}_{jt}a_{it})^{-1})$, $t = 1, 2, \ldots, m$, on the main diagonal. By our later result (20), each of these block matrices is positive semidefinite, so $\widetilde{H}_m$ is positive semidefinite. The computation of the determinant of $\widetilde{H}_m$ can be done through a Cauchy matrix (see, e.g., [3, p. 30]), since for scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, all less than 1 in absolute value,

$$\frac{1}{1 - \alpha_i\alpha_j} = \frac{1}{\alpha_i} \cdot \frac{1}{\lambda_i + \mu_j}, \quad \text{where } \lambda_i = \frac{1}{\alpha_i},\ \mu_j = -\alpha_j.$$

Referring to (6) and (7), we define

$$H_m = \begin{pmatrix} (I - A_1^*A_1)^{-1} & (I - A_1^*A_2)^{-1} & \ldots & (I - A_1^*A_m)^{-1} \\ (I - A_2^*A_1)^{-1} & (I - A_2^*A_2)^{-1} & \ldots & (I - A_2^*A_m)^{-1} \\ \ldots & \ldots & \ldots & \ldots \\ (I - A_m^*A_1)^{-1} & (I - A_m^*A_2)^{-1} & \ldots & (I - A_m^*A_m)^{-1} \end{pmatrix}, \tag{17}$$

where all A_i are strictly contractive matrices of the same size, $p \times q$, say. The following example shows that H_m is not positive semidefinite in general. Let

$$A_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Then the smallest eigenvalue of H_3 in (17) is -0.0221. So $H_3 \ngeqslant 0$.

Although H_m is not positive semidefinite in general, under certain circumstances its positive semidefiniteness does hold. We first present a lemma which is of interest in its own right and then show that H_m is positive semidefinite when A_i are vectors. At end, we give a result of Bellman's type. We note here that when all $A_i^*A_j$ are scalars, Ando's $\widetilde{H}_m$ is the transpose of H_m. As is well-known, a matrix is positive semidefinite if and only if its transpose is positive semidefinite. Therefore $\widetilde{H}_m$ and H_m are essentially the same when numerical aspects such as determinant or trace are in the consideration. (See (7) and later Theorem 9.)

Lemma 2. *Let V be a finite-dimensional inner product space over the complex number field. If $u_1, u_2, \ldots, u_m$ are vectors in V such that $\|u_i\| < 1$ for $i = 1, 2, \ldots, m$, then*

$$U_m = \begin{pmatrix} \frac{1}{1-\langle u_1,u_1\rangle} & \frac{1}{1-\langle u_1,u_2\rangle} & \cdots & \frac{1}{1-\langle u_1,u_m\rangle} \\ \frac{1}{1-\langle u_2,u_1\rangle} & \frac{1}{1-\langle u_2,u_2\rangle} & \cdots & \frac{1}{1-\langle u_2,u_m\rangle} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{1-\langle u_m,u_1\rangle} & \frac{1}{1-\langle u_m,u_2\rangle} & \cdots & \frac{1}{1-\langle u_m,u_m\rangle} \end{pmatrix} \geqslant 0. \tag{18}$$

Proof. Let $\alpha_{ij} = \langle u_i, u_j\rangle$ and $B = (\alpha_{ij})$. Then B (or its transpose, to be precise) is a Gram matrix, so $B \geqslant 0$. Since all $\|u_i\| < 1$, we have $|\alpha_{ij}| < 1$. By writing $\frac{1}{1-\alpha_{ij}}$ as a convergent power series

$$\sum_{k=0}^{\infty} \alpha_{ij}^k,$$

we see that

$$U_m = \sum_{k=0}^{\infty} B^{[k]} \geqslant 0. \qquad \square$$

This approach does not work for the general block matrix case (17), even though each $(I - A_i^* A_j)^{-1}$ can be written as a convergent power series of a matrix. In addition, Marcus' (13) follows at once when $\|u\| < 1$ and $\|v\| < 1$. Note that in (13), the condition $\|u + v\| < 2$ is a bit weaker.

Two variations of this lemma come handy: If $a > 0$ and all $\|u_i\| < \sqrt{a}$, then

$$\begin{pmatrix} \frac{1}{a-\langle u_1,u_1\rangle} & \frac{1}{a-\langle u_1,u_2\rangle} & \cdots & \frac{1}{a-\langle u_1,u_m\rangle} \\ \frac{1}{a-\langle u_2,u_1\rangle} & \frac{1}{a-\langle u_2,u_2\rangle} & \cdots & \frac{1}{a-\langle u_2,u_m\rangle} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{a-\langle u_m,u_1\rangle} & \frac{1}{a-\langle u_m,u_2\rangle} & \cdots & \frac{1}{a-\langle u_m,u_m\rangle} \end{pmatrix} \geqslant 0 \tag{19}$$

and if $x_1, x_2, \ldots, x_m$ are complex numbers such that all $|x_i| < 1$ then

$$\begin{pmatrix} \frac{1}{1-|x_1|^2} & \frac{1}{1-\bar{x}_1 x_2} & \cdots & \frac{1}{1-\bar{x}_1 x_m} \\ \frac{1}{1-\bar{x}_2 x_1} & \frac{1}{1-|x_2|^2} & \cdots & \frac{1}{1-\bar{x}_2 x_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{1-\bar{x}_m x_1} & \frac{1}{1-\bar{x}_m x_2} & \cdots & \frac{1}{1-|x_m|^2} \end{pmatrix} \geqslant 0. \tag{20}$$

Theorem 8. *Let $u_1, u_2, \ldots, u_m$ be strictly contractive vectors; precisely, they are complex row vectors of n components all having norm less than 1. Then*

$$V_m = \begin{pmatrix} (I - u_1^* u_1)^{-1} & (I - u_1^* u_2)^{-1} & \cdots & (I - u_1^* u_m)^{-1} \\ (I - u_2^* u_1)^{-1} & (I - u_2^* u_2)^{-1} & \cdots & (I - u_2^* u_m)^{-1} \\ \cdots & \cdots & \cdots & \cdots \\ (I - u_m^* u_1)^{-1} & (I - u_m^* u_2)^{-1} & \cdots & (I - u_m^* u_m)^{-1} \end{pmatrix} \geqslant 0. \tag{21}$$

Proof. Notice that

$$(I - u_i^* u_j)^{-1} = I + u_i^*(1 - u_j u_i^*)^{-1} u_j = I + (1 - u_j u_i^*)^{-1} u_i^* u_j.$$

It follows that

$$V_m = J_m \otimes I + (\Lambda \otimes J_n) \circ D,$$

where J_k is the k-square matrix all whose entries are 1, Λ is an m-square matrix with (i, j)-entry $(1 - u_j u_i^*)^{-1}$, i.e., the transpose of the matrix $\left(\frac{1}{1-\langle u_i, u_j\rangle}\right)$, and D is an (mn)-square matrix with (i, j)-block $u_i^* u_j$. Note that $u_i^* u_j$ is an n-square matrix and that $D \geqslant 0$.

By Lemma 2, Λ is positive semidefinite. Therefore $V_m \geqslant 0$. $\square$

We conclude the paper by showing a result resembling Bellman's (7) with trace in place of the determinant.

Theorem 9. *Let $A_1, A_2, \ldots, A_m$ be strictly contractive $p \times q$ matrices. Then*

$$T_m = \begin{pmatrix} \frac{1}{\operatorname{tr}(I-A_1^*A_1)} & \frac{1}{\operatorname{tr}(I-A_1^*A_2)} & \cdots & \frac{1}{\operatorname{tr}(I-A_1^*A_m)} \\ \frac{1}{\operatorname{tr}(I-A_2^*A_1)} & \frac{1}{\operatorname{tr}(I-A_2^*A_2)} & \cdots & \frac{1}{\operatorname{tr}(I-A_2^*A_m)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\operatorname{tr}(I-A_m^*A_1)} & \frac{1}{\operatorname{tr}(I-A_m^*A_2)} & \cdots & \frac{1}{\operatorname{tr}(I-A_m^*A_m)} \end{pmatrix} \geqslant 0.$$

Proof. Since $\operatorname{tr}(I - A_i^*A_j) = q - \operatorname{tr}(A_i^*A_j)$ and trace can be regarded as an inner product of the matrix space, by (19) with $q \geqslant \langle A_i, A_i \rangle$ as $A_i \in \mathcal{SC}_{p\times q}$, $T_m \geqslant 0$. $\square$

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