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On the pseudohermitian sectional curvature of a strictly pseudoconvex CR manifold

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Abstract

We show that the pseudohermitian sectional curvature $H_\theta(\sigma)$ of a contact form θ on a strictly pseudoconvex CR manifold M measures the difference between the lengths of a circle in a plane tangent at a point of M and its projection on M by the exponential map associated to the Tanaka–Webster connection of (M,θ). Any Sasakian manifold (M,θ) whose pseudohermitian sectional curvature $K_\theta(\sigma)$ is a point function is shown to be Tanaka–Webster flat, and hence a Sasakian space form of φ-sectional curvature $c=-3$. We show that the Lie algebra $\mathfrak{i}(M,\theta)$ of all infinitesimal pseudohermitian transformations on a strictly pseudoconvex CR manifold M of CR dimension n has dimension $\leqslant (n+1)^2$ and if $\dim_{\mathbb{R}} \mathfrak{i}(M,\theta) = (n+1)^2$ then $H_\theta(\sigma) = \text{constant}$.

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Keywords: CR manifold; Tanaka–Webster connection; Pseudohermitian sectional curvature; Jacobi field

1. Introduction

In his famous 1987 paper S.M. Webster introduced (cf. [18]) the notion of pseudohermitian sectional curvature H_θ of a nondegenerate CR manifold, associated to a fixed contact form θ, and exhibited a class of spherical nondegenerate real hypersurfaces $M \subset \mathbb{C}^{n+1}$ with $H_\theta(\sigma) = \pm 1/(2c)$, for each $c \in (0,+\infty)$. If M is a nondegenerate CR manifold and θ a contact form on M then let R be the curvature 4-tensor field of the Tanaka–Webster connection ∇ of (M,θ). Let $\sigma \subset H(M)_x$ be a *holomorphic* 2-*plane* tangent at $x \in M$ i.e. $J_x(\sigma) = \sigma$. Here $H(M)$ is the maximal complex distribution of M and J its complex structure. If $\{X, J_x X\}$ is a linear basis of σ then we set

$$H_\theta(\sigma) = \frac{1}{4} G_{\theta,x}(X,X)^{-2} R_x(X, J_x X, X, J_x X). \tag{1}$$

The definition of $H_\theta(\sigma)$ doesn't depend upon the choice of basis in σ because of $R(X,Y,Z,W) = -R(X,Y,W,Z)$ (as the curvature is a 2-form) and $R(X,Y,Z,W) = -R(Y,X,Z,W)$ (as the Levi form is parallel with respect to ∇). Then H_θ is a $\mathbb{R}$-valued function on the total space of the Grassmann bundle $G_2(\mathbb{C}^n) \to G_2(H(M)) \xrightarrow{\pi} M$ of all

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holomorphic 2-planes tangent to M. We also set $H_\theta = \mathrm{Sect}(M, \theta)$. The coefficient 1/4 in (1) is chosen such that the standard sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ together with the canonical contact form $\theta_0 = \frac{i}{2}(\bar\partial - \partial)|z|^2$ has constant pseudohermitian sectional curvature $\mathrm{Sect}(S^{2n+1}, \theta_0) \equiv 1$. Clearly (1) is a (pseudohermitian) analog of the *holomorphic* sectional curvature of a Hermitian manifold (cf. e.g. [12, vol. II, p. 168]) rather than an analog of the sectional curvature of a Riemannian manifold (cf. [12, vol. I, p. 202]). Yet if $G_2(\mathbb{R}^{2n+1}) \to G_2(T(M)) \xrightarrow{\pi} M$ is the Grassmann bundle of all 2-planes tangent at M then (1) is the restriction to $G_2(H(M))$ of the function $K_\theta : G_2(T(M)) \to \mathbb{R}$ given by

$$K_\theta(\sigma) = R_x(u, v, u, v), \quad \sigma \subset T_x(M), \tag{2}$$

where $\{u, v\}$ is a $g_{\theta,x}$-orthonormal basis of σ and g_θ is the Webster metric of (M, θ) (cf. Section 2 for definitions) and (2) may be referred to as the (*pseudohermitian*) *sectional curvature* determined by the (arbitrary) 2-plane σ.

A number of fundamental questions remain unanswered. First, what is the geometric interpretation of $K_\theta(\sigma)$? Precisely, if $\sigma \in G_2(T(M))_x$ and $rw(s) = r(\cos s)u + r(\sin s)v$ is a circle in σ and $\beta_r(s) = \exp_x rw(s)$, $0 \leqslant s \leqslant 2\pi$, then is $K_\theta(\sigma)$ a "measure" of the difference $2\pi r - L(\beta_r)$? Here $\exp_x$ is the exponential map associated to the Tanaka–Webster connection ∇ of (M, θ) and $L(\beta_r)$ the length of β_r. Another fundamental question is whether the algebraic machinery in [12] (cf. vol. I, pp. 198–203, and vol. II, pp. 165–169) applies, eventually leading to a meaningful concept of pseudohermitian space form. Moreover, as pseudohermitian transformations are (within pseudohermitian geometry) analogs to isometries between Riemannian manifolds, it is a natural question whether manifolds (M, θ) whose Lie algebra $\mathfrak{i}(M, \theta)$ of infinitesimal pseudohermitian transformations has maximal dimension have constant pseudohermitian sectional curvature.

Our findings are that the pseudohermitian sectional curvature (1) satisfies

$$L(\beta_r) = 2\pi r - \frac{\pi r^3}{12}\big(16 H_\theta(\sigma) - 3\big) + \mathrm{O}(r^4) \tag{3}$$

(cf. Theorem 1 below for the precise statement) providing the geometric interpretation mentioned above. Also we prove a Schur like result, cf. Theorem 2 below. Combining Theorem 2 with a result by Y. Kamishima, [10], we obtain

Corollary 1. *Let (M, θ) be a compact connected Sasakian manifold of CR dimension $n \geqslant 2$. If there is a C^∞ function $f : M \to \mathbb{R}$ such that $K_\theta = f \circ \pi$ then M is isometric to the Heinsenberg infranilmanifold $\mathbb{H}_n/\Gamma$ (with $\Gamma = \rho(\pi_1(M)) \subset \mathbb{H}_n \rtimes \mathrm{U}(n)$).*

Here $\mathbb{H}_n$ is the Heisenberg group endowed with the standard strictly pseudoconvex CR structure and canonical contact form (cf. e.g. [9, Chapter 1]).

The paper is organized as follows. Section 1 is devoted to a remainder of CR and pseudohermitian geometry and to the proof of Theorem 1. The main technical ingredient are Jacobi fields of the Tanaka–Webster connection, on the line of thought in [2]. A Schur like result for the sectional curvature (2) and the proof of Corollary 1 form the object of Section 3. In Section 4 we show (cf. Theorem 3 below) that for any strictly pseudoconvex CR manifold $\dim_{\mathbb{R}} \mathfrak{i}(M, \theta) \leqslant (n + 1)^2$ and if $\dim_{\mathbb{R}} \mathfrak{i}(M, \theta) = (n + 1)^2$ then (M, θ) has a constant pseudohermitian sectional curvature (1). The proof of Theorem 3 relies on standard techniques in the theory of (infinitesimal) affine transformations. The explicit expression of the curvature tensor of a pseudohermitian space form (i.e. a pseudohermitian manifold whose sectional curvature (1) is constant) is derived in Section 5 (cf. (19) in Theorem 4 below) paving the road towards a study of the geometry of the second fundamental form of a CR submanifold of a pseudohermitian space form (in the spirit of [19, pp. 76–136]). The computational details (leading, as a byproduct, to a Sasakian version of the Kählerian Schur theorem) are provided in Appendix A to this paper. A classification result of E. Musso, [15], and our Theorem 4 lead to

Corollary 2. *Let (M, θ) be a G-homogeneous pseudohermitian space form of pseudohermitian sectional curvature $H_\theta(\sigma) = c$, $c \in \mathbb{R}$, with L_θ positive definite.* (i) *If $c > 0$ then (M, θ) is contact homothetic to the canonical pseudohermitian manifold of index k over B.* (ii) *If $c < 0$ then (M, θ) is contact homothetic to either $B \times S^1$ or $B \times \mathbb{R}$.* (iii) *If $c = 0$ then (M, θ) is contact homothetic to either $\mathbb{C}^n \times S^1$ or $\mathbb{C}^n \times \mathbb{R}$.*

The description of the pseudohermitian structures on the model spaces (i)–(iii) in Corollary 2 is provided in Section 5. Finally, in Section 6 we show that given a pseudohermitian immersion $f : M \to M'$ between two strictly pseudoconvex CR manifolds the sectional curvature (1) of M doesn't exceed the sectional curvature (1) of the ambient space. Theorem 5 in Section 6 is suitable for several applications. For instance

Corollary 3. *There is no pseudohermitian immersion of the standard sphere S^{2m+1} into an ellipsoid $\{(z,w): g_{\alpha\bar\beta} z^\alpha \bar z^\beta - w\overline{w} + c = 0\} \subset \mathbb{C}^{n+1}$, with $c \in (0,+\infty)$ and $[g_{\alpha\bar\beta}] \in \mathrm{GL}(n,\mathbb{C})$ Hermitian.*

Corollary 4. *For any compact Sasakian manifold (M,θ) there are $n \geqslant 1$ and $A = \{0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{n+1}\}$ such that $H_\theta(\sigma) \leqslant \mathrm{Sect}(S^{2n+1}, \theta_A)$ where $\theta_A = (\sum_{j=1}^{n+1} a_j |z_j|^2)^{-1}\theta_0$.*

2. The geometric interpretation of pseudohermitian sectional curvature

2.1. The Tanaka–Webster connection

Let us start by recalling the notions of CR and pseudohermitian geometry needed through this paper. Let $(M, T_{1,0}(M))$ be a $(2n+d)$-dimensional CR manifold, of CR dimension n, where $T_{1,0}(M)$ denotes the CR structure. The maximal complex distribution is $H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$. It carries the complex structure $J : H(M) \to H(M)$ given by $J(Z + \overline{Z}) = i(Z - \overline{Z})$, for any $Z \in T_{1,0}(M)$. Throughout $T_{0,1}(M) = \overline{T_{1,0}(M)}$ and overlines denote complex conjugates. The standard example of a CR manifold is that of a real submanifold $M \subset \mathbb{C}^N$ such that $\dim_{\mathbb{C}}[T_x(M) \otimes_{\mathbb{R}} \mathbb{C}] \cap T^{1,0}(\mathbb{C}^N)_x =$ constant, $x \in M$. This is of course always true for real hypersurfaces in $\mathbb{C}^N$.

On each CR manifold M there is a natural first order differential operator $\bar\partial_b$ given by $(\bar\partial_b f)\overline{Z} = \overline{Z}(f)$ for any C^1 function $f : M \to \mathbb{C}$ and any $Z \in T_{1,0}(M)$. Then $\bar\partial_b f = 0$ are the *tangential Cauchy–Riemann equations* and a C^1 solution is a *CR function* on M.

Let $H(M)^\perp \subset T^*(M)$ be the conormal bundle associated to $H(M)$. When M has hypersurface type (i.e. $d = 1$) and M is orientable, which we shall always assume, $H(M)^\perp$ is a trivial line bundle hence M admits globally defined nowhere zero differential 1-forms θ such that $\mathrm{Ker}(\theta) = H(M)$. These are referred to as *pseudohermitian structures*. With each pseudohermitian structure θ one may associate the Levi form $L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W})$, $Z, W \in T_{1,0}(M)$, and M is *nondegenerate* (respectively *strictly pseudoconvex*) if L_θ is nondegenerate (respectively positive definite) for some θ. Two pseudohermitian structures θ and $\hat\theta$ are related by $\hat\theta = f\theta$ for some C^∞ function $f : M \to \mathbb{R} \setminus \{0\}$ and a simple calculation shows that $L_{\hat\theta} = f L_\theta$. Nondegeneracy is a *CR invariant* property i.e. it is invariant under a transformation $\hat\theta = f\theta$. Clearly, strict pseudoconvexity is not a CR invariant property (e.g. if L_θ is positive definite and $\hat\theta = -\theta$ then $L_{\hat\theta}$ is negative definite). If M is a nondegenerate CR manifold of CR dimension n then each pseudohermitian structure is a *contact form* i.e. $\theta \wedge (d\theta)^n$ is a volume form on M.

Let M be a nondegenerate CR manifold and θ a contact form on M. The pair (M,θ) is commonly referred to as a *pseudohermitian manifold*. There is a unique nowhere zero globally defined tangent vector field T on M, transverse to $H(M)$, determined by $\theta(T) = 1$ and $(d\theta)(T, X) = 0$ for any $X \in T(M)$ (T is the *characteristic direction* of $d\theta$). On any pseudohermitian manifold (M,θ) there is a unique linear connection ∇ (the *Tanaka–Webster connection* of (M,θ)) such that (i) $H(M)$ is parallel with respect to ∇, (ii) $\nabla J = 0$ and $\nabla g_\theta = 0$, and (iii) the torsion T_∇ of ∇ is *pure* i.e.

$$T_\nabla(Z, W) = 0, \quad T_\nabla(Z, \overline{W}) = 2i L_\theta(Z, \overline{W}) T, \quad Z, W \in T_{1,0}(M),$$

$$\tau \circ J + J \circ \tau = 0.$$

Cf. N. Tanaka, [17], S.M. Webster, [18], or Chapter I of [9]. Here g_θ is the *Webster metric* i.e. the semi-Riemannian metric on M defined by

$$g_\theta(X, Y) = (d\theta)(X, JY), g_\theta(T, X) = 0, g_\theta(T, T) = 1,$$

for any $X, Y \in H(M)$. Also τ is the *pseudohermitian torsion* i.e. the vector-valued 1-form $\tau(X) = T_\nabla(T, X)$, $X \in T(M)$. The complex structure $J : H(M) \to H(M)$ appearing in axiom (ii) is thought of as extended to a $(1,1)$-tensor field on M by requesting that $JT = 0$. When M is strictly pseudoconvex and L_θ is positive definite the Webster metric is a Riemannian metric on M and (J, T, θ, g_θ) is a contact metric structure (in the sense of D.E. Blair, [3]) which is normal if and only if $\tau = 0$. If this is the case then g_θ is a Sasakian metric on M. Therefore Sasakian manifolds are precisely the strictly pseudoconvex CR manifolds with a fixed contact form θ such that the Levi form L_θ is positive definite and the pseudohermitian torsion of the Tanaka–Webster connection vanishes. By a result of G. Marinescu et al., [14], for any Sasakian manifold M there is a CR embedding $M \to \mathbb{C}^N$ for some $N \geqslant 2$.

2.2. Jacobi fields

A study of Jacobi fields of the Tanaka–Webster connection on a nondegenerate CR manifold was started in [2]. Let M be a strictly pseudoconvex CR manifold, of CR dimension n, and θ a contact form with L_θ positive definite. Let $x \in M$ and let $\exp_x$ be the exponential mapping, associated to the Tanaka–Webster connection ∇ of (M, θ). Here we use a few facts from the general theory of linear connections on manifolds e.g. by Proposition 8.2 in [12, vol. I, p. 147], there is $r_0 > 0$ such that $\exp_x : B(x, r_0) \to M$ is a C^∞ diffeomorphism on some neighborhood U of x in M. Here $B(x, r_0) = \{v \in T_x(M) : \|v\| < r_0\}$ and $\|v\|^2 = g_{\theta,x}(v, v)$. Let

$$L(\beta_r) = \int_0^{2\pi} \|\dot\beta_r(s)\|\, ds$$

be the length of the curve β_r (defined in the Introduction) in (M, g_θ). Let $\gamma_v(t) = \exp_x tv$ denote the geodesic of ∇ of initial conditions (x, v), $v \in T_x(M)$. Given $0 < r < r_0$ we consider the geodesics $\gamma_{w(s)} : [-r, r] \to U$ and set $\beta_t(s) = \gamma_{w(s)}(t)$. Next let X_s be the vector field along $\gamma_{w(s)}$ defined by

$$X_{s,\gamma_{w(s)}(t)} = \dot\beta_t(s), \quad 0 \leqslant s \leqslant 2\pi, \ |t| \leqslant r.$$

Then $L(\beta_r) = \int_0^{2\pi} \|X_s\|_{\gamma_{w(s)}(r)}\, ds$. Once again a general fact within connection theory (cf. Theorem 1.2 in [12, vol. II, p. 64]) guarantees that X_s is a Jacobi field of the Tanaka–Webster connection i.e. X_s satisfies the Jacobi equation

$$\nabla^2_{\dot\gamma_{w(s)}} X_s + \nabla_{\dot\gamma_{w(s)}} T_\nabla(X_s, \dot\gamma_{w(s)}) + R(X_s, \dot\gamma_{w(s)})\dot\gamma_{w(s)} = 0 \tag{4}$$

along $\gamma_{w(s)}$. Let us set $X'_s = \nabla_{\dot\gamma_{w(s)}} X_s$ for simplicity. An elementary calculation shows that X_s satisfies the initial conditions

$$X_{s,x} = 0, \quad X'_{s,x} = w\left(s + \frac{\pi}{2}\right). \tag{5}$$

We wish to write the Taylor development of $f(r) = \|X_s\|^2_{\gamma_{w(s)}(r)}$ (with $0 \leqslant s \leqslant 2\pi$ fixed) up to order 4. This is the classical approach to the geometric interpretation of sectional curvature in Riemannian geometry, except that we must deal with the presence of torsion terms. The first of the initial conditions (5) gives $f(0) = 0$. Next, as $\nabla g_\theta = 0$

$$f'(r) = 2g_\theta(X'_s, X_s)_{\gamma_{w(s)}(r)} \tag{6}$$

hence $f'(0) = 0$. Differentiating in (6) we obtain

$$f''(r) = 2g_\theta\big(\nabla^2_{\dot\gamma_{w(s)}} X_s, X_s\big)_{\gamma_{w(s)}(r)} + 2\|X'_s\|^2_{\gamma_{w(s)}(r)} \tag{7}$$

hence (by (5))

$$f''(0) = 2\left\| w\left(s + \frac{\pi}{2}\right)\right\|^2 = 2.$$

Let us set $P_s = \nabla_{\dot\gamma_{w(s)}} T_\nabla$ for simplicity. Similarly we may differentiate in (7) so that to get

$$f'''(r) = 2g_\theta\big(\nabla^3_{\dot\gamma_{w(s)}} X_s, X_s\big)_{\gamma_{w(s)}(r)} + 6g_\theta\big(\nabla^2_{\dot\gamma_{w(s)}} X_s, X'_s\big)_{\gamma_{w(s)}(r)} \tag{8}$$

hence (by the Jacobi equation (4))

$$\begin{aligned} f'''(0) &= 6g_\theta\big(\nabla^2_{\dot\gamma_{w(s)}} X_s, X'_s\big)_x \\ &= -6g_\theta\big(\nabla_{\dot\gamma_{w(s)}} T_\nabla(X_s, \dot\gamma_{w(s)}), X'_s\big)_x - 6g_\theta\big(R(X_s, \dot\gamma_{w(s)})\dot\gamma_{w(s)}, X'_s\big)_x \end{aligned}$$

$$= -6 g_\theta\big(P_s(X_s, \dot\gamma_{w(s)}), X_s'\big)_x + g_\theta\big(T_\nabla(X_s', \dot\gamma_{w(s)}), X_s'\big)_x$$
$$= -6\left\langle T_{\nabla,x}\left(w\left(s+\frac{\pi}{2}\right), w(s)\right), w\left(s+\frac{\pi}{2}\right)\right\rangle,$$

where $g_{\theta,x} = \langle\, ,\rangle$. Thus

$$f'''(0) = 6\left\langle T_{\nabla,x}(u,v), w\left(s+\frac{\pi}{2}\right)\right\rangle.$$

Finally we may differentiate in (8) to obtain

$$f^{(4)}(r) = 2 g_\theta(\nabla^4_{\dot\gamma_{w(s)}} X_s, X_s)_{\gamma_{w(s)}(r)} + 8 g_\theta(\nabla^3_{\dot\gamma_{w(s)}} X_s, X_s')_{\gamma_{w(s)}(r)} + 6\|\nabla^2_{\dot\gamma_{w(s)}} X_s\|^2_{\gamma_{w(s)}(r)}.$$

Let us evaluate the terms in the right hand side at $r = 0$. The first term vanishes (by (5)). To compute the second term note first that (by (4))

$$\begin{aligned}
\nabla^2_{\dot\gamma_{w(s)}} T_\nabla(X_s, \dot\gamma_{w(s)}) &= \nabla_{\dot\gamma_{w(s)}}\big\{P_s(X_s, \dot\gamma_{w(s)}) + T_\nabla(X_s', \dot\gamma_{w(s)})\big\}\\
&= (\nabla_{\dot\gamma_{w(s)}} P_s)(X_s, \dot\gamma_{w(s)}) + 2P_s(X_s', \dot\gamma_{w(s)}) + T_\nabla(\nabla^2_{\dot\gamma_{w(s)}} X_s, \dot\gamma_{w(s)})\\
&= (\nabla_{\dot\gamma_{w(s)}} P_s)(X_s, \dot\gamma_{w(s)}) + 2P_s(X_s', \dot\gamma_{w(s)})\\
&\quad - T_\nabla\big(\nabla_{\dot\gamma_{w(s)}} T_\nabla(X_s, \dot\gamma_{w(s)}), \dot\gamma_{w(s)}\big) - T_\nabla\big(R(X_s, \dot\gamma_{w(s)})\dot\gamma_{w(s)}, \dot\gamma_{w(s)}\big)\\
&= (\nabla_{\dot\gamma_{w(s)}} P_s)(X_s, \dot\gamma_{w(s)}) + 2P_s(X_s', \dot\gamma_{w(s)}) - T_\nabla\big(P_s(X_s, \dot\gamma_{w(s)}), \dot\gamma_{w(s)}\big)\\
&\quad - T_\nabla\big(T_\nabla(X_s', \dot\gamma_{w(s)}), \dot\gamma_{w(s)}\big) - T_\nabla\big(R(X_s, \dot\gamma_{w(s)})\dot\gamma_{w(s)}, \dot\gamma_{w(s)}\big)
\end{aligned}$$

hence

$$\big(\nabla^2_{\dot\gamma_{w(s)}} T_\nabla(X_s, \dot\gamma_{w(s)})\big)_x = T_{\nabla,x}\big(T_{\nabla,x}(u,v), w(s)\big) - 2P_{s,x}(u,v). \tag{9}$$

Similarly

$$\nabla_{\dot\gamma_{w(s)}} R(X_s, \dot\gamma_{w(s)})\dot\gamma_{w(s)} = (\nabla_{\dot\gamma_{w(s)}} R)(X_s, \dot\gamma_{w(s)})\dot\gamma_{w(s)} + R(X_s', \dot\gamma_{w(s)})\dot\gamma_{w(s)}$$

hence

$$\big(\nabla_{\dot\gamma_{w(s)}} R(X_s, \dot\gamma_{w(s)})\dot\gamma_{w(s)}\big)_x = R_x\left(w\left(s+\frac{\pi}{2}\right), w(s)\right)w(s). \tag{10}$$

Therefore (by (4) and (9)–(10))

$$\begin{aligned}
g_\theta(\nabla^3_{\dot\gamma_{w(s)}} X_s, X_s')_x &= -g_\theta\big(\nabla_{\dot\gamma_{w(s)}} T_\nabla(X_s, \dot\gamma_{w(s)}), X_s'\big)_x - g_\theta\big(\nabla_{\dot\gamma_{w(s)}} R(X_s, \dot\gamma_{w(s)})\dot\gamma_{w(s)}, X_s'\big)_x\\
&= 2\left\langle P_{s,x}(u,v), w\left(s+\frac{\pi}{2}\right)\right\rangle - \left\langle T_{\nabla,x}\big(T_{\nabla,x}(u,v), w(s)\big), w\left(s+\frac{\pi}{2}\right)\right\rangle\\
&\quad - \left\langle R_x\left(w\left(s+\frac{\pi}{2}\right), w(s)\right)w(s), w\left(s+\frac{\pi}{2}\right)\right\rangle.
\end{aligned}$$

Finally

$$\begin{aligned}
\|\nabla^2_{\dot\gamma_{w(s)}} X_s\|^2_x &= \big\|\nabla_{\dot\gamma_{w(s)}} T_\nabla(X_s, \dot\gamma_{w(s)}) + R(X_s, \dot\gamma_{w(s)})\dot\gamma_{w(s)}\big\|^2_x\\
&= \big\|P_s(X_s, \dot\gamma_{w(s)}) + T_\nabla(X_s', \dot\gamma_{w(s)})\big\|^2_x = \big\|T_{\nabla,x}(u,v)\big\|^2
\end{aligned}$$

and we may conclude that

$$\begin{aligned}
f^{(4)}(0) &= 6\big\|T_{\nabla,x}(u,v)\big\|^2 - 32K_\theta(\sigma)\\
&\quad + 16\left\langle P_{s,x}(u,v), w\left(s+\frac{\pi}{2}\right)\right\rangle - 8\left\langle T_{\nabla,x}\big(T_{\nabla,x}(u,v), w(s)\big), w\left(s+\frac{\pi}{2}\right)\right\rangle.
\end{aligned}$$

We obtain the following

Theorem 1. *Let M be a strictly pseudoconvex CR manifold and θ a contact form on M such that L_θ is positive definite. Then*

$$H_\theta(\sigma) = \frac{3}{16} + \lim_{r\to 0} \frac{3}{4\pi r^3}\bigl\{2\pi r - L(\beta_r)\bigr\} \tag{11}$$

for any holomorphic 2-plane $\sigma \subset H(M)_x$ and $x \in M$.

Theorem 1 provides the geometric interpretation we seek for. The constant $3/16$ (absent in the Riemannian counterpart of (11)) is due to the nonvanishing of $T_\nabla(X,Y)$ for $X, Y \in H(M)$ i.e. of the torsion component proportional to the Levi form. If in turn $\sigma \subset T_x(M)$ is a 2-plane tangent to $u = T_x$ and $\{T_x, v\} \subset \sigma$ is a $g_{\theta,x}$-orthonormal basis of σ then we shall show that

$$L(\beta_r) = 2\pi r - \frac{\pi r^3}{12}\left\{16K_\theta(\sigma) + \frac{3}{2}A_x(v,v)^2 + 2\Omega_x(\tau_x v, v) - \|\tau_x v\|^2\right\} + \mathrm{O}(r^4), \tag{12}$$

where $A(X,Y) = g_\theta(X, \tau Y)$ and $\Omega = -d\theta$. So an interpretation similar to that in Theorem 1 is not available unless (M,θ) is a Sasakian manifold. Indeed if this is the case ($\tau = 0$) then we obtain $K_\theta(\sigma) = \lim_{r\to 0}(3/(4\pi r^3))\{2\pi r - L(\beta_r)\}$.

Proof of Theorem 1. Let $\sigma \subset H(M)_x$ be a holomorphic 2-plane and $v = J_x u$ where $u \in \sigma$, $\|u\| = 1$. Recall (as a consequence of the purity axioms, cf. also Chapter I in [9]) that

$$T_\nabla(X,Y) = -\Omega(X,Y)T, \quad X, Y \in H(M), \tag{13}$$

hence $f'''(0) = 0$. On the other hand

$$(\nabla_X T_\nabla)(Y,Z) = -(\nabla_X \Omega)(Y,Z)T = 0$$

for any $X \in T(M)$, $Y, Z \in H(M)$, hence $P_s(u,v) = 0$. Also (again by (13)) $\|T_{\nabla,x}(u,v)\| = 1$ and

$$\left\langle T_{\nabla,x}\bigl(T_{\nabla,x}(u,v), w(s)\bigr), w\left(s + \frac{\pi}{2}\right)\right\rangle = \left\langle \tau_x\bigl(w(s)\bigr), w\left(s+\frac{\pi}{2}\right)\right\rangle = A_x\left(w(s), w\left(s + \frac{\pi}{2}\right)\right) = -g(s),$$

where $g(s) = (\sin 2s)A_x(u,u) - (\cos 2s)A_x(u,v)$, because of $A_x(v,v) = -A_x(u,u)$ (itself a consequence of $\tau \circ J = -J \circ \tau$). It follows that

$$f^{(4)}(0) = 6 - 32H_\theta(\sigma) + 8g(s).$$

Summing up $f(r) = \sum_{j=0}^4 \frac{f^{(j)}(0)}{j!} r^j + \mathrm{O}(r^5) = r^2(1-\delta)$, where $\delta = (r^2/12)\{16H_\theta(\sigma) - 3 - 4g(s)\} + \mathrm{O}(r^3)$, hence

$$\|X_s\|_{\gamma_{w(s)}(r)} = r\sqrt{1-\delta} = r\Bigl(1 - \frac{\delta}{2} + \mathrm{O}(\delta^2)\Bigr) = r - \frac{r^3}{24}\bigl\{16H_\theta(\sigma) - 3 - 4g(s)\bigr\} + \mathrm{O}(r^4).$$

Finally, by integration we obtain (as $\int_0^{2\pi} g(s)\,ds = 0$) the identity (3) and the proof of Theorem 1 is complete. For 2-planes tangent to $u = T_x$ we have

$$f'''(0) = 6A_x\left(v, w\left(s + \frac{\pi}{2}\right)\right)$$

and

$$\left\langle T_{\nabla,x}\bigl(T_{\nabla,x}(u,v), w(s)\bigr), w\left(s+\frac{\pi}{2}\right)\right\rangle = \|\tau_x v\|^2 \cos^2 s + \Omega_x(\tau_x v, v)\sin^2 s.$$

Also $(\nabla_X T_\nabla)(T,Y) = (\nabla_X \tau)Y$ implies

$$\left\langle P_{s,x}(u,v), w\left(s + \frac{\pi}{2}\right)\right\rangle = (\nabla_{\dot\gamma_{w(s)}} A)_x(v,v)\cos s$$

hence

$$f^{(4)}(0) = 6\|\tau_x v\|^2 - 32K_\theta(\sigma) + 16(\nabla_{\dot\gamma_{w(s)}} A)_x(v,v)\cos s - 8\big\{\|\tau_x v\|^2\cos^2 s + \Omega_x(\tau_x v, v)\sin^2 s\big\}.$$

Similar to the above we set

$$\delta = -rA_x(v,v)\cos s + \frac{r^2}{12}\big\{16K_\theta(\sigma) + (4\cos^2 s - 3)\|\tau_x v\|^2 - 8(\nabla_{\dot\gamma_{w(s)}} A)_x(v,v)\cos s + 4\Omega_x(\tau_x v, v)\sin^2 s\big\} + O(r^3)$$

hence

$$\|X_s\|_{\gamma_{w(s)}(r)} = r\sqrt{1-\delta} = r\left(1 - \frac{\delta}{2} - \frac{\delta^2}{8} + O(\delta^3)\right)$$

and integration over $0 \leqslant s \leqslant 2\pi$ leads to (12).

3. A Schur-like result

The scope of this section is to establish the following

Theorem 2. *Let M be a connected strictly pseudoconvex CR manifold of CR dimension $n \geqslant 2$ and θ a contact form on M with L_θ positive definite. Let $S(X,Y) = (\nabla_X \tau)Y - (\nabla_Y \tau)X$. Assume that the pseudohermitian sectional curvature is a point function only i.e. $K_\theta = f \circ \pi$ for some C^∞ function $f : M \to \mathbb{R}$. If $S = 0$ then $\nabla f = \theta(\nabla f)T$. Moreover if (M,θ) is a Sasakian manifold ($\tau = 0$) then $f = 0$; consequently $R = 0$ and (M,θ) is a Sasakian space form of sectional curvature $c = -3$.*

Here ∇f is the ordinary gradient of f with respect to the Webster metric i.e. $g_\theta(\nabla f, X) = X(f)$ for any $X \in T(M)$. As a byproduct of Theorem 2 there are no "pseudohermitian space forms" except for those with $K_\theta = 0$. Moreover (as argued in [2]) these aren't Tanaka–Webster flat unless $\tau = 0$. So the term *pseudohermitian space form* should be reserved for pseudohermitian manifolds (M,θ) such that the sectional curvature (1) (rather than (2)) is constant and then examples abound. For instance (cf. [18] or Section 1.5 in [9]) if $[g_{\alpha\bar\beta}] \in \mathrm{GL}(n,\mathbb{C})$ is a Hermitian matrix and $c \in (0,+\infty)$ then let $Q_\pm(c) \subset \mathbb{C}^{n+1}$ be the real hypersurface defined by $r_\pm(z,w) \equiv g_{\alpha\bar\beta} z^\alpha \bar z^\beta \pm (w\bar w - c) = 0$, where $(z^1,\dots,z^n,w)$ are the natural complex coordinates on $\mathbb{C}^{n+1}$. Then $Q_\pm(c)$ is a nondegenerate CR manifold and the contact form $\theta_\pm = i g_{\alpha\bar\beta}(z^\alpha\, d\bar z^\beta - \bar z^\beta\, dz^\alpha) \pm i(w\, d\bar w - \bar w\, dw)$ has constant sectional curvature $\mathrm{Sect}(Q_\pm(c),\theta_\pm) = \pm 1/(2c)$. To prove Theorem 2 let us set

$$R_1(X,Y,Z,W) = g_\theta(X,Z)g_\theta(Y,W) - g_\theta(Y,Z)g_\theta(X,W)$$

for any $X,Y,Z,W \in T(M)$. If $L := R - 4fR_1$ then (by hypothesis)

$$L(X,Y,X,Y) = 0, \quad X,Y \in T(M).$$

Thus (by a result in [2, Appendix A])

$$\begin{aligned} R(X,Y,Z,W) = {} & 4fR_1(X,Y,Z,W) + \Omega(Y,W)A(X,Z) \\ & - \Omega(Y,Z)A(X,W) + \Omega(X,Z)A(Y,W) - \Omega(X,W)A(Y,Z) \\ & + g_\theta\big(S(Z_H,W_H),(\theta\wedge I)(X,Y)\big) - g_\theta\big(S(X_H,Y_H),(\theta\wedge I)(Z,W)\big), \end{aligned} \tag{14}$$

where $X_H = \pi_H X$ and $\pi_H : T(M) \to H(M)$ is the projection associated to the decomposition $T(M) = H(M) \oplus \mathbb{R}$. Also I is the identical transformation and $(\theta\wedge I)(X,Y) = \frac{1}{2}\{\theta(X)Y - \theta(Y)X\}$. Note that $\nabla g_\theta = 0$ yields $\nabla R_1 = 0$ hence (by computing the covariant derivative of (14) and using $\nabla\Omega = 0$)

$$\begin{aligned} (\nabla_U R)(X,Y,Z,W) = {} & U(f)R_1(X,Y,Z,W) \\ & + \Omega(Y,W)(\nabla_U A)(X,Z) - \Omega(Y,Z)(\nabla_U A)(X,W) \\ & + \Omega(X,Z)(\nabla_U A)(Y,W) - \Omega(X,W)(\nabla_U A)(Y,Z) \end{aligned}$$

$$+ g_\theta\big((\nabla_U S)(Z_H, W_H), (\theta \wedge I)(X, Y)\big)$$
$$- g_\theta\big((\nabla_U S)(X_H, Y_H), (\theta \wedge I)(Z, W)\big) \tag{15}$$

for any $X, Y, Z, W, U \in T(M)$. Let us take the cyclic sum over (U, Z, W) and use the second Bianchi identity (cf. Theorem 5.3 in [12, vol. I, p. 135])

$$\sum_{UZW} (\nabla_U R)(X, Y, Z, W) = -\sum_{UZW} g_\theta(R(T_\nabla(U, Z), W)Y, X)$$

so that to obtain

$$\begin{aligned}
-\sum_{UZW} R\big(T_\nabla(U, Z), W\big)Y = U(f)\big\{&g_\theta(Y, W)Z - g_\theta(Y, Z)W\big\} \\
&+ Z(f)\big\{g_\theta(Y, U)W - g_\theta(Y, W)U\big\} + W(f)\big\{g_\theta(Y, Z)U - g_\theta(Y, U)Z\big\} \\
&+ \Omega(Y, W)S(U, Z) + \Omega(Y, U)S(Z, W) + \Omega(Y, Z)S(W, U) \\
&+ g_\theta\big(Y, S(U, W)JZ + S(Z, U)JW + S(W, Z)JU\big) \\
&- g_\theta\big((\nabla_U S)(\pi_H\cdot, Y_H), (\theta \wedge I)(Z, W)\big)^\sharp \\
&- g_\theta\big((\nabla_Z S)(\pi_H\cdot, Y_H), (\theta \wedge I)(W, U)\big)^\sharp \\
&- g_\theta\big((\nabla_W S)(\pi_H\cdot, Y_H), (\theta \wedge I)(U, Z)\big)^\sharp \\
&- \frac{1}{2}\theta(Y)\big\{(\nabla_U S)(Z_H, W_H) + (\nabla_Z S)(W_H, U_H) + (\nabla_W S)(U_H, Z_H)\big\} \\
&+ \frac{1}{2}g_\theta\big(Y, (\nabla_U S)(Z_H, W_H) + (\nabla_Z S)(W_H, U_H) + (\nabla_W S)(U_H, Z_H)\big)T
\end{aligned} \tag{16}$$

for any $Y, Z, W, U \in T(M)$. Here $\sharp$ denotes raising of indices with respect to g_θ i.e. $g_\theta(\omega^\sharp, X) = \omega(X)$ for any $\omega \in T^*(M)$ and any $X \in T(M)$. In particular for $Y, Z, W, U \in H(M)$ the left hand member of (16) becomes (by (13)) $\sum_{UZW} \Omega(U, Z)R(T, W)Y$. To compute terms of the form $R(T, Y)Z$ we need to recall the identity (cf. Section 1.4.2 in [9])

$$\begin{aligned}
g_\theta\big(R(X, Y)Z, W\big) = {}& g_\theta\big(R(W, Z)Y, X\big) - g_\theta\big((LX \wedge LY)Z, W\big) \\
&+ g_\theta\big((LW \wedge LZ)Y, X\big) + g_\theta\big(S(X, Y), Z\big)\theta(W) - g_\theta\big(S(W, Z), Y\big)\theta(X) \\
&- \theta(Z)g_\theta\big(S(X, Y), W\big) + \theta(Y)g_\theta\big(S(W, Z), X\big) \\
&+ 2g_\theta\big((\theta \wedge \mathcal{O})(X, Y), Z\big)\theta(W) - 2g_\theta\big((\theta \wedge \mathcal{O})(W, Z), Y\big)\theta(X) \\
&- 2\theta(Z)g_\theta\big((\theta \wedge \mathcal{O})(X, Y), W\big) + 2\theta(Y)g_\theta\big((\theta \wedge \mathcal{O})(W, Z), X\big)
\end{aligned} \tag{17}$$

for any $X, Y, Z, W \in T(M)$. Here

$$L = \tau + J, \qquad \mathcal{O} = \tau^2 + 2J\tau - I.$$

Also $(X \wedge Y)Z = g_\theta(Z, X)Y - g_\theta(Z, Y)X$. The lack of symmetry of $R(X, Y, Z, W)$ in the pairs (X, Y) and (Z, W) (in contrast with the case of Riemannian curvature, cf. Proposition 2.1 in [12], p. 201) is the consequence of the presence of torsion terms in the first Bianchi identity. Let us set $X = T$ and $Y, Z, W \in H(M)$ in (17). We obtain (as $LT = 0$)

$$g_\theta\big(R(T, Y)Z, W\big) = g_\theta\big(Y, S(Z, W)\big). \tag{18}$$

Next for any vector field $Z \in H(M)$ we may choose $Y \in H(M)$ such that $g_\theta(Y, Z) = 0$ and $\|Y\| = 1$. Also let $U = Y$ and $W = JY$. Then (16) becomes (by (18))

$$g_\theta\big(Z, S(Y, JY)\big) = Z(f) - g_\theta\big(JY, S(Y, Z)\big) + g_\theta\big(Y, S(JY, Z)\big)$$

or $Z(f) = 2g_\theta(S(Y, JY), Z)$ yielding the first statement in Theorem 2. Similarly we may use (16) for $Z = T$ and $W \in H(M)$ chosen such that $\|W\| = 1$ together with $U = Y$ and $Y = JW$ so that to obtain (when $\tau = 0$) $T(f)W -$

$W(f)T = 0$. As M is connected f is constant and then by Theorem 5 in [2] it follows that $R = 0$ (and in particular M is a spherical CR manifold i.e. the Chern–Moser tensor vanishes identically (cf. [10, p. 187])). If this is the case then (by Proposition 4 in [2]) $(M, (J, -T, -\theta, g_\theta))$ is a Sasakian space form of (constant) φ-sectional curvature $c = -3$. Finally, if M is compact let $(\rho, \mathrm{dev}) : (\mathrm{Aut}_{\mathrm{CR}}(\tilde{M}), \tilde{M}) \to (\mathrm{PU}(n+1, 1), S^{2n+1})$ be the developing pair for M as a spherical CR manifold (cf. [10, p. 195]) where $\tilde{M}$ is the universal covering space of M. Then (cf. [10, p. 205]) $\mathrm{dev} : \tilde{M} \to S^{2n+1} \setminus \{\infty\} \approx \mathbb{H}_n$ is an isometry (where $\mathbb{H}_n$ is thought of as carrying the left invariant Webster metric associated to the contact form $\theta_0 = dt + i \sum_{j=1}^{n} (z^j d\bar{z}^j - \bar{z}^j dz^j)$) thus proving Corollary 1 (cf. also Theorem 6.1 in [10, p. 207]).

4. Pseudohermitian manifolds of maximal $\dim_{\mathbb{R}} \mathfrak{i}(M, \theta)$

4.1. *Infinitesimal pseudohermitian transformations*

The purpose of this section is to prove the following

Theorem 3. *Let (M, θ) be a connected pseudohermitian manifold of CR dimension n with L_θ positive definite. Then* (a) $\dim_{\mathbb{R}} \mathfrak{i}(M, \theta) \leqslant (n+1)^2$. (b) *If* $\dim_{\mathbb{R}} \mathfrak{i}(M, \theta) = (n+1)^2$ *then (M, θ) has constant pseudohermitian sectional curvature $H_\theta(\sigma)$.*

Let (M, θ) a $(2n+1)$-dimensional pseudohermitian manifold of CR dimension n. A *CR isomorphism* is a C^∞ diffeomorphism $f : M \to M$ and a CR map i.e. $(d_x f) T_{1,0}(M)_x = T_{1,0}(M)_{f(x)}$ for any $x \in M$. A CR isomorphism $f : M \to M$ is a *pseudohermitian transformation* if $f^*\theta = \theta$. Let $\mathrm{Psh}(M, \theta)$ be the group of all pseudohermitian transformations. By a result of S.M. Webster, [18], (i) $\mathrm{Psh}(M, \theta)$ is a Lie group of dimension $\leqslant (n+1)^2$ with isotropy groups of dimension $\leqslant n^2$. Moreover (ii) if M is strictly pseudoconvex then the isotropy groups are compact and if M is compact then $\mathrm{Psh}(M, \theta)$ is compact. The statement (i) also follows from part (a) in Theorem 3. Indeed each 1-parameter subgroup of $\mathrm{Psh}(M, \theta)$ induces an infinitesimal pseudohermitian transformation which is complete and conversely, so that the Lie algebra of $\mathrm{Psh}(M, \theta)$ is isomorphic to the Lie subalgebra of $\mathfrak{i}(M, \theta)$ consisting of all complete infinitesimal pseudohermitian transformations. In particular, if $\dim_{\mathbb{R}} \mathfrak{i}(M, \theta) = (n+1)^2$ then $\dim \mathrm{Psh}(M, \theta) = (n+1)^2$ hence one may apply the classification (up to contact homotheties) result Theorem 4.10 in [15, p. 236]. Another proof of S. M. Webster's result (i)–(ii) above was given by E. Musso, cf. op. cit., p. 225.

Let $\mathrm{GL}(m, \mathbb{R}) \to L(M) \xrightarrow{\Pi} M$ be the principal bundle of all linear frames tangent to M, where $m = 2n+1$. Any diffeomorphism $f : M \to M$ induces in a natural manner an automorphism $\tilde{f}$ of $\mathrm{GL}(m, \mathbb{R}) \to L(M) \to M$ (cf. e.g. [12, vol. I, p. 226]). Assume from now on that M is strictly pseudoconvex and θ is chosen such that L_θ is positive definite. Let $U(M, \theta)_x$ consist of all linear frames $b \in L(M)_x$ such that

$$b(e_0) = T_x, \quad b(e_\alpha) \in H(M)_x, \quad b(e_{\alpha+n}) = J_x b(e_\alpha), \quad 1 \leqslant \alpha \leqslant n,$$

$$g_{\theta,x}\big(b(e_i), b(e_j)\big) = \delta_{ij}, \quad 0 \leqslant i, j \leqslant 2n.$$

This construction gives rise to a principal subbundle $\mathrm{U}(n) \to U(M, \theta) \to M$ of $L(M)$. By a result of S. Nishikawa et al. (cf. Proposition 10 in [7, p. 1065]) a diffeomorphism $f : M \to M$ is a pseudohermitian transformation if and only if $\tilde{f}(U(M, \theta)) = U(M, \theta)$. Also for any fibre-preserving diffeomorphism $F : U(M, \theta) \to U(M, \theta)$ leaving invariant the canonical form ν ($\nu_b = b^{-1} \circ (d_b \Pi)$, $b \in U(M, \theta)$) there is $f \in \mathrm{Psh}(M, \theta)$ such that $F = \tilde{f}$.

A tangent vector field X on M is an *infinitesimal pseudohermitian transformation* of (M, θ) if the local 1-parameter group of local transformations induced by X consists of local pseudohermitian transformations of (M, θ). Let X be a vector field on M and $\{\varphi_t\}_{|t|<\epsilon}$ the local 1-parameter group of local transformations induced by X. Let $\tilde{X}$ be the *natural lift* of X to $L(M)$ (cf. [12, vol. I, pp. 229–230]) i.e. the vector field $\tilde{X}$ on $L(M)$ induced by the local 1-parameter group $\{\tilde{\varphi}_t\}_{|t|<\epsilon}$ of local transformations of $L(M)$. Let $\mathfrak{i}(M, \theta)$ denote the set of all infinitesimal pseudohermitian transformations of (M, θ). By Proposition 11 in [7, p. 1066], the following statements are equivalent 1) $X \in \mathfrak{i}(M, \theta)$, 2) $\tilde{X}_b \in T_b(U(M, \theta))$ for any $b \in U(M, \theta)$, 3) $\mathcal{L}_X \theta = 0$ and $\mathcal{L}_X \theta^\alpha = f^\alpha_\beta \theta^\beta$ for any local frame $\{\theta^\alpha \colon 1 \leqslant \alpha \leqslant n\}$ of $T_{1,0}(M)^*$ defined on the open subset $U \subseteq M$ and some C^∞ functions $f^\alpha_\beta : U \to \mathbb{R}$. Here $\mathcal{L}_X$ denotes the Lie derivative. As a corollary of $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$ and the previous characterization of $\mathfrak{i}(M, \theta)$ it follows that $\mathfrak{i}(M, \theta)$ is a Lie algebra.

4.2. Affine transformations

To prove the first statement in Theorem 3 it suffices to show that, for a fixed linear frame $b \in U(M,\theta)$, the linear map

$$\Phi_b : \mathfrak{i}(M,\theta) \to T_b\big(U(M,\theta)\big), \quad \Phi_b(X) = \tilde{X}_b, \quad X \in \mathfrak{i}(M,\theta),$$

is injective. Indeed

$$\dim_{\mathbf{R}} T_b\big(U(M,\theta)\big) = \dim M + \dim \mathrm{U}(n) = (n+1)^2.$$

Let ∇ be the Tanaka–Webster connection of (M,θ). An *affine transformation* of (M,∇) is a diffeomorphism $f : M \to M$ such that $\nabla_{\dot{\gamma}_f} X = 0$ along $\gamma_f := f \circ \gamma$, for any tangent vector field X along γ such that $(\nabla_{\dot{\gamma}} X)_{\gamma(t)} = 0$, and for any curve γ in M. Let $\mathfrak{U}(M,\nabla)$ be the group of all affine transformations of (M,∇). If $f : M \to M$ is a diffeomorphism and X is a vector field on M we set $(f_*X)_y = (d_{f^{-1}(y)} f) X_{f^{-1}(y)}$ for any $y \in M$. By a result of J. Masamune et al. (cf. the proof of Lemma 1 in [6, p. 357]) if we set

$$\nabla^f_X Y := (f_*)^{-1} \nabla_{f_*X} f_* Y$$

and f is a pseudohermitian transformation then $\nabla^f = \nabla$. Therefore we may apply Proposition 1.4 in [12, vol. I, p. 228], to conclude that f is an affine transformation, hence $\mathrm{Psh}(M,\theta)$ is a subgroup of $\mathfrak{U}(M,\nabla)$.

A tangent vector field X on M is an *infinitesimal affine transformation* of (M,∇) if the local 1-parameter group induced by X consists of local affine transformations of (M,∇). Let $\mathfrak{a}(M,\nabla)$ be the Lie algebra of all affine transformations of (M,∇).

Let $\omega \in \Gamma^\infty(T^*(L(M)) \otimes \mathfrak{gl}(m,\mathbb{R}))$ be the connection 1-form associated to the Tanaka–Webster connection ∇ and let us denote by $\mathfrak{a}(\omega)$ the Lie algebra of all tangent vector fields $\mathcal{X}$ on $L(M)$ such that 1) $(d_u R_a)\mathcal{X}_u = \mathcal{X}_{ua}$, $u \in L(M)$, $a \in \mathrm{GL}(m,\mathbb{R})$, 2) $\mathcal{L}_{\mathcal{X}} \nu = 0$, and 3) $\mathcal{L}_{\mathcal{X}} \omega = 0$. Here $\nu_b = b^{-1} \circ (d_b \Pi)$ for any $b \in L(M)$. It is a well known fact (cf. e.g. [12, vol. I, p. 232]) of general connection theory that the map $X \mapsto \tilde{X}$ gives a Lie algebra isomorphism $\mathfrak{a}(M,\nabla) \approx \mathfrak{a}(\omega)$.

Now we may prove Theorem 3. To this end let $X \in \mathrm{Ker}(\Phi_u)$. Then $X \in \mathfrak{i}(M,\theta) \subset \mathfrak{a}(M,\nabla)$ hence $\tilde{X} \in \mathfrak{a}(\omega)$ and $\tilde{X}_u = 0$ hence one may apply the lemma in [12, vol. I, p. 232] (in the proof of Theorem 2.3, cf. op. cit.) to conclude that $\tilde{X} = 0$ identically on $L(M)$. Yet (by Proposition 2.1 in [12, vol. I, p. 229]) $\tilde{X}$ is Π-related to X so $X = 0$ everywhere on M.

To prove the second statement in Theorem 3 let $\sigma \in G_2(H(M))_x$ and let $b \in U(M,\theta)$ such that $\Pi(b) = x$. Let $u \in \sigma$ such that $\|u\| = 1$ and $\xi \in \mathbb{C}^n$ given by $\xi = b^{-1}(u)$. Here $\mathbb{C}^n \approx \mathbb{R}^{2n} \times \{0\} \subset \mathbb{R}^m$. Let $B(\xi)$ and $B(J_0\xi)$ be the standard horizontal vector fields associated (in the sense of [12, vol. I, p. 119]) to ξ and $J_0\xi$, where J_0 is the standard complex structure on $\mathbb{C}^n$. Let $\Omega = D\omega$ be the curvature 2-form of the Tanaka–Webster connection. Then, again by a general fact within connection theory (cf. [12, vol. I, p. 133])

$$H_\theta(\sigma) = g_{\theta,x}\big(R_x(u, J_x u) J_x u, u\big) = 2\big(\Omega\big(B(\xi), B(J_0\xi)\big)_b \cdot J_0\xi, \xi\big),$$

where $(\,,\,)$ is the Euclidean inner product on $\mathbb{R}^m$ and $A \cdot \xi$ is the matrix product ($A \in \mathfrak{gl}(m,\mathbb{R}) \approx \mathbb{R}^{m^2}$).

We wish to show that $H_\theta(\sigma)$ is a point function only. To this end let $\sigma' \in G_2(H(M))_x$ be another holomorphic frame tangent at $x \in M$ and $v \in \sigma'$ such that $\|v\| = 1$. We set $\eta = b^{-1}(v) \in \mathbb{C}^n$. There is $g \in \mathrm{U}(n)$ such that $\eta = g\xi$. Then (by Proposition 2.2 in [12, vol. I, p. 119])

$$\begin{aligned}
\Omega\big(B(\eta), B(J_0\eta)\big)_b &= \Omega\big(B(g\xi), B(J_0 g\xi)\big)_b \\
&= \Omega_b\big((d_{bg} R_{g^{-1}}) B(\xi)_{bg}, (d_{bg} R_{g^{-1}}) B(J_0\xi)_{bg}\big) \\
&= \mathrm{ad}(g)\Omega_{bg}\big(B(\xi)_{bg}, B(J_0\xi)_{bg}\big) = g \cdot \Omega_{bg}\big(B(\xi)_{bg}, B(J_0\xi)_{bg}\big) \cdot g^{-1},
\end{aligned}$$

where $R_g : U(M,\theta) \to U(M,\theta)$ is the right translation by g and ad denotes the adjoint representation of $\mathrm{GL}(m,\mathbb{R})$ in its Lie algebra. Moreover

$$\begin{aligned}
H_\theta(\sigma') &= 2\big(\Omega\big(B(\eta), B(J_0\eta)\big)_b \cdot J_0\eta, \eta\big) = 2\big(\big(g \cdot \Omega_{bg}\big(B(\xi)_{bg}, B(J_0\xi)_{bg}\big) \cdot g^{-1}\big) \cdot g J_0\xi, g\xi\big) \\
&= 2\big(\Omega_{bg}\big(B(\xi)_{bg}, B(J_0\xi)_{bg}\big) \cdot J_0\xi, \xi\big)
\end{aligned}$$

and it remains to be shown that the function $F_x := F|_{U(M,\theta)_x}$ is constant, where

$$F\colon U(M,\theta)\to\mathbb{R},\quad F(b)=\Omega\big(B(\xi),\ B(J_0\xi)\big)_b,\quad b\in U(M,\theta).$$

Let $X\in\mathfrak{i}(M,\theta)\subset\mathfrak{a}(M,\nabla)$. Then (by Proposition 2.2 in [12, vol. I, p. 230])

$$\begin{aligned}\tilde X(F)&=\tilde X\big(\Omega\big(B(\xi),B(J_0\xi)\big)\big)=(\mathcal{L}_{\tilde X}\Omega)\big(B(\xi),B(J_0\xi)\big)\\&\quad+\Omega\big(\big[\tilde X,B(\xi)\big],B(J_0\xi)\big)+\Omega\big(B(\xi),\big[\tilde X,B(J_0\xi)\big]\big)=0.\end{aligned}$$

This simple fact has two consequences. First, let V be an arbitrary tangent vector on $U(M,\theta)_x$ i.e.

$$V\in T_b\big(U(M,\theta)_x\big)=\mathrm{Ker}(d_b\Pi)\subset T_b\big(U(M,\theta)\big)$$

for some $b\in U(M,\theta)$ with $\Pi(b)=x$. As Φ_b is assumed to be on-to there is $X\in\mathfrak{i}(M,\theta)$ such that $\tilde X_b=V$ hence $V(F)=\tilde X(F)_b=0$. As $U(n)$ is connected (and $U(M,\theta)_x\approx U(n)$, a diffeomorphism) it follows that F is constant i.e. $\Omega_{bg}(B(\xi)_{bg},B(J_0\xi)_{bg})=\Omega_b(B(\xi)_b,B(J_0\xi)_b)$ hence there is a smooth function $f\colon M\to\mathbb{R}$ such that $H_\theta=f\circ\pi$. At this point we may apply Theorem 6 in Appendix A provided that $n\geqslant 3$ and $\tau=0$. However one may prove Theorem 3 in full generality as follows. Let $W\in T_b(U(M,\theta))$ be an arbitrary tangent vector and $Y\in\mathfrak{i}(M,\theta)$ such that $\tilde Y_b=W$. Then $W(F)=\tilde Y(F)_b=0$ hence for any fixed $\xi\in\mathbb{C}^n$ the function $(\Omega(B(\xi),B(J_0\xi))\cdot J_0\xi,\xi)$ is constant in a neighborhood of b, so that f follows to be locally constant, and then constant on M.

5. Pseudohermitian space forms

A pseudohermitian manifold (M,θ) with $H_\theta(\sigma)=\text{const.}$ is said to be a *pseudohermitian space form*. Similarly to Theorem 5 in [2] (giving the precise form of the curvature tensor field R of (M,θ) when $K_\theta(\sigma)=\text{const.}$) we establish

Theorem 4. *Let (M,θ) be a pseudohermitian manifold of CR dimension n. If $H_\theta(\sigma)=c$ (with $c\in\mathbb{R}$) for any $\sigma\in G_2(H(M))$ then*

$$\begin{aligned}R(X,Y,Z,W)&=c\big\{2\Omega(X,Y)\Omega(Z,W)+g_\theta(X,Z)g_\theta(Y,W)-g_\theta(X,W)g_\theta(Y,Z)\\&\quad+\Omega(X,Z)\Omega(Y,W)-\Omega(X,W)\Omega(Y,Z)\big\}+g_\theta(X,Z)A(Y,JW)-g_\theta(X,W)A(Y,JZ)\\&\quad+g_\theta(Y,W)A(X,JZ)-g_\theta(Y,Z)A(X,JW)+\Omega(X,Z)A(Y,W)-\Omega(X,W)A(Y,Z)\\&\quad+\Omega(Y,W)A(X,Z)-\Omega(Y,Z)A(X,W)\end{aligned}\tag{19}$$

for any $X,Y,Z,W\in H(M)$. In particular

$$\mathrm{Ric}(X,Y)=2c(n+1)g_\theta(X,Y)+2(n-1)A(X,JY)\tag{20}$$

for any $X,Y\in H(M)$ hence each pseudohermitian space form (M,θ) is a pseudo-Einstein manifold of constant pseudohermitian scalar curvature $\rho=2cn(n+1)$.

Here $\mathrm{Ric}(X,Y)=\mathrm{trace}\{Z\mapsto R(Z,Y)X\}$. If $\{T_\alpha\colon 1\leqslant\alpha\leqslant n\}$ is a local frame of $T_{1,0}(M)$ on the open set $U\subseteq M$ then we set $g_{\alpha\bar\beta}=L_\theta(T_\alpha,T_{\bar\beta})$ and $R_{\alpha\bar\beta}=\mathrm{Ric}(T_\alpha,T_{\bar\beta})$. Then $R_{\alpha\bar\beta}$ is the *pseudohermitian Ricci tensor* and $\rho=g^{\alpha\bar\beta}R_{\alpha\bar\beta}$ is the *pseudohermitian scalar curvature*. Cf. J.M. Lee, [13] (or [9, Chapter 5]) a contact form θ is *pseudo-Einstein* if $R_{\alpha\bar\beta}=(\rho/n)g_{\alpha\bar\beta}$. To prove Theorem 4 we consider the 4-tensor field

$$\begin{aligned}R_0(X,Y,Z,W)&=\frac{1}{4}\big\{g_\theta(X,Z)g_\theta(Y,W)-g_\theta(X,W)g_\theta(Y,Z)\\&\quad+\Omega(X,Z)\Omega(Y,W)-\Omega(X,W)\Omega(Y,Z)+2\Omega(X,Y)\Omega(Z,W)\big\}\end{aligned}\tag{21}$$

for any $X,Y,Z,W\in H(M)$ and set $L=R-4cR_0$. Then we exploit the symmetries of L to establish (19) (using the algebraic machinery in the proof of Proposition 7.1 in [12, vol. II, p. 166]). Details are given in Appendix A where we also prove a Sasakian version of the complex Schur theorem.

E. Musso has classified (cf. [15]) up to contact homotheties the G-homogeneous pseudo-Einstein manifolds (M,θ) with L_θ positive definite. The same problem when L_θ is but nondegenerate is open. We recall that a pseudohermitian manifold is G-*homogeneous* if there is a closed subgroup $G \subset \mathrm{Psh}(M,\theta)$ such that G acts transitively on M. Also a *contact homothety* among two pseudohermitian manifolds (M,θ) and (M',θ') is a CR diffeomorphism $f: M \to M'$ such that $f^*\theta' = r\theta$ for some $r \in (0,+\infty)$.

Let (M,θ) be a G-homogeneous pseudohermitian manifold with G connected and L_θ positive definite. As usual we fix a point $x_0 \in M$ and let $H \subset G$ be the isotropy subgroup at x_0 and $H \to G \to M = G/H$ the corresponding principal bundle. Let V be the left invariant vector field on G determined by T_{x_0}. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. We consider a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ where $\mathfrak{p}$ is identified with $T_{x_0}(M)$. Due to this identification one has a direct sum decomposition $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{v}$ where the H-invariant subspaces $\mathfrak{m}$ and $\mathfrak{v}$ correspond to $H(M)_{x_0}$ and $\mathbb{R}T_{x_0}$, respectively. Let η be the left invariant differential 1-form on G determined by

$$(\mathfrak{h} \oplus \mathfrak{m}) \rfloor \eta = 0, \qquad \eta(V) = 1,$$

and let us set $K = \{a \in G\colon \mathrm{ad}(a)^*\eta = \eta\}$. Finally let $K' = K_0 H$, where K_0 is the connected component of the identity in K, and $B = G/K'$. Then the natural projection $p: M \to B$ organizes M as a principal bundle (with S^1 or $\mathbb{R}$ as a structure group) over B (and the fibres of p are maximal integral curves of T). Combining Theorem 4 above with Theorems 4.5, 4.7 and 4.8 in [15, pp. 233–236], we may conclude that Corollary 2 holds.

Let us briefly describe the pseudohermitian structures on the model spaces in (i)–(iii) of Corollary 2. Under the assumptions of Corollary 2 it follows (by (20)) that B is a simply-connected compact homogeneous Kähler–Einstein manifold. Therefore, by a result in [15, pp. 230–232], there is a principle S^1-bundle $\pi_{(1)}: B_{(1)} \to B$ and a canonical contact form $\theta_{(1)}$ on $B_{(1)}$ such that $(B_{(1)}, \theta_{(1)})$ is a pseudohermitian manifold with $L_{\theta_{(1)}}$ positive definite and $\pi_{(1)}$ is a Riemannian submersion of $(B_{(1)}, g_{\theta_{(1)}})$ onto B. Moreover if $c_1(M)$ is the integral first Chern class of $T(M)$ then $c_1(M) = kc_1(B_{(1)})$ for some $k \in \mathbb{Z}$, $k > 0$. Let $\pi_{(k)}: B_{(k)} \to B$ be the kth tensor power of $\pi_{(1)}: B_{(1)} \to B$. Again by a result in [15, p. 232], there is a unique pseudohermitian structure $\theta_{(k)}$ on $B_{(k)}$ such that $(B_{(k)}, \theta_{(k)})$ is a pseudohermitian manifold with $L_{\theta_{(k)}}$ positive definite and $\pi_{(k)}$ is a Riemannian submersion of $(B_{(k)}, g_{\theta_{(k)}})$ onto $(B, \sqrt{k}g)$, where g is the Kähler–Einstein metric of B. Then $(B_{(k)}, \theta_{(k)})$ is referred to as the *canonical pseudohermitian manifold of index* k over B. The contact form of the model space $B \times S^1$ in (ii) is given by $\theta' = a\,d\gamma + i(\bar\partial - \partial)\log K(z,z)$ for some $a \in (0,+\infty)$, where γ is a local fibre coordinate (i.e. $\partial/\partial\gamma$ is tangent to the S^1-action on $B \times S^1$) and $K(z,\zeta)$ is the Bergman kernel of B (thought of as an affinely homogeneous Siegel domain of the second kind, cf. Theorem 4.7 in [15, p. 235]). Similarly $B \times \mathbb{R}$ is endowed with the contact form $\theta'' = a\,dt + i(\bar\partial - \partial)\log K(z,z)$ for some $a \in (0,+\infty)$. As to the model spaces in (iii), $\mathbb{C}^n \times S^1$ is endowed with the contact form $\theta' = a\,d\gamma + 2\sum_{j=1}^n y^j\,dx^j$ while $\mathbb{C}^n \times \mathbb{R}$ carries $\theta'' = a\,dt + 2\sum_{j=1}^n y^j\,dx^j$.

6. Pseudohermitian immersions

Let M and M' be two CR manifolds of CR dimensions n and $n+k$ respectively, with $k \geqslant 1$. A *CR immersion* is a C^∞ immersion $f: M \to M'$ and a CR map. Given pseudohermitian structures θ and θ' on M and M' respectively, a CR immersion is *isopseudohermitian* if $f^*\theta' = \theta$. Assume that M and M' are nondegenerate and let T' be the characteristic direction of $d\theta'$. A *pseudohermitian immersion* is an isopseudohermitian CR immersion such that $T'^{\perp} = 0$. If $V \in T(M')$ then $V^\perp = \mathrm{nor}(V)$ and $\mathrm{nor}: T(M') \to E(f)$ is the projection associated to the decomposition $T(M') = [f_*T(M)] \oplus E(f)$ while $E(f) \to M$ denotes the normal bundle of the given immersion. Here we assume that $(d_x f)T_x(M)$ is nondegenerate in $(T_{f(x)}(M'), g_{\theta', f(x)})$ and then $E(f)_x$ is the $g_{\theta', f(x)}$-orthogonal complement of $(d_x f)T_x(M)$. A theory of pseudohermitian immersions has been started by S. Dragomir [5]. Cf. also [1]. Assume from now on that both M and M' are strictly pseudoconvex and θ, θ' are chosen such that L_θ, $L_{\theta'}$ are positive definite. We shall need the pseudohermitian analogs of the Gauss and Weingarten formulae

$$\nabla'_{f_*X} f_*Y = f_*\nabla_X Y + \alpha(f)(X,Y), \tag{22}$$

$$\nabla'_{f_*X}\xi = -f_* a_\xi X + \nabla^\perp_X \xi, \tag{23}$$

for any $X, Y \in T(M)$ and any $\xi \in \Gamma^\infty(E(f))$. Cf. (41)–(42) in [5, p. 185]. Here ∇' is the Tanaka–Webster connection of (M',θ') while $\alpha(f)$ is a $E(f)$-valued $C^\infty(M)$-bilinear form, a_ξ is an endomorphism of $T(M)$, and $\nabla^\perp$ is a connection in $E(f) \to M$ (the pseudohermitian analogs to the second fundamental form, Weingarten operator and normal connection of an isometric immersion). Let R' be the curvature tensor field of ∇'. We recall (cf. (61) in [5,

p. 191])

$$\tan\{R'(f_*X, f_*Y)f_*Z\} = R(X,Y)Z + a_{\alpha(f)(X,Z)}Y - a_{\alpha(f)(Y,Z)}X$$

for any $X, Y, Z \in T(M)$, where $\tan : T(M') \to T(M)$ is the natural projection. Let us take the inner product with $W \in T(M)$ and use

$$g_{\theta'}\big(\alpha(f)(X,Y), \xi\big) = g_\theta(a_\xi X, Y)$$

(cf. (50) in [5, p. 188]) so that to get

$$\begin{aligned} R'(f_*W, f_*Z, f_*X, f_*Y) = R(W,Z,X,Y) &+ g_{\theta'}\big(\alpha(f)(X,Z), \alpha(f)(Y,W)\big) \\ &- g_{\theta'}\big(\alpha(f)(Y,Z), \alpha(f)(X,W)\big). \end{aligned} \tag{24}$$

Lemma 1. *For any $X, Y \in T(M)$*

$$\alpha(f)(X, JY) = J'\alpha(f)(X,Y), \tag{25}$$

$$\alpha(f)(JX, Y) = J'\alpha(f)(X,Y) - \theta(X)J'QY, \tag{26}$$

where J' is the complex structure on $H(M')$ (extended to an endomorphism of $T(M')$ by requiring that $J'T' = 0$) and $Q(X) = \alpha(f)(T,X)$. Consequently

$$\alpha(JX, JY) = -\alpha(f)(X,Y) + \theta(X)QY \tag{27}$$

for any $X, Y \in T(M)$.

Replacing (W, Z, X, Y) by (X, JX, X, JX) in (24) and using Lemma 1 leads to the following

Theorem 5. *Let $f : M \to M'$ be a pseudohermitian immersion between two pseudohermitian manifolds (M, θ) and (M', θ') such that L_θ and $L_{\theta'}$ are positive definite. Then*

$$R'(f_*X, J'f_*X, f_*X, J'f_*X) = R(X, JX, X, JX) + 2\big\|\alpha(f)(X,X)\big\|^2 - 2\theta(X)g_{\theta'}\big(\alpha(f)(X,X), QX\big)$$

for any $X \in T(M)$. In particular $H_\theta(\sigma) \leqslant H_{\theta'}((d_x f)\sigma)$ for any $\sigma \in G_2(H(M))_x$ and any $x \in M$.

It remains that we prove Lemma 1. The identity (25) is a consequence of $\nabla' J' = 0$ and the Gauss formula (22). Cf. also (43) in [5, p. 187]. Moreover the identity

$$T_{\nabla'} = 2(\theta' \wedge \tau' - \Omega' \otimes T')$$

(cf. e.g. [9, Chapter 1]) leads to

$$\alpha(f)(Y,X) = \alpha(f)(X,Y) - 2(\theta \wedge Q)(X,Y), \tag{28}$$

where $Q(X) = \alpha(f)(T,X)$ for any $X \in T(M)$. Finally (25) and (28) imply (26)–(27).

The proof of Corollary 3 is immediate. Corollary 4 follows from a result by L. Ornea & M. Verbitsky, cf. Theorem 6.1 in [16, p. 141]. Indeed let (M, θ) be a compact Sasakian manifold and $V = M \times S^1$. Then V is a Vaisman manifold (cf. e.g. [8] for the relevant notions) admitting (cf. Theorem 5.1 in [16, p. 138]) an immersion $\phi : M \to H_\Lambda$ into a primary Hopf manifold $H_\Lambda = (\mathbb{C}^{n+1} \setminus \{0\})/\Gamma_\Lambda$ for some $n \geqslant 1$ and some $\Lambda = (\lambda_1, \ldots, \lambda_{n+1}) \in \mathbb{C}^{n+1}$ such that $0 < |\lambda_{n+1}| \leqslant \cdots \leqslant |\lambda_1| < 1$. Here Γ_Λ is the discrete group of complex analytic transformations of $\mathbb{C}^{n+1} \setminus \{0\}$ generated by $(z_1, \ldots, z_{n+1}) \mapsto (\lambda_1 z_1, \ldots, \lambda_{n+1} z_{n+1})$. See also [11, p. 202]. Moreover ϕ descends to a pseudohermitian immersion $M \to (S^{2n+1}, \theta_A)$ with $\lambda_j = e^{-a_j}$ hence (by Theorem 5 above) the upper bound on $H_\theta(\sigma)$ in Corollary 4.

Proposition 1. *Let (M, θ) be a pseudohermitian manifold with L_θ positive definite. If $\hat\theta = e^{2u}\theta$, $u \in C^\infty(M)$, then*

$$e^{2u} H_{\hat\theta}(\sigma) = H_\theta(\sigma) + 2iu_0 - 2u_\alpha u^\alpha - 2(\nabla_{\bar\beta} u_\alpha)\eta^\alpha \eta^{\bar\beta} \tag{29}$$

for any $\sigma \in G_2(H(M))_x$ and $x \in M$, where $X = Z + \overline{Z} \in \sigma$, $Z = \xi^\alpha T_\alpha$, and $\eta^\alpha = \|\xi\|^{-1}\xi^\alpha$, $\|\xi\|^2 = g_{\alpha\bar\beta}\xi^\alpha\xi^{\bar\beta}$. Consequently the pseudohermitian sectional curvature is not a CR invariant. In particular if $\hat\theta = (1/a)\theta$ $(a > 0)$ then $H_{\hat\theta}(\sigma) = aH_\theta(\sigma)$.

Proof. Let $\{T_\alpha\colon 1 \leqslant \alpha \leqslant n\}$ be a local frame of $T_{1,0}(M)$. Let $\hat{\nabla}$ be the Tanaka–Webster connection of $(M, e^{2u}\theta)$ and $\hat{\Gamma}^A_{BC}$ the connection coefficients with respect to $\{T_A\colon A \in \{0, 1, \dots, n, \bar{1}, \dots, \bar{n}\}\}$ (with the convention $T_0 = T$). We set $R_C{}^D{}_{AB} T_D = R(T_A, T_B) T_C$ so that

$$R_\alpha{}^\beta{}_{\lambda\bar{\mu}} = T_\lambda(\Gamma^\beta_{\bar{\mu}\alpha}) - T_{\bar{\mu}}(\Gamma^\beta_{\lambda\alpha}) + 2i\Gamma^\beta_{0\alpha} g_{\lambda\bar{\mu}} + \Gamma^\gamma_{\bar{\mu}\alpha}\Gamma^\beta_{\lambda\gamma} - \Gamma^\gamma_{\lambda\alpha}\Gamma^\beta_{\bar{\mu}\gamma} + \Gamma^\gamma_{\bar{\mu}\lambda}\Gamma^\beta_{\gamma\alpha} - \Gamma^{\bar{\gamma}}_{\lambda\bar{\mu}}\Gamma^\beta_{\bar{\gamma}\alpha}.$$

The proof of Proposition 1 is to replace in $\hat{R}_\alpha{}^\beta{}_{\lambda\bar{\mu}}$ from the identities

$$\hat{\Gamma}^\alpha_{\gamma\beta} = \Gamma^\alpha_{\gamma\beta} + 2(u_\gamma \delta^\alpha_\beta + u_\beta \delta^\alpha_\gamma), \tag{30}$$

$$\hat{\Gamma}^\alpha_{\bar{\gamma}\beta} = \Gamma^\alpha_{\bar{\gamma}\beta} - 2u^\alpha g_{\beta\bar{\gamma}}, \tag{31}$$

$$e^{2u}\hat{\Gamma}^\gamma_{\hat{0}\alpha} = \Gamma^\gamma_{0\alpha} + 2u_0\delta^\gamma_\alpha + i(\nabla^\gamma u_\alpha - 2u_\alpha u^\gamma + u^{\bar{\rho}}\Gamma^\gamma_{\bar{\rho}\alpha} - u^\rho \Gamma^\gamma_{\rho\alpha}), \tag{32}$$

where $u_A = T_A(u)$. Also $\nabla_{\bar{\beta}} u_\alpha = T_{\bar{\beta}}(u_\alpha) - \Gamma^\mu_{\bar{\beta}\alpha} u_\mu$ and $\nabla^\gamma u_\alpha = g^{\gamma\bar{\beta}}\nabla_{\bar{\beta}} u_\alpha$, etc. For a proof of (30)–(32) one may see Proposition 1 in [4, pp. 39–40], or [9, Chapter 2]. One obtains

$$\begin{aligned}\hat{R}_\alpha{}^\beta{}_{\lambda\bar{\mu}} = {} & R_\alpha{}^\beta{}_{\lambda\bar{\mu}} + 4ig_{\lambda\bar{\mu}}u_0\delta^\beta_\alpha - 4(\delta^\beta_\lambda g_{\bar{\mu}\alpha} + \delta^\beta_\alpha g_{\bar{\mu}\lambda})u^\gamma u_\gamma - 2g_{\alpha\bar{\mu}}\nabla_\lambda u^\beta - 2g_{\lambda\bar{\mu}}\nabla^\beta u_\alpha \\ & - 2\delta^\beta_\alpha \nabla_{\bar{\mu}} u_\lambda - 2\delta^\beta_\lambda \nabla_{\bar{\mu}} u_\alpha\end{aligned}$$

hence (by the commutation formula $\nabla_{\bar{\beta}} u_\alpha = \nabla_\alpha u_{\bar{\beta}} + 2ig_{\alpha\bar{\beta}} u_0$)

$$e^{-2u}\hat{R}_{\alpha\bar{\beta}\lambda\bar{\mu}}\xi^\alpha\xi^{\bar{\beta}}\xi^\lambda\xi^{\bar{\mu}} = R_{\alpha\bar{\beta}\lambda\bar{\mu}}\xi^\alpha\xi^{\bar{\beta}}\xi^\lambda\xi^{\bar{\mu}} - 8\|\xi\|^4\big\{(\nabla_{\bar{\beta}} u_\alpha)\eta^\alpha\eta^{\bar{\beta}} + u^\alpha u_\alpha - iu_0\big\}$$

implying (29).

Corollary 5. *Let (M, θ) be a compact Sasakian manifold such that the Vaisman manifold $V = M \times S^1$ admits an immersion ϕ into an ordinary complex Hopf manifold H_Λ, $\Lambda = (\lambda, \dots, \lambda)$, $0 < \lambda < 1$, and ϕ commutes with the Lee flows. Then $H_\theta(\sigma) \leqslant -\log\lambda$ for any $\sigma \in G_2(H(M))$.*

It should be observed that ϕ is obtained (cf. Theorem 5.1 in [16]) by first building an immersion $\tilde{V} \to H^0(V', L^k_{\mathbb{C}})$ (cf. op. cit., p. 139) of the universal covering $\tilde{V}$ into a suitable space of holomorphic sections and the problem of the effective computability of n and Λ (in terms of the given data, i.e. the locally conformal Kähler structure on V) is an open problem.

Corollary 6. *Let u be a CR-pluriharmonic function on M i.e. there is a C^∞ function $v : M \to \mathbb{R}$ such that $u + iv$ is a CR function. Then $e^{2u} H_{\hat{\theta}}(\sigma) \leqslant H_\theta(\sigma) + 2v_0$ for any $\sigma \in G_2(H(M))$.*

Proof. By a result of J.M. Lee, [13], if u is CR-pluriharmonic and v is conjugate to u then the complex Hessian of u is given by

$$\nabla_{\bar{\beta}} u_\alpha = (iu_0 - v_0) g_{\alpha\bar{\beta}}$$

hence (by (29)) $e^{2u} H_{\hat{\theta}}(\sigma) = H_\theta(\sigma) - 2u^\alpha u_\alpha + 2v_0 \leqslant H_\theta(\sigma) + 2v_0$. □

Appendix A. The Sasakian Schur theorem

As a first purpose of Appendix A we prove Theorem 4. First note that (by Proposition 7.2 in [12, vol. II, p. 167]) the 4-tensor (21) satisfies

$$R_0(X, Y, Z, W) = -R_0(Y, X, Z, W) = -R_0(X, Y, W, Z), \tag{33}$$

$$R_0(X, Y, Z, W) = R_0(Z, W, X, Y), \tag{34}$$

$$\sum_{YZW} R_0(X, Y, Z, W) = 0, \tag{35}$$

$$R_0(JX, JY, Z, W) = R_0(X, Y, JZ, JW) = R_0(X, Y, Z, W), \tag{36}$$

for any $X, Y, Z, W \in H(M)$. On the other hand (cf. e.g. the Appendix A in [2])

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = -R(X, Y, W, Z), \tag{37}$$

$$\begin{aligned} R(X, Y, Z, W) = {} & R(Z, W, X, Y) - 2\Omega(Y, Z)A(X, W) + 2\Omega(Y, W)A(X, Z) - 2\Omega(X, W)A(Y, Z) \\ & + 2\Omega(X, Z)A(Y, W), \end{aligned} \tag{38}$$

$$\sum_{YZW} R(X, Y, Z, W) = -2 \sum_{YZW} \Omega(Y, Z)A(W, X), \tag{39}$$

for any $X, Y, Z, W \in H(M)$. Moreover (by $\nabla J = 0$)

$$R(JX, JY, Z, W) = R(X, Y, Z, W). \tag{40}$$

Let us assume that $H_\theta = f \circ \pi$ for some $f : M \to \mathbb{R}$ and set $L = R - 4f R_0$. The properties (33)–(36) and (37)–(40) imply

$$L(X, Y, Z, W) = -L(Y, X, Z, W) = -L(X, Y, W, Z), \tag{41}$$

$$\begin{aligned} L(X, Y, Z, W) = {} & L(Z, W, X, Y) - 2\Omega(Y, Z)A(X, W) + 2\Omega(Y, W)A(X, Z) \\ & - 2\Omega(X, W)A(Y, Z) + 2\Omega(X, Z)A(Y, W), \end{aligned} \tag{42}$$

$$\sum_{YZW} L(X, Y, Z, W) = -2 \sum_{YZW} \Omega(Y, Z)A(W, X), \tag{43}$$

$$L(JX, JY, Z, W) = L(X, Y, Z, W). \tag{44}$$

As to the analog of the second equality in (36) for the 4-tensor L one has (by (42) and (44))

$$\begin{aligned} L(X, Y, JZ, JW) &= L(JZ, JW, X, Y) - 2\Omega(Y, JZ)A(X, JW) + 2\Omega(Y, JW)A(X, JZ) \\ &\quad - 2\Omega(X, JW)A(Y, JZ) + 2\Omega(X, JZ)A(Y, JW) \\ &= L(Z, W, X, Y) + 2g_\theta(Y, Z)A(X, JW) - 2g_\theta(Y, W)A(X, JZ) \\ &\quad + 2g_\theta(X, W)A(Y, JZ) - 2g_\theta(X, Z)A(Y, JW) \end{aligned}$$

and applying once more (42) leads to

$$\begin{aligned} L(X, Y, JZ, JW) = {} & L(X, Y, Z, W) \\ & + 2\Omega(Y, Z)A(X, W) - 2\Omega(Y, W)A(X, Z) \\ & + 2\Omega(X, W)A(Y, Z) - 2\Omega(X, Z)A(Y, W) \\ & + 2g_\theta(Y, Z)A(X, JW) - 2g_\theta(Y, W)A(X, JZ) \\ & + 2g_\theta(X, W)A(Y, JZ) - 2g_\theta(X, Z)A(Y, JW). \end{aligned} \tag{45}$$

Note that (by the very definition of H_θ)

$$L(X, JX, X, JX) = 0 \tag{46}$$

for any $X \in H(M)$. We shall need the 4-tensor K defined by

$$K(X, Y, Z, W) = L(X, JY, Z, JW) + L(X, JZ, Y, JW) + L(X, JW, Y, JZ).$$

As an immediate consequence of (46)

$$K(X, X, X, X) = 0. \tag{47}$$

Using repeatedly (41)–(42) and (44)–(45) together with

$$\Omega(JX, JY) = \Omega(X, Y), \; A(JX, JY) = -A(X, Y),$$

we derive the identities

$$\begin{aligned}
K(Y,X,Z,W) = {} & K(X,Y,Z,W) + 4\Omega(X,Y)A(Z,W) \\
& - 2g_\theta(X,Z)A(Y,JW) + 2g_\theta(Y,W)A(X,JZ) \\
& - 2g_\theta(X,W)A(Y,JZ) + 2g_\theta(Y,Z)A(X,JW),
\end{aligned} \tag{48}$$

$$\begin{aligned}
K(X,Y,W,Z) = {} & K(X,Y,Z,W) + 2g_\theta(Y,Z)A(X,JW) - 2\Omega(Y,W)A(X,Z) \\
& - 2g_\theta(X,W)A(Y,JZ) + 2\Omega(X,Z)A(Y,W) \\
& + 2\Omega(Y,Z)A(X,W) - 2g_\theta(Y,W)A(X,JZ) \\
& - 2\Omega(X,W)A(Y,Z) + 2g_\theta(X,Z)A(Y,JW),
\end{aligned} \tag{49}$$

$$\begin{aligned}
K(Z,Y,X,W) = {} & K(X,Y,Z,W) + 2\Omega(Z,W)A(X,Y) + 2\Omega(X,Y)A(Z,W) \\
& + 2\Omega(W,X)A(Y,Z) + 2\Omega(Y,Z)A(X,W),
\end{aligned} \tag{50}$$

$$\begin{aligned}
K(X,W,Z,Y) = {} & K(X,Y,Z,W) + 4\Omega(X,Y)A(Z,W) \\
& - 4\Omega(Y,W)A(X,Z) - 4\Omega(X,W)A(Y,Z) \\
& + 4g_\theta(X,Y)A(Z,JW) - 4g_\theta(Z,W)A(X,JY) \\
& + 4g_\theta(Y,Z)A(X,JW) - 4g_\theta(X,W)A(Y,JZ),
\end{aligned} \tag{51}$$

$$\begin{aligned}
K(W,Y,Z,X) = {} & K(X,Y,Z,W) - 4\Omega(X,W)A(Y,Z) \\
& + 2g_\theta(X,Y)A(Z,JW) + 2g_\theta(X,Z)A(Y,JW) \\
& - 2g_\theta(Y,W)A(X,JZ) - 2g_\theta(Z,W)A(X,JY),
\end{aligned} \tag{52}$$

$$\begin{aligned}
K(X,Z,Y,W) = {} & K(X,Y,Z,W) + 2g_\theta(Y,W)A(X,JZ) + 2\Omega(Z,W)A(X,Y) \\
& - 2g_\theta(X,Z)A(Y,JW) + 2\Omega(X,Y)A(Z,W) \\
& - 2\Omega(Y,W)A(X,Z) - 2g_\theta(Z,W)A(X,JY) \\
& - 2\Omega(X,Z)A(Y,W) + 2g_\theta(X,Y)A(Z,JW).
\end{aligned} \tag{53}$$

Let us replace X by $X+Y$ in (47) and use (48)–(53) so that to obtain

$$\begin{aligned}
& 2K(X,X,X,Y) + 3K(X,Y,X,Y) + 2K(X,Y,Y,Y) \\
& \quad = 8\big\{g_\theta(X,Y)A(X,JX) - g_\theta(X,X)A(X,JY) + \Omega(X,Y)A(X,X)\big\}.
\end{aligned} \tag{54}$$

Next we replace Y by $Y+Z$ in (54) and use again (48)–(53). We obtain

$$\begin{aligned}
& K(X,Y,X,Z) + K(X,Y,Y,Z) + K(X,Y,Z,Z) \\
& \quad = 2\Omega(Y,Z)A(X,X) - 2\Omega(X,Y)A(X,Z) + 2\Omega(X,Z)A(X,Y) \\
& \qquad - 2\Omega(X,Y)A(Y,Z) + 2\Omega(Y,Z)A(X,Y) - 2\Omega(X,Y)A(Z,Z) \\
& \qquad + 2\Omega(Y,Z)A(X,Z) + 2\Omega(X,Z)A(Y,Y) + 2\Omega(X,Z)A(Y,Z) \\
& \qquad + 4g_\theta(X,Z)A(X,JY) - 4g_\theta(X,Y)A(X,JZ) \\
& \qquad + 2g_\theta(X,Z)A(Y,JZ) - 2g_\theta(X,Y)A(Y,JZ) \\
& \qquad + 2g_\theta(X,Z)A(Y,JY) - 2g_\theta(X,Y)A(Z,JZ) \\
& \qquad + 2g_\theta(Y,Z)A(X,JY) - 2g_\theta(Y,Z)A(X,JZ) \\
& \qquad + 2g_\theta(Z,Z)A(X,JY) - 2g_\theta(Y,Y)A(X,JZ).
\end{aligned} \tag{55}$$

Finally let us replace Z by $Z+W$ in (55) and derive the expression of the 4-tensor K

$$\begin{aligned}
& L(X,JY,Z,JW) + L(X,JZ,Y,JW) + L(X,JW,Y,JZ) \\
& \quad = 2\Omega(Y,W)A(X,Z) + 2\Omega(X,W)A(Y,Z) + 2\Omega(Y,X)A(Z,W) \\
& \qquad + 2g_\theta(X,W)A(Y,JZ) - 2g_\theta(Y,Z)A(X,JW)
\end{aligned}$$

$$+ 2g_\theta(Z, W)A(X, JY) - 2g_\theta(X, Y)A(Z, JW). \tag{56}$$

Setting $Z = X$ and $W = Y$ in (56) gives

$$L(X, JY, X, JY) + L(X, JX, Y, JY) - L(X, JY, JX, Y) = 0.$$

Let us apply (45) to the last term. We obtain

$$\begin{aligned} 2L(X, JY, X, JY) + L(X, JX, Y, JY) &= 4\Omega(X, Y)A(X, Y) + 2g_\theta(Y, Y)A(X, JX) \\ &\quad - 2g_\theta(X, X)A(Y, JY). \end{aligned} \tag{57}$$

On the other hand we replace (Y, Z, W) in (43) by (JX, Y, JZ) so that to get

$$\begin{aligned} &L(X, JX, Y, JY) + L(X, Y, JY, JX) + L(X, JY, JX, Y) \\ &\quad = -2\big\{\Omega(JX, Y)A(JY, X) + \Omega(Y, JY)A(JX, X) + \Omega(JY, JX)A(Y, X)\big\} \end{aligned}$$

or (again by (45))

$$\begin{aligned} &L(X, JX, Y, JY) - L(X, Y, X, Y) - L(X, JY, X, JY) \\ &\quad = 2g_\theta(X, Y)A(X, JY) - 2g_\theta(Y, Y)A(X, JX) - 2\Omega(X, Y)A(X, Y). \end{aligned} \tag{58}$$

Let us subtract (58) from (57). This gives

$$\begin{aligned} &3L(X, JY, X, JY) + L(X, Y, X, Y) \\ &\quad = 6\Omega(X, Y)A(X, Y) - 2g_\theta(X, X)A(Y, JY) + 4g_\theta(Y, Y)A(X, JX) - 2g_\theta(X, Y)A(X, JY). \end{aligned} \tag{59}$$

Replacing Y by JY in (59) leads to the identity

$$\begin{aligned} &3L(X, Y, X, Y) + L(X, JY, X, JY) \\ &\quad = -6g_\theta(X, Y)A(X, JY) + 2g_\theta(X, X)A(Y, JY) + 4g_\theta(Y, Y)A(X, JX) + 2\Omega(X, Y)A(X, Y). \end{aligned} \tag{60}$$

Solving for $L(X, Y, X, Y)$ in the linear system (59)–(60) gives

$$L(X, Y, X, Y) = g_\theta(X, X)A(Y, JY) - 2g_\theta(X, Y)A(X, JY) + g_\theta(Y, Y)A(X, JX). \tag{61}$$

Once $L(X, Y, X, Y)$ is known one may apply (41)–(43) and the algebraic scheme in the proof of Proposition 1.2 in [12, vol. I, p. 198], to compute the whole of $L(X, Y, Z, W)$. Precisely we replace Y by $Y + W$ in (61) and use (43) so that to get

$$\begin{aligned} L(X, Y, X, W) = {} & \Omega(X, Y)A(X, W) + \Omega(Y, W)A(X, X) - \Omega(X, W)A(X, Y) \\ & + g_\theta(X, X)A(Y, JW) - g_\theta(X, Y)A(X, JW) \\ & - g_\theta(X, W)A(X, JY) + g_\theta(Y, W)A(X, JX). \end{aligned}$$

Next we replace X by $X + Z$ and derive the identity

$$\begin{aligned} L(X, Y, Z, W) = {} & L(X, W, Y, Z) + 2\Omega(X, Z)A(Y, W) - \Omega(X, Y)A(Z, W) + \Omega(Z, W)A(X, Y) \\ & - \Omega(Y, Z)A(X, W) + \Omega(W, X)A(Y, Z) \\ & + 2g_\theta(X, Z)A(Y, JW) + 2g_\theta(Y, W)A(X, JZ) \\ & - g_\theta(X, Y)A(Z, JW) - g_\theta(Y, Z)A(X, JW) \\ & - g_\theta(X, W)A(Y, JZ) - g_\theta(Z, W)A(X, JY). \end{aligned} \tag{62}$$

Another identity of the kind is got by replacing (Y, Z, W) in (62) by (Z, W, Y) i.e.

$$\begin{aligned} L(X, Z, W, Y) = {} & L(X, Y, Z, W) + 2\Omega(X, W)A(Z, Y) - \Omega(X, Z)A(W, Y) + \Omega(W, Y)A(X, Z) \\ & - \Omega(Z, W)A(X, Y) + \Omega(Y, X)A(Z, W) \\ & + 2g_\theta(X, W)A(Z, JY) + 2g_\theta(Z, Y)A(X, JW) \end{aligned}$$

$$
\begin{aligned}
&- g_\theta(X,Z)A(W,JY) - g_\theta(Z,W)A(X,JY)\\
&- g_\theta(X,Y)A(Z,JW) - g_\theta(W,Y)A(X,JZ).
\end{aligned}
\tag{63}
$$

Finally let us compute $3L(X,Y,Z,W)$ by expressing the second and third copy of $L(X,Y,Z,W)$ from (62)–(63), respectively. Then (by (43))

$$
\begin{aligned}
L(X,Y,Z,W) = {} & \Omega(X,Z)A(Y,W) - \Omega(Y,Z)A(X,W)\\
&+ \Omega(Y,W)A(X,Z) - \Omega(X,W)A(Y,Z)\\
&+ g_\theta(X,Z)A(Y,JW) - g_\theta(Y,Z)A(X,JW)\\
&+ g_\theta(Y,W)A(X,JZ) - g_\theta(X,W)A(Y,JZ)
\end{aligned}
$$

and (19) in Theorem 4 is proved. The identity (20) follows from (19) by contraction. Another scope of Appendix A is to establish the following Sasakian analog to the complex Schur theorem in [12, vol. II, p. 168].

Theorem 6. *Let (M,θ) be a connected pseudohermitian manifold of CR dimension $n \geqslant 3$. Assume that $H_\theta = f \circ \pi$ for some C^∞ function $f : M \to \mathbb{R}$. If $S = 0$ then $\nabla f = \theta(\nabla f)T$. Moreover if $\tau = 0$ then f is constant.*

Therefore each Sasakian manifold of CR dimension $\geqslant 3$ whose pseudohermitian sectional curvature (1) is but a point function is actually a pseudohermitian space form. The proof of the complex Schur theorem is to show that each Kählerian manifold whose holomorphic sectional curvature is a point function f is an Einstein manifold. Yet, if the manifold dimension is $\geqslant 3$, the Einstein condition together with the second Bianchi identity imply that f is constant (cf. Note 3 in [12, vol. I, p. 292–294]). As argued in [13], the pseudo-Einstein condition together with the second Bianchi identity (for the Tanaka–Webster connection) does not imply in general that the pseudohermitian scalar curvature is constant (due to the presence of torsion terms in the second Bianchi identity). Therefore we use the full curvature tensor

$$
\begin{aligned}
R(X,Y,Z,W) = {} & f\{2\Omega(X,Y)\Omega(Z,W) + g_\theta(X,Z)g_\theta(Y,W) - g_\theta(X,W)g_\theta(Y,Z)\\
&+ \Omega(X,Z)\Omega(Y,W) - \Omega(X,W)\Omega(Y,Z)\}\\
&+ g_\theta(X,Z)A(Y,JW) - g_\theta(X,W)A(Y,JZ)\\
&+ g_\theta(Y,W)A(X,JZ) - g_\theta(Y,Z)A(X,JW)\\
&+ \Omega(X,Z)A(Y,W) - \Omega(X,W)A(Y,Z)\\
&+ \Omega(Y,W)A(X,Z) - \Omega(Y,Z)A(X,W)
\end{aligned}
$$

for any $X,Y,Z,W \in H(M)$. Rather than contracting we take the covariant derivative of the previous identity. A rather lengthy calculation (based on $\nabla g_\theta = 0$ and $\nabla\Omega = 0$) leads to

$$
\begin{aligned}
(\nabla_U R)(X,Y,Z,W) = {} & U(f)\{2\Omega(X,Y)\Omega(Z,W) + g_\theta(X,Z)g_\theta(Y,Z) - g_\theta(X,W)g_\theta(Y,Z)\\
&+ \Omega(X,Z)\Omega(Y,W) - \Omega(X,W)\Omega(Y,Z)\}\\
&+ g_\theta(X,Z)(\nabla_U A)(Y,JW) - g_\theta(X,W)(\nabla_U A)(Y,JZ)\\
&+ g_\theta(Y,W)(\nabla_U A)(X,JZ) - g_\theta(Y,Z)(\nabla_U A)(X,JW)\\
&+ \Omega(X,Z)(\nabla_U A)(Y,W) - \Omega(X,W)(\nabla_U A)(Y,Z)\\
&+ \Omega(Y,W)(\nabla_U A)(X,Z) - \Omega(Y,Z)(\nabla_U A)(X,W)
\end{aligned}
\tag{64}
$$

for any $U \in T(M)$ and any $X,Y,Z,W \in H(M)$. From now on let $U \in H(M)$ and let us take the cyclic permutation $U \to Z \to W \to U$ in (64) to get two more identities of the kind. Next let us add up the three resulting identities and express $\sum_{UZW}(\nabla_U R)(X,Y,Z,W)$ from the second Bianchi identity

$$
\begin{aligned}
&(\nabla_U R)(X,Y,Z,W) + (\nabla_Z R)(X,Y,W,U) + (\nabla_W R)(X,Y,U,Z)\\
&\quad = g_\theta\big(\Omega(U,Z)W + \Omega(Z,W)U + \Omega(W,U)Z, S(Y,X)\big)
\end{aligned}
\tag{65}
$$

for any $X, Y, Z, W, U \in H(M)$. A calculation based on

$$(\nabla_X A)(Y, Z) - (\nabla_Y A)(X, Z) = g_\theta\big(S(X, Y), Z\big),$$
$$(\nabla_X A)(JY, Z) - (\nabla_Y A)(JX, Z) = g_\theta\big(S(X, Y), JZ\big),$$

furnishes

$$\begin{aligned}
&g_\theta\big(\Omega(U, Z)W + \Omega(Z, W)U + \Omega(W, U)Z, S(Y, \cdot)\big)^\sharp \\
&\quad = U(f)\big\{g_\theta(Y, W)Z - g_\theta(Y, Z)W + 2\Omega(Z, W)JY + \Omega(Y, W)JZ - \Omega(Y, Z)JW\big\} \\
&\qquad + Z(f)\big\{g_\theta(Y, U)W - g_\theta(Y, W)U + 2\Omega(W, U)JY + \Omega(Y, U)JW - \Omega(Y, W)JU\big\} \\
&\qquad + W(f)\big\{g_\theta(Y, Z)U - g_\theta(Y, U)Z + 2\Omega(U, Z)JY + \Omega(Y, Z)JU - \Omega(Y, U)JZ\big\} \\
&\qquad + g_\theta(S(U, W), JY)Z - g_\theta(S(U, Z), JY)W - g_\theta(S(Z, W), JY)U \\
&\qquad + g_\theta(Y, Z)JS(U, W) - g_\theta(Y, W)JS(U, Z) - g_\theta(Y, U)JS(Z, W) \\
&\qquad + g_\theta(S(U, W), Y)JZ - g_\theta(S(U, Z), Y)JW - g_\theta(S(Z, W), Y)JU \\
&\qquad + \Omega(Y, W)S(U, Z) - \Omega(Y, Z)S(U, W) + \Omega(Y, U)S(Z, W).
\end{aligned} \tag{66}$$

Let $U \in H(M)$ be arbitrary and let us choose $Y, Z \in H(M)$ such that $g_\theta(Y, Z) = \Omega(Y, Z) = 0$, $g_\theta(Y, U) = \Omega(Y, U) = 0$ and $g_\theta(Z, U) = \Omega(Z, U) = 0$. Also we assume $\|Y\| = 1$ and set $W = Y$. Then (66) gives

$$\begin{aligned}
&U(f)Z - Z(f)U - JS(U, Z) + \Omega\big(S(Z, U), Y\big)Y + g_\theta\big(S(Z, U), Y\big)JY + \Omega\big(S(U, Y), Y\big)Z \\
&\quad + g_\theta\big(S(U, Y), Y\big)JZ + \Omega\big(S(Y, Z), Y\big)U + g_\theta\big(S(Y, Z), Y\big)JU = 0.
\end{aligned}$$

Hence $S = 0$ yields $U(f) = 0$ and the first statement in Theorem 6 is proved. Note that (as $S = 0$)

$$(\nabla_Z R)(X, Y, W, T) = 0.$$

Hence, by the second Bianchi identity

$$\begin{aligned}
&(\nabla_T R)(X, Y, Z, W) + (\nabla_Z R)(X, Y, W, T) + (\nabla_W R)(X, Y, T, Z) \\
&\quad = g_\theta\big(R\big(\tau(W), Z\big)Y, X\big) - g_\theta\big(R\big(\tau(Z), W\big)Y, X\big) - g_\theta\big(R\big(T_\nabla(Z, W), T\big)Y, X\big)
\end{aligned}$$

we obtain

$$(\nabla_T R)(X, Y, Z, W) = g_\theta\big(R\big(\tau(W), Z\big)Y, X\big) - g_\theta\big(R\big(\tau(Z), W\big)Y, X\big) \tag{67}$$

for any $X, Y, Z, W \in H(M)$. Let us set $U = T$ in (64) and substitute from (67) in the resulting identity. We obtain

$$\begin{aligned}
R\big(\tau(W), Z\big)Y - R\big(\tau(Z), W\big)Y = {}& T(f)\big\{g_\theta(Y, W)Z - g_\theta(Y, Z)W \\
& + 2\Omega(Z, W)JY + \Omega(Y, W)JZ - \Omega(Y, Z)JW\big\} \\
& + (\nabla_T A)(Y, JW)Z - (\nabla_T A)(Y, JZ)W + g_\theta(Y, W)(\nabla_T \tau)JZ \\
& - g_\theta(Y, Z)(\nabla_T \tau)JW + (\nabla_T A)(Y, W)JZ - (\nabla_T A)(Y, Z)JW \\
& + \Omega(Y, W)(\nabla_T \tau)Z - \Omega(Y, Z)(\nabla_T \tau)W.
\end{aligned} \tag{68}$$

Let $Y, Z \in H(M)$ such that $\|Y\| = 1$ and $g_\theta(Y, Z) = \Omega(Y, Z) = 0$ and set $W = Y$ in (68). Together with the assumption $\tau = 0$ this yields $T(f) = 0$. Theorem 6 is proved.

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