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Research problems

The Research Problems section presents unsolved problems in discrete mathematics. In issues devoted to particular conferences, these typically are problems collected by the guest editors. In regular issues, the Research Problems generally consist of problems submitted on an individual basis.

Older problems are acceptable if they are not as widely known as they should be or if the exposition features a new partial result. Concise definitions and commentary (such as motivation and known partial results) should be provided to make the problems accessible and interesting to a broad cross-section of the readership. Problems are solicited from all readers. Ideally, they should be presented in the style below, occupy at most one journal page, and be sent to

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Comments and questions of a technical nature about a particular problem should be sent to the correspondent for that problem. Other comments and information about partial or full solutions should be sent to Professor West (for possible later updates on the status of published problems).

PROBLEM 412. Steiner extension of undirected graphs

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A *Steiner extension* of a finite undirected graph G is a Steiner triple system S such that the points of S are the vertices of G and such that each triple in S contains a unique edge of G. Note that if a graph with n vertices and m edges has a Steiner extension, then the Steiner extension is a Steiner triple system with n points and m triples, and so n is congruent to 1 or 3 modulo 6 and $m = n(n-1)/6$.

Conjecture 1. No finite graph with more than three vertices has a unique Steiner extension.

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Conjecture 2. Every finite graph with more than three vertices has an even number of Steiner extensions.

Comment. The stronger Conjecture 2 has been verified, by a computer search, for all graphs with seven vertices and all graphs with nine vertices.

Reference

V. Chvátal, Sylvester-Gallai theorem and metric betweenness, DIMACS Technical Report 2002-19, http://www.cs.rutgers.edu/~chvatal/sg.ps

PROBLEM 413. Fibonacci index of natural numbers

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A *Fibonacci-type sequence* is a sequence satisfying the Fibonacci recurrence $a_n = a_{n-1} + a_{n-2}$ for some initial values a_1 and a_2 that are natural numbers. Every natural number appears in some such sequence (it may be the initial term).

Question. Given a natural number m, let $g(m)$ be the maximum n such that m is the nth term in some Fibonacci-type sequence. What is the behavior of $g(n)$?

Comment. Note that the maximum index for the appearance of m occurs in a sequence with $a_1 \geqslant a_2$. Among all Fibonacci-type sequences, the nth term is smallest when $a_1 = a_2 = 1$, which is the Fibonacci sequence itself where $a_n = F_n$. Since F_n is approximately $\phi^n/\sqrt{5}$, where $\phi = (1+\sqrt{5})/2$, always $g(m) \leqslant \log_\phi(\sqrt{5}m)$, approximately. How much smaller than this can it be, and when?

The values for small m are listed below. It appears that the greatest deviations from the upper bound occur immediately after Fibonacci numbers. Also, $g(F_n) = n$ for $n \geqslant 2$. Furthermore, $g(F_n + F_{n-2}) = n$, using the sequence with $a_1 = 2$ and $a_2 = 1$. Is this the only value of m between F_n and F_{n+1} such that $g(m) = n$?

The data suggest further conjectures. The question has applications in cryptography.

m	1	2	3	4	5	6	7	8	9	10	11	12
$g(m)$	2	3	4	4	5	4	5	6	4	5	6	5

m	13	14	15	16	17	18	19	20	21	22	23	24
$g(m)$	7	5	5	6	6	7	6	6	8	6	6	6

PROBLEM 414. Permutations with many patterns

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The *pattern* formed by k positions in a permutation is the permutation of $\{1,\dots,k\}$ specifying the relative order of the elements in those positions. Recent problems and results about patterns in permutations are discussed in [1].

Let $f(n)$ denote the largest number of different patterns that can occur in a permutation of n letters. When $n=2$, the permutation (12) has one pattern of length 1 and one of length 2, as does the permutation (21), so $f(2)=2$. For n up to 7, the values of f are $1,2,4,8,15,28,55$, and examples of maximizing permutations of those lengths are (1), (12), (132), (2413), (25314), (253614), (2574163).

Evidently, a permutation of n letters cannot contain more than 2^n different patterns, one for each subset of its letters. Also it cannot contain more than $k!$ different patterns of length k, for each k. Hence we have the upper bound

$$f(n) \leqslant \sum_{k=1}^{n} \min\left(k!, \binom{n}{k}\right).$$

To find a lower bound, consider the permutation

$$p_n = (n,\ 1\ , n-1,\ 2,\ n-2,\ 3, \dots)$$

of n letters. We claim that p_n contains exactly F_{n+1} distinct patterns, where F_n is the nth Fibonacci number. Letting g_n be the number of distinct patterns in p_n, it suffices to show that $g_n = g_{n-1} + g_{n-2}$ for $n \geqslant 3$, with $g_1 = 1$ and $g_2 = 2$.

For every pattern in the permutation p_n, the first letter is either the largest letter of the pattern or the smallest letter of the pattern. Let S be the set of distinct patterns that occur in the last $n-1$ letters of p_n. Note that $|S| = g_{n-1}$. Prefix each such pattern with a new highest letter. This gives a set S' of g_{n-1} different patterns, each contained in p_n. Likewise, when T is the set of distinct patterns occurring among the last $n-2$ letters of p_n, we have $|T| = g_{n-2}$. Prefix each such pattern with a new lowest letter. This gives a set T' of g_{n-2} different patterns, each contained in p_n. The only pattern in both S' and T' is (1). The only pattern contained in p_n that is not in S' or T' is the empty pattern. Therefore $g_n = g_{n-1} + g_{n-2}$.

Question. Is $\lim f(n)^{1/n} < 2$?

Reference

[1] H. Wilf, The patterns of permutations, Discrete Math. 257 (2002) 575–583.

PROBLEM 415. Zero-sum Ramsey numbers

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Given a graph G and an integer k dividing the number of edges in G, the *zero-sum Ramsey number* $R(G,\mathbb{Z}_k)$ is the smallest n such that in every edge-coloring of K_n with integers modulo k, there is a copy of G on which the colors sum to a multiple of k. Let $n(G)$ and $e(G)$ denote the numbers of vertices and edges in G.

Of course, the parameter is not defined when k does not divide $e(G)$, since one can then assign color 1 to every edge and have no such copy. Otherwise, $R(G,\mathbb{Z}_k)$ is well-defined, because it is bounded by the ordinary k-color Ramsey number of G. The notion of zero-sum Ramsey number was first introduced in [1] for the special case $k=e(G)$.

Question 1. Is it true for every k that $R(G,\mathbb{Z}_k)$ is bounded by a polynomial in $n(G)$? In particular, do there exist constants a_k,b_k,c_k such that $R(G,\mathbb{Z}_k)\leqslant a_k n(G)^{b_k}+c_k$ whenever k divides $e(G)$?

Question 2. What is the value of $R(K_n,\mathbb{Z}_3)$ when $n\equiv 7 \bmod 9$?

Comments. The value of $R(G,\mathbb{Z}_2)$ is known exactly for every graph. The first proof appears in [2], with another in [3] and related material in [4,5]. In particular, $R(G,\mathbb{Z}_2)\leqslant n(G)+2$, which motivates Question 1.

Concerning Question 2, the value of $R(K_n,\mathbb{Z}_3)$ is already known for all other cases. If $n\equiv 2 \bmod 3$, then $3\nmid\binom{n}{2}$. If $n\equiv 0 \bmod 3$, then $R(K_n,\mathbb{Z}_3)=n+4$, except that $R(K_3,\mathbb{Z}_3)=11$. If $n\equiv 1 \bmod 3$, then $R(K_n,\mathbb{Z}_3)=n+3$ when $n\equiv 1 \bmod 9$ or $n\equiv 4 \bmod 9$. This leaves the case $n\equiv 7 \bmod 9$, where the value is known to be $n+3$ or $n+4$, and for $n=7$ it is exactly 11. In general, zero-sum Ramsey numbers are *not* monotone.

The state of the art for zero-sum Ramsey numbers is described in [6,7]. The questions posed here would be the next step in furthering knowledge in this area. For additional zero-sum type problems, see [8,9].

In particular, there are other ways to force zero-sum copies of desired graphs. The appealing conjecture below concerns trees. In a sense it is trivial for $k=1$, since every graph with minimum degree at least m contains every tree with m edges. Caro and Roditty [10] proved this conjecture for $k=2$.

Conjecture 3. If m is a multiple of k, T is a tree with m edges, and G is a graph with minimum degree at least $m+k-1$, then every mod-k-coloring of $E(G)$ contains a copy of T on which the colors sum to a multiple of k.

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PROBLEM 416. Weak pancyclicity of locally connected graphs

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Let G be a finite simple undirected graph, and let $g(G)$ and $c(G)$ denote the girth and the circumference of G (that is, the length of a shortest cycle in G and the length of a longest cycle in G), respectively. We say that G is *weakly pancyclic* if G contains cycles of every length ℓ such that $g(G) \leqslant \ell \leqslant c(G)$. A graph G is *locally connected* if for every vertex v in G, the subgraph induced by the neighborhood of v is a connected graph.

Conjecture. Every locally connected graph is weakly pancyclic.

Comments. The concept of locally connected graphs was introduced by Chartrand and Pippert [1]. Among other applications, it plays an important role in the closure concept for claw-free graphs [2]. More information about weakly pancyclic graphs appears in [3], for example.

The present conjecture is based on a result by Clark [4], who proved that every connected, locally connected graph is vertex pancyclic (having cycles of all lengths from 3 to $|V(G)|$ through every vertex. Without the claw-free assumption, it is easy to construct locally connected graphs that are non-Hamiltonian. Nevertheless, all known examples are weakly pancyclic, and indeed [4] proved the conjecture for claw-free graphs.

In a chordal graph, every block is locally connected, and for every cycle of length at least 4 there is a cycle with length one less that is obtained by skipping one vertex. Thus the conjecture holds for chordal graphs.

It is easy to show that the square of every graph is locally connected. (The *square* adds edges making vertices at distance 2 in the original graph adjacent.) Fleischner [5]

(Theorem 6) proved that the square of every graph is weakly pancyclic, thus verifying the conjecture for squares of graphs.

The lexicographic product of graphs is another way to obtain locally connected graphs (the lexicographic product $G[H]$ consists of disjoint copies of H corresponding to the vertices of G, plus edges from every vertex of H_u to every vertex of H_v when $uv \in E(G)$). Kaiser and Kriesell [6] proved that the lexicographic product $G[H]$ is weakly pancyclic provided that G is connected and H has at least one edge.

Also, Kriesell [7] verified the conjecture for graphs with maximum degree at most 4.

Finally, planar triangulations are locally connected. Balister [8] proved the conjecture for this class as follows. Let C be a cycle in a planar triangulation G. By induction on the number of faces inside, we prove that the interior (with boundary) contains cycles of all shorter lengths. If some face inside has two edges on C, then using the third edge yields a cycle C' with length one less and fewer faces inside. Otherwise, there is a face with one edge on C and the third vertex inside. Detouring from C to include this vertex forms a longer cycle C', but again it has fewer regions inside and the induction hypothesis applies.

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