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ScienceDirect
J. Math. Anal. Appl. 343 (2008) 743–751
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Invariant subspaces of positive strictly singular operators on Banach lattices ☆

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Received 12 June 2007

Available online 31 January 2008

Submitted by G. Corach

Abstract

It is shown that every positive strictly singular operator T on a Banach lattice satisfying certain conditions is AM-compact and has invariant subspaces. Moreover, every positive operator commuting with T has an invariant subspace. It is also proved that on such spaces the product of a disjointly strictly singular and a regular AM-compact operator is strictly singular. Finally, we prove that on these spaces the known invariant subspace results for compact-friendly operators can be extended to strictly singular-friendly operators.

Keywords: Strictly singular operator; Disjointly strictly singular operator; Positive operator; AM-compact operator; Dunford–Pettis operator; Banach lattice; Invariant subspace; Invariant ideal

Introduction

Read [14] presented an example of a strictly singular operator with no (closed non-zero proper) invariant subspaces. It remains an open question whether every positive strictly singular operator on a Banach lattice has an invariant subspace. The present paper contains several results in this direction.

Recall that a bounded operator T on a Banach space X is said to be *strictly singular* if its restriction to any infinite-dimensional subspace is not an isomorphism. We say that T is ℓ_2-*singular* if the preceding definition is true for every subspace isomorphic to ℓ_2. Furthermore, if X is a Banach lattice, we say that T is *disjointly strictly singular* if the preceding is true for every subspace spanned by a pairwise disjoint infinite sequence. Disjointly strictly singular operators were introduced in [9] as a lattice version of strictly singular operators. Unlike strictly singular operators, they do not form an operator ideal. Clearly, every strictly singular operator is ℓ_2-singular and disjointly

☆ The first author was partially supported by grants MTM2005-00082 and CS-PR27/05-14045. The second author was partially supported by grants PR27/05-14045, MTM2005-00082, and AP-2004-4841. The third author was supported by NSERC.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.01.067

strictly singular. An operator between Banach spaces is called *Dunford–Pettis* if it takes weakly null sequences to norm null sequences. An operator from a Banach lattice to a Banach space is *AM-compact* if it takes order intervals into relatively compact sets. If E is a Banach lattice and $T \in \mathcal{L}(E)$, we say that T is *regular* if it can be written as the difference of two positive operators. We write $\mathcal{L}(E)$ for the space of all (linear bounded) operators on E, and $\mathcal{L}_r(E)$ and $\mathcal{L}(E)_+$ for the collections of all regular or positive operators on E.

Throughout this paper, E will be a fixed order continuous Banach lattice with a weak order unit. We can assume by [11, Proposition 1.b.14] that there is a probability space (Ω, Σ, μ) such that E is an order ideal of $L_1(\mu)$ such that $L_\infty(\mu) \subseteq E$ and $\|x\|_1 \leqslant \|x\|_E \leqslant 2\|x\|_\infty$. We fix (Ω, Σ, μ) throughout the paper. We will also make use of the following fact due to Weis [17, Theorem 2.2]: every regular operator $T : E \to E$ can be extended to a bounded operator $\widetilde{T} : L_1(\mu) \to L_1(\mu)$.

Consider a sequence (f_n) in E which is bounded in $\|\cdot\|_\infty$ and which, viewed as a sequence in $L_1(\mu)$, is equivalent to the unit vector basis of ℓ_2. We say that E satisfies the *R-condition* if every such sequence has a subsequence which remains equivalent to the unit vector basis of ℓ_2 when viewed as a sequence in E. We show in Section 1 that every p-concave Banach lattice with $1 \leqslant p < \infty$ satisfies the R-condition. In particular, if E contains $L_p(\mu)$, then E satisfies the R-condition.

In Section 2 we establish certain connections between various special classes of operators on Banach lattices. We show that a regular operator T on E is AM-compact iff its extension to $L_1(\mu)$ is a Dunford–Pettis operator. We prove that the product of a disjointly strictly singular and a regular AM-compact operators is strictly singular. We also show that if E satisfies the R-condition, then every regular ℓ_2-singular operator is AM-compact. In Section 3 we use the results of Section 2 to show that certain strictly singular operators as well as many other operators related to them have invariant subspaces.

In [2] the authors introduced the concept of a *compact-friendly* operator, and proved the existence of invariant subspaces for these operators under certain additional assumptions. In Section 4 we define *strictly singular-friendly* operators in a similar fashion, and show that under the same assumptions plus the R-condition strictly singular-friendly operators have invariant subspaces.

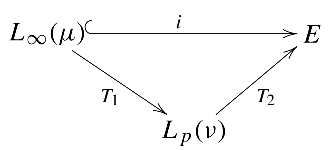
We will also use the following standard notations. We say that E contains a copy of ℓ_1 if there is a subspace Z of E such that Z is isomorphic to ℓ_1. If, in addition, Z is a sublattice of E, we say that Z is a lattice copy of ℓ_1 in E. It is well known that E contains a copy of ℓ_1 if and only if it contains a lattice copy of ℓ_1 if and only if E^* is not order continuous [3, pp. 190, 238]. We say that an operator $T \in \mathcal{L}(E)$ preserves a (lattice) copy of ℓ_1 if E contains a (lattice) copy of ℓ_1 such that the restriction of T to this subspace is an isomorphism. For more information on Banach lattices we refer the reader to [1,3,11,12].

1. *R*-condition

We will say that E satisfies the *R-condition* if every bounded sequence (f_n) in $L_\infty(\mu)$ which is equivalent in $L_1(\mu)$ to the unit basis of ℓ_2, has a subsequence which is equivalent in E to the unit basis of ℓ_2. In this section we show that many Banach lattices enjoy the R-condition.

Lemma 1.1. *Suppose that the inclusion $i : L_\infty(\mu) \hookrightarrow E$ factorizes through $L_p(\nu)$ for some probability measure ν and $1 \leqslant p < \infty$ with positive factors. Then E satisfies the R-condition.*

Proof. Let (f_n) be a bounded sequence in $L_\infty(\mu)$ which is equivalent in $L_1(\mu)$ to the unit vector basis of ℓ_2. By hypothesis, we have the following factorization:



Since (f_n) viewed as a sequence in $L_1(\mu)$ is equivalent to the unit vector basis of ℓ_2, it is weakly null in $L_1(\mu)$. Since (f_n) is order bounded in E, (f_n) is weakly null in E by Amemiya's Theorem [12, 2.4.8].

The sequence $(T_1 f_n)$ has a subsequence which converges weakly to some $g \in L_p(\nu)$; therefore $T_2 g = 0$. Consider the sequence $y_n = T_1 f_n - g$; it has a weakly null subsequence. It cannot be null in $L_p(\nu)$ because (f_n) is not null in E.

Therefore, by passing to a subsequence, we may assume that (y_n) is weakly null, seminormalized, and $(y_n) \subset [-y, y]$ for some y in $L_p(\nu)$. Since $L_p(\nu)$ has an unconditional basis, we can extract a subsequence (y_{n_k}) which is unconditional with constant K. For every $n \in \mathbb{N}$, let $r_n : [0,1] \to [-1,1]$ be the nth Rademacher function $r_n(t) = \operatorname{sign} \sin 2^n \pi t$. By [11, Theorem 1.d.6], there exists $C > 0$ such that

$$\left\| \sum_{k=1}^m a_k y_{n_k} \right\|_p \leqslant K \int_0^1 \left\| \sum_{k=1}^m r_k(s) a_k y_{n_k} \right\|_p ds \leqslant KC \left\| \left(\sum_{k=1}^m |a_k y_{n_k}|^2 \right)^{\frac{1}{2}} \right\|_p \leqslant KC \|y\|_p \left\| (a_i) \right\|_{\ell_2^m}$$

for all $(a_k)_{k=1}^m$. Then, for some $c > 0$ we have

$$\begin{aligned} c \left\| (a_i) \right\|_{\ell_2^m} &\leqslant \left\| \sum_{k=1}^m a_k f_{n_k} \right\|_1 \leqslant \left\| \sum_{k=1}^m a_k f_{n_k} \right\|_E = \left\| \sum_{k=1}^m a_k T_2 y_{n_k} \right\|_E \leqslant \|T_2\| \left\| \sum_{k=1}^m a_k y_{n_k} \right\|_p \\ &\leqslant KC \|T_2\| \|y\|_p \left\| (a_i) \right\|_{\ell_2^m}. \end{aligned}$$

Therefore (f_{n_k}) is equivalent in E to the unit vector basis of ℓ_2. □

We claim that if $L_p(\mu) \subseteq E$ for some $1 \leqslant p < \infty$, then E satisfies R-condition. This is a special case of a more general fact: *if E is p-concave for some $1 \leqslant p < \infty$, then E has R-condition.* This follows from Lemma 1.1 together with Krivine's factorization theorem [11, Theorem 1.d.11] since the inclusion map $i : L_\infty(\mu) \to E$ is p-convex (an inspection of the proof of [11, Theorem 1.d.11] reveals that in our setting the factors are positive).

The last statement can be extended as follows: we will show that if E has property (U_2), then it satisfies R-condition. Property (U_2) was introduced by Räbiger in [13]: a Banach lattice F has *property* (U_2) if for every seminormalized weakly null order bounded sequence (x_n) in F there is a subsequence (x_{n_i}) and a constant $C > 0$ such that

$$\left\| \sum_{i=1}^m a_i x_{n_i} \right\| \leqslant C \left\| (a_i) \right\|_{\ell_2^m}$$

for any coefficients $a_1, \ldots, a_m$. It was proved in [13] that every Banach lattice which is p-concave for some $1 \leqslant p < \infty$ has property (U_2). However, the following example shows that the converse is false.

Example. The Banach lattice $\ell_2(\ell_\infty^{2^n})$ has property (U_2) but it is not p-concave for any $1 \leqslant p < \infty$.

Proof. Let us write every element $x \in \ell_2(\ell_\infty^{2^n})$ as a sequence $(x_i)_{i=1}^\infty$, with $x_1 \in \ell_\infty^1$, $(x_2, x_3) \in \ell_\infty^2$, $(x_4, x_5, x_6, x_7) \in \ell_\infty^4$, and so on. Therefore,

$$\|x\| = \left(\sum_{n=0}^\infty \max_{2^n \leqslant i < 2^{n+1}} |x_i|^2 \right)^{\frac{1}{2}}.$$

For $x \in \ell_2(\ell_\infty^{2^n})$ we consider $\bar{x} = (\bar{x}_i)_{i=1}^\infty$, defined by $\bar{x}_i = \max\{|x_j|:\ 2^n \leqslant j < 2^{n+1}\}$ whenever $2^n \leqslant i < 2^{n+1}$. Clearly, $\|\bar{x}\| = \|x\|$, and $\bar{x}$ belongs to the closed linear span of (e^n), where

$$e_k^n = \begin{cases} 1 & \text{if } 2^n \leqslant k < 2^{n+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that (e^n) is equivalent to the unit vector basis of ℓ_2.

Let $(x^{(n)})$ be a seminormalized order bounded weakly null sequence in $\ell_2(\ell_\infty^{2^n})$. For each n, put $y^{(n)} = \overline{x^{(n)}}$. We can write $y^{(n)} = \sum_{i=1}^\infty \alpha_{n,i} e^i$. Since $(x^{(n)})$ is weakly null, both $(x^{(n)})$ and $(y^{(n)})$ converge to zero coordinate-wise, hence $\lim_n \alpha_{n,i} = 0$ for every i. It follows from [10, Proposition 1.a.12] that there is a subsequence $(y^{(n_k)})$ equivalent to the unit vector basis of ℓ_2. Therefore, we have:

$$\left\| \sum_{k=1}^{\infty} a_k x^{(n_k)} \right\| \leqslant \left\| \sum_{k=1}^{\infty} |a_k| y^{(n_k)} \right\| \leqslant C \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{\frac{1}{2}}$$

for some constant $C > 0$ independent of the sequence $(a_k)_{k=1}^{\infty}$. Therefore, $\ell_2(\ell_\infty^{2^n})$ has property (U_2). On the other hand, this space contains ℓ_∞^n's uniformly, so that it is not p-concave for any $1 \leqslant p < \infty$. □

Proposition 1.2. *If E has property (U_2), then it satisfies the R-condition.*

Proof. Let (x_n) be a bounded sequence in $L_\infty(\mu)$ equivalent in $L_1(\mu)$ to the unit vector basis of ℓ_2. Again, (x_n) is weakly null in E. By property (U_2), there exists a constant $C > 0$ such that, after passing to a subsequence, we have

$$\left\| \sum_{i=1}^{m} a_i x_{n_i} \right\|_E \leqslant C \left\| (a_i) \right\|_{\ell_2^m}$$

for any $(a_i)_{i=1}^m$. On the other hand, by our choice of (x_n) there exists another constant c such that

$$c \left\| (a_i) \right\|_{\ell_2^m} \leqslant \left\| \sum_{i=1}^{m} a_i x_{n_i} \right\|_1 \leqslant \left\| \sum_{i=1}^{m} a_i x_{n_i} \right\|_E$$

for any coefficients $a_1, \ldots, a_m$. Hence (x_{n_i}) is also equivalent to the unit vector basis of ℓ_2 in E. □

2. Strictly singular and AM-compact operators

We start by showing that a regular operator T on E is AM-compact iff its extension $\widetilde{T}$ to $L_1(\mu)$ is Dunford–Pettis. This result is related to Theorem 19.18 of [3] which asserts that a regular operator from E to $L_1(\mu)$ is Dunford–Pettis iff it maps order intervals onto norm compact sets. We will use the following observation of Uhl [5,16].

Theorem 2.1 *(Uhl). An operator $T: L_1(\mu) \to L_1(\mu)$ is Dunford–Pettis iff its restriction to $L_\infty(\mu)$ is compact as an operator from $L_\infty(\mu)$ to $L_1(\mu)$.*

Theorem 2.2. *Let T be a regular operator on E. Then T is AM-compact iff $\widetilde{T}: L_1(\mu) \to L_1(\mu)$ is Dunford–Pettis.*

Proof. Suppose that $\widetilde{T}$ is Dunford–Pettis. It suffices to show that if (x_n) is a sequence in $[0, x]$ for some $x \in E_+$, then (Tx_n) has a convergent subsequence. Without loss of generality we can take (x_n) normalized. Since order intervals in $L_1(\mu)$ are weakly compact, there exists a subsequence (x_{n_k}) which converges weakly to some g in $L_1(\mu)$. We then have $Tx_{n_k} = \widetilde{T}x_{n_k} \xrightarrow{\|\cdot\|_1} \widetilde{T}g$ because $\widetilde{T}$ is Dunford–Pettis. Since T is regular, then (Tx_n) is contained in $[-|T|x, |T|x]$. Also, $\widetilde{T}g \in [-|T|x, |T|x]$ since order intervals are closed. By Amemiya's Theorem [12, 2.4.8] it follows that $\widetilde{T}g \in E$ and $Tx_{n_k} \to \widetilde{T}g$ in E.

Conversely, suppose that T is AM-compact. Then T is AM-compact as an operator from $L_\infty(\mu)$ to $L_1(\mu)$, because $L_\infty(\mu)$ is an ideal in E and the inclusion $E \hookrightarrow L_1(\mu)$ is continuous. Since the unit ball in $L_\infty(\mu)$ is an order interval, it follows that this operator is in fact compact. Now Theorem 2.1 implies that $\widetilde{T}$ is Dunford–Pettis. □

Remark 2.3. It was shown in [16] that an operator $S: L_1(\mu) \to L_1(\mu)$ is Dunford–Pettis iff it is ℓ_2-strictly singular. Moreover, if S is not Dunford–Pettis, then one can find a sequence (f_n) bounded in $L_\infty(\mu)$ such that (f_n) viewed as a sequence in $L_1(\mu)$ is weakly null and equivalent to the unit vector basis of ℓ_2, and the restriction of S to the span of (f_n) in $L_1(\mu)$ is an isomorphism.

Proposition 2.4. *Suppose that E satisfies the R-condition and $T \in \mathcal{L}_r(E)$ is ℓ_2-strictly singular. Then T is AM-compact.*

Proof. In view of Theorem 2.2 it suffices to show that $\widetilde{T}: L_1(\mu) \to L_1(\mu)$ is Dunford–Pettis. Suppose it is not. Let (f_n) be as in Remark 2.3 for $S = \widetilde{T}$. Since E satisfies the R-condition, after passing to a subsequence we have the following chain of inequalities with appropriate constants:

$$\left\| T\left(\sum_{n=1}^{\infty}\alpha_n f_n\right)\right\|_E \geqslant \left\| T\left(\sum_{n=1}^{\infty}\alpha_n f_n\right)\right\|_1 \geqslant C_1\left\|\sum_{n=1}^{\infty}\alpha_n f_n\right\|_1 \geqslant C_2\left\|(\alpha_n)\right\|_2 \geqslant C_3\left\|\sum_{n=1}^{\infty}\alpha_n f_n\right\|_E$$

for any $\sum_{n=1}^{\infty}\alpha_n f_n$ in E. This contradicts T being ℓ_2-singular. □

Corollary 2.5. *Suppose that X is an arbitrary Banach lattice, $T: X \to X$ is strictly singular and factors with positive factors through E, and E satisfies the R-condition. Then T^3 is AM-compact.*

Proof. Suppose that we can factor $T = RS$ where $X \xrightarrow{S} E \xrightarrow{R} X$ and $S, R \geqslant 0$. Then $STR: E \to E$ is positive and strictly singular, hence AM-compact by Proposition 2.4. Since AM-compact operators form an algebraic ideal among regular operators, it follows from $T^3 = R(STR)S$ that T^3 is AM-compact □

Theorem 2.6. *Suppose that $S, T \in \mathcal{L}(E)$ such that S is disjointly strictly singular and T is regular and AM-compact. Then ST is strictly singular.*

The proof of this theorem will be based on the following two facts. The first one is an observation which follows easily from the results of [6].

Theorem 2.7. *Let T be an operator on an order continuous Banach lattice. If T preserves a copy of ℓ_1, then T preserves a lattice copy of ℓ_1.*

Proof. If T preserves a copy of ℓ_1, then there is a normalized sequence (x_n) such that (x_n) and (Tx_n) are both equivalent to the unit vector basis of ℓ_1. Therefore, (Tx_n) has no weakly Cauchy subsequence. Thus, T is not weakly sequentially precompact (see [6, Section 1]) and the required conclusion follows from [6, Theorem 1.1]. □

The second fact needed for the proof of Theorem 2.6 is related to the well-known Kadec–Pełczyński sets. Recall that, given $\varepsilon > 0$, the Kadec–Pełczyński set $M(\varepsilon)$ is defined as follows:

$$M(\varepsilon) = \{x \in E\colon\ \mu(\sigma(x,\varepsilon)) \geqslant \varepsilon\}$$

where $\sigma(x,\varepsilon) = \{t \in \Omega\colon\ |x(t)| \geqslant \varepsilon\|x\|_E\}$. It is known [11, Proposition 1.c.8] that $\|x\|_1 \geqslant \varepsilon^2\|x\|_E$ for all $x \in M(\varepsilon)$; hence the norms $\|\cdot\|_E$ and $\|\cdot\|_1$ are equivalent on every subspace of E contained in $M(\varepsilon)$ for some $\varepsilon > 0$. On the other hand, if a normalized sequence (x_n) in E is not contained in any $M(\varepsilon)$, then there is a subsequence (x_{n_k}) and a normalized disjoint (unconditional basic) sequence (y_k) in E equivalent to (x_{n_k}) with $\|x_{n_k} - y_k\|_E \to 0$.

Lemma 2.8. *Suppose that $T \in \mathcal{L}_r(E)$ is AM-compact and (x_n) is a normalized weakly null sequence in E such that (Tx_n) is not null. Then there is a subsequence (x_{n_k}) and a disjoint seminormalized basic sequence (y_k) in E equivalent to (Tx_{n_k}) and $\|Tx_{n_k} - y_k\|_E \to 0$.*

Proof. By Theorem 2.2, $\widetilde{T}$ is Dunford–Pettis. Clearly, (x_n) is still weakly null viewed as a sequence in $L_1(\mu)$, so that $\|Tx_n\|_1 \to 0$. It follows that (Tx_n) cannot be entirely contained in some $M(\varepsilon)$ as this would imply $\|Tx_n\|_E \to 0$. The conclusion follows from the preceding remark. □

Proof of Theorem 2.6. Let Y be an infinite-dimensional subspace of E such that ST is an isomorphism on Y. Suppose first that Y contains no isomorphic copy of ℓ_1. Applying Rosenthal's theorem [15] to any bounded sequence in Y with no convergent subsequences, we conclude that Y contains a normalized weakly null sequence (x_n). Since T is an isomorphism on Y, we may assume by Lemma 2.8 that (Tx_n) is equivalent to a disjoint seminormalized basic sequence (y_n) and $\|Tx_n - y_n\|_E \to 0$. Since S is disjointly strictly singular, we can choose a normalized block sequence (w_n) of (y_n) such that $\|Sw_n\|_E \to 0$. If (v_n) is the corresponding block sequence of (Tx_n) with the same coefficients, then (v_n) is seminormalized and $\|Sv_n\|_E \to 0$. This contradicts ST being an isomorphism on Y.

Now suppose that Y contains a copy of ℓ_1. It follows that S preserves a copy of ℓ_1. Theorem 2.7 yields that S preserves a lattice copy of ℓ_1. This contradicts S being disjointly strictly singular. □

Corollary 2.9. *If $T \in \mathcal{L}_r(E)$ is disjointly strictly singular and AM-compact, then T^2 is strictly singular.*

3. Invariant subspaces of positive strictly singular operators

In this section we apply the results of the preceding sections to the Invariant Subspace Problem. Invariant subspaces are always assumed to be non-zero and proper. A subspace is said to be *hyperinvariant* under an operator T if it is invariant under every operator commuting with T. Recall that T is said to be *quasinilpotent* if its spectrum is $\{0\}$ or, equivalently, if $\lim_n \sqrt[n]{\|T^n\|} = 0$. We use the following standard lemma.

Lemma 3.1. *Suppose that T is an operator on a Banach space. If T is not quasinilpotent and some power of T is strictly singular, then T has a finite-dimensional hyperinvariant subspace.*

Proof. Suppose that T is an operator on a Banach space X such that T is not quasinilpotent and T^n is strictly singular for some n. Clearly, T^n is not quasinilpotent.

Suppose first that X is a Banach space over $\mathbb{C}$. Then T^n has non-trivial eigenspaces by [1, Theorem 7.11]. Let Z be a non-trivial eigenspace of T^n. Since T^n is strictly singular, we have $\dim Z < \infty$. It is easy to see that Z is hyperinvariant under T.

Now suppose that X is a Banach space over $\mathbb{R}$. The complexification $T^n_{\mathbb{C}}$ of T^n is still strictly singular by [1, p. 177]. Again, let Z be a non-trivial eigenspace of $T^n_{\mathbb{C}}$ in $X_{\mathbb{C}}$. Then $\dim Z < \infty$ and Z is hyperinvariant under $T_{\mathbb{C}}$. Let $(x_1 + iy_1), \dots, (x_m + iy_m)$ be a basis of Z, put $M = \operatorname{span}\{x_1, \dots, x_m, y_1, \dots, y_m\}$ in X. Clearly, $0 < \dim M \leqslant 2m$. We claim that M is hyperinvariant under T. Indeed, suppose that S is an operator such that $ST = TS$. Then $S_{\mathbb{C}} T_{\mathbb{C}} = T_{\mathbb{C}} S_{\mathbb{C}}$. Since Z in hyperinvariant under $T_{\mathbb{C}}$, for every $k \leqslant m$ we have $S_{\mathbb{C}}(x_k + iy_k) \in Z$, so that Sx_k and Sy_k are both in M. Hence, $S(M) \subseteq M$. □

We make use of the following statement, which is a special case of Theorem 10.26 of [1].

Theorem 3.2. *(See [1].) Every quasinilpotent AM-compact positive operator on a Banach lattice has an invariant subspace.*

Combining Theorem 3.2 with Lemma 3.1 we obtain the following result.

Proposition 3.3. *Suppose that T is a positive AM-compact operator on a Banach lattice. If T^n is strictly singular for some n, then T has an invariant subspace.*

Together with Theorem 2.4, this yields the following.

Corollary 3.4. *If E satisfies the R-condition and $T \in \mathcal{L}(E)_+$ is strictly singular, then T is AM-compact and has an invariant subspace.*

Corollary 3.5. *Suppose that E satisfies the R-condition and $S, T \in \mathcal{L}(E)$ are such that $0 \leqslant S \leqslant T$. If T is strictly singular, then S has an invariant subspace.*

Proof. It was shown in [7] that every operator on a Banach lattice with order continuous norm which is dominated by a strictly singular operator has strictly singular square; hence S^2 is strictly singular. Theorem 2.4 yields that T is AM-compact, hence $0 \leqslant S \leqslant T$ implies that S is AM-compact. The conclusion now follows from Proposition 3.3. □

Proposition 3.6. *Every positive disjointly strictly singular AM-compact operator on E has an invariant subspace.*

Proof. Follows from Corollary 2.9 and Proposition 3.3. □

Propositions 2.4 and 3.6 immediately yield the following result.

Theorem 3.7. *If E satisfies the R-condition and $T \in \mathcal{L}(E)_+$ is ℓ_2-singular and disjointly strictly singular, then T is AM-compact and has an invariant subspace.*

Recall that if T is a positive operator on a Banach lattice, then its left and right semi-commutants are defined as follows:

$$\langle T] = \{S \geqslant 0:\ ST \leqslant TS\} \quad \text{and} \quad [T\rangle = \{S \geqslant 0:\ ST \geqslant TS\}.$$

Theorem 3.8. *(See [4,8].) Suppose that T is a positive quasinilpotent operator on a Banach lattice X. Suppose that there exists a closed ball $B(x_0, r)$ in X centered at some $x_0 \geqslant 0$, of positive radius $r < \|x_0\|$ such that for every sequence (x_n) in $B(x_0, r) \cap [0, x_0]$ there exists a subsequence (x_{n_i}) and a sequence of operators (K_i) such that $0 \leqslant K_i \leqslant T$ for each i and $(K_i x_{n_i})$ converges to a non-zero vector. Then $\langle T]$ has a (common) invariant closed order ideal. In particular, T has an invariant subspace.*

Theorem 3.9. *Suppose that E satisfies R-condition and T is a positive quasinilpotent strictly singular operator on E. Then $\langle T]$ has an invariant closed ideal.*

Proof. Choose $x_0 \in X_+$ and $r > 0$ so that $B(x_0, r) \cap \ker T = \emptyset$. Suppose that (x_n) is a sequence in $B(x_0, r) \cap [0, x_0]$. We claim that there is a subsequence (x_{n_i}) such that (Tx_{n_i}) converges in norm to a non-zero vector; then the result will follow from Theorem 3.8. We may assume without loss of generality that (x_n) has no norm convergent subsequences. Since order intervals are weakly compact in E, we may assume by passing to a subsequence that (x_n) converges weakly to some x. Since $B(x_0, r)$ is weakly closed and $B(x_0, r) \cap \ker T = \emptyset$, we have $x \neq 0$ and $Tx \neq 0$. Notice that $\|Tx_n - Tx\|_1 \to 0$ since the extension $\widetilde{T} : L_1(\mu) \to L_1(\mu)$ is Dunford–Pettis by Theorem 2.2 and Corollary 3.4. Again, since the sequence (Tx_n) is order bounded (T is positive) we obtain $\|Tx_n - Tx\|_E \to 0$ by Amemiya's Theorem. □

To produce a similar result for $[T\rangle$ we use the following version of a theorem of Drnovšek [1, Theorem 10.50]. We start by recalling a few definitions. Let X be a Banach lattice. An operator T on X is *locally quasinilpotent* at a point x if $\lim_n \sqrt[n]{\|T^n x\|} = 0$. A point w in X is called *quasi-interior* if E_w is norm dense in X, where

$$E_w = \{x \in X:\ |x| \leqslant \lambda |w| \text{ for some } \lambda > 0\},$$

the *principal ideal* generated by w. Suppose that S and T are two operators on X, we say that T *dominates* S if $|Sx| \leqslant T|x|$ for every $x \in X$. Please see Section 10.4 of [1] for the other terminology used in the proof.

Theorem 3.10. *Suppose that T is a positive operator on a Banach lattice X with a quasi-interior point w such that*

(i) *T is locally quasinilpotent at some $x_0 > 0$, and*
(ii) *there is $S \in [T\rangle$ such that S dominates a non-zero AM-compact operator K.*

Then $[T\rangle$ has an invariant closed ideal.

Proof. Since the null ideal N_T of T is $[T\rangle$-invariant, we may assume that $N_T = \{0\}$. Let $z \in X$ such that $Kz \neq 0$. We may assume that $|Kz| \leqslant w$ as otherwise we can replace w with $w \vee |Kz|$. By Lemma 4.16(1) of [1] there exists an operator M dominated by the identity operator such that $MKz > 0$. Put $K_1 = MK$. It follows from $N_T = 0$ that $TK_1 z \neq 0$, hence $TK_1 \neq 0$. It is easy to see that TK_1 is AM-compact and is dominated by TS.

Let $\mathcal{J}$ be the semigroup ideal in $[T\rangle$ generated by TS, i.e., $\mathcal{J} = \{ATSB:\ A, B \in [T\rangle\}$. It can be verified directly that $\mathcal{J}$ is finitely quasinilpotent at x_0. Since $TS \in \mathcal{J}$ and TS dominates a non-zero AM-compact operator, $\mathcal{J}$ has an invariant closed ideal by Theorem 10.44 of [1]. Now Theorem 10.49 of [1] yields that $[T\rangle$ has an invariant closed ideal. □

Corollary 3.11. *Suppose that E has R-condition and T is a positive quasinilpotent strictly singular operator on E. Then $[T\rangle$ has an invariant closed ideal.*

Proof. By Corollay 3.4, T is AM-compact. Now apply Theorem 3.10. □

Corollary 3.12. *Suppose that E has R-condition and T is a positive strictly singular operator on E. Then every positive operator commuting with T has an invariant subspace.*

Proof. If T is quasinilpotent, then the conclusion follows from Theorem 3.9 or Corollary 3.11. If T is not quasinilpotent, then the result follows from Lemma 3.1. □

Proposition 3.13. *Suppose that X is an arbitrary Banach lattice, $T : X \to X$ is strictly singular and factors with positive factors through E, and E satisfies the R-condition. If T is locally quasinilpotent at a positive vector, then $[T\rangle$ has an invariant closed ideal.*

Proof. Corollary 2.5 yields that T^3 is AM-compact. The result now follows from Theorem 3.10. □

4. Invariant subspaces of SS-friendly operators

It is well known that compact operators enjoy good properties concerning the Invariant Subspace Problem. The compactness properties were relaxed in [2], where the authors introduced the class of compact-friendly operators and showed that these operators also have invariant subspaces. We present here the analogous concept for strictly singular operators.

We call an operator $B \in \mathcal{L}(E)_+$ *strictly singular-friendly* if there is a positive operator that commutes with B and dominates a non-zero operator which is dominated by a strictly singular positive operator. In particular, every operator dominating a positive strictly singular operator is strictly singular-friendly.

Theorem 4.1. *Suppose that E satisfies the R-condition. If $B \in \mathcal{L}(E)_+$ is a non-zero strictly singular-friendly operator which is locally quasinilpotent at some $x_0 > 0$, then B has a non-trivial closed invariant ideal. Moreover, if (T_n) is a sequence in $[B\rangle$, then there exist a non-trivial closed ideal that is invariant under B and each T_n.*

Proof. Our argument is similar to the one in [2]. We can suppose without loss of generality that $\|B\| < 1$. Pick small enough scalars $\alpha_n > 0$ such that the positive operator $T = \sum_{n=1}^{\infty} \alpha_n T_n$ exists and $\|B + T\| < 1$. It is clear that $T \in [B\rangle$, and $(B + T)^n \in [B\rangle$ for every n, so that the positive operator $A = \sum_{n=0}^{\infty} (B + T)^n$ also belongs to $[B\rangle$.

For each $x > 0$, let J_x be the principal ideal generated by Ax, that is,

$$J_x = \bigl\{y \in E\colon\ |y| \leqslant \lambda Ax \text{ for some } \lambda > 0\bigr\}.$$

Since $x \leqslant Ax$, we have that $x \in J_x$ so this is a non-zero ideal.

Note that J_x is $(B + T)$-invariant. Indeed, if $y \in J_x$, then $|y| \leqslant \lambda Ax$ for some $\lambda > 0$ so

$$\bigl|(B+T)y\bigr| \leqslant (B+T)|y| \leqslant \lambda(B+T)\sum_{n=0}^{\infty}(B+T)^n x = \lambda\sum_{n=1}^{\infty}(B+T)^n x \leqslant \lambda Ax.$$

Clearly J_x is also invariant under B and T, since $0 \leqslant B, T \leqslant B + T$, so it is also T_n-invariant for each n.

Therefore, for our purposes, it suffices to prove that there exists a positive $x \in E$ such that the ideal J_x is not norm dense in E. Suppose the contrary, that is, Ax is a quasi-interior point in E for each $x > 0$. By assumption, there exist operators R, S, and C in $\mathcal{L}(E)$ such that R and S are positive, S is strictly singular, $C \neq 0$, $RB = BR$, and C is dominated by both R and S.

Since $C \neq 0$, there exists some $x_1 > 0$ such that $Cx_1 \neq 0$. Then $A|Cx_1|$ is a quasi-interior point satisfying $A|Cx_1| \geqslant |Cx_1|$. By [1, Lemma 4.16] there exists an operator $M_1 \in \mathcal{L}(E)$ dominated by the identity operator such that $x_2 = M_1 C x_1 > 0$. Let $U_1 = M_1 C$. Note that U_1 is dominated by S and by R.

Now we have $\overline{J_{x_2}} = E$. Therefore, since $C \neq 0$, there exists $0 < y < Ax_2$ such that $Cy \neq 0$. Because $A|Cy|$ is a quasi-interior point and $|Cy| \leqslant A|Cy|$, then, as before, there exists $M_2 \in \mathcal{L}(E)$ dominated by the identity operator such that $x_3 = M_2 C y > 0$. Since $|y| \leqslant Ax_2$ and Ax_2 is a quasi-interior point, it follows that there is an operator $M \in \mathcal{L}(E)$ dominated by the identity such that $MAx_2 = y$. So $x_3 = M_2 C y = M_2 C M A x_2$. And the operator $U_2 = M_2 C M A$ is dominated by SA and by RA.

Consider the operator U_2U_1. From $U_2U_1x_1 = x_3 > 0$, we see that U_2U_1 is a non-zero operator. Since both U_1 and U_2 are dominated by strictly singular positive operators, U_2U_1 is strictly singular by [7]. Moreover

$$|U_2U_1x| = |M_2CMAM_1Cx| \leqslant RAR|x|$$

for each $x \in E$.

Let $V = RAR$. Since A and R belong to $[B\rangle$, then V also belongs to $[B\rangle$. Observe that V dominates U_2U_1 which is strictly singular, therefore AM-compact. The result now follows from Theorem 3.10. □

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