An abstract form of a theorem of Helson and applications to sets of synthesis and sets of uniqueness✩

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Abstract

Let $E$ be a compact perfect subset of the real line $\mathbb{R}$ such that the restriction of the Fourier transform $a \mapsto \hat{a}|_E$ from $L^1(\mathbb{R})$ into $C(E)$ is onto. Helson proved that then, for $\mu \in M(E)$, $\lim_{|y| \to \infty} |\hat{\mu}(y)| = 0$ is possible only if $\mu = 0$. In this paper we present an abstract version of this theorem of Helson and provide some applications of it to the study of sets of spectral synthesis and sets of uniqueness.

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0. Introduction

Let $G$ be an infinite locally compact commutative group equipped with its Haar measure, $L^1(G)$ its group algebra and $M(G)$ its measure algebra [42]. We shall denote by $\widehat{G}$ the dual group of $G$. The group $G$ being infinite, the Fourier transform $\phi : L^1(G) \to C_0(\widehat{G})$ is never onto but there are closed subsets $E$ of $\widehat{G}$ for which the restriction $\phi_E : L^1(G) \to C_0(E)$ of the Fourier transform to the set $E$, $\phi_E(a) = \hat{a}|_E$, is onto. Such a set, because of Helson’s Theorem [14] cited in the abstract, is said to be a Helson set. The kernel of the homomorphism $\phi_E$ is the closed ideal

$$k(E) = \{ a \in L^1(G) : \hat{a} = 0 \text{ on } E \}$$

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so that, for a Helson set $E$, the Banach algebras $L^1(G)/k(E)$ and $C_0(E)$ are isomorphic. Rudin in [42, Theorem 5.6.7], extending Helson’s Theorem to all locally compact commutative groups, proved that, for a Helson set $E \subseteq \widehat{G}$, the intersection $k(E)\perp \cap C_0(G)$ is trivial. That is, $k(E)\perp \cap C_0(G) = \{0\}$. Here $k(E)\perp$ is the annihilator of the ideal $k(E)$ in the dual space $L^\infty(G)$ of $L^1(G)$. The reader can find a shorter proof of this theorem in Doss’ paper [8] and another proof in Hewitt and Ross [18, Section 41.18]. As a search in MathSciNet shows, Helson’s Theorem in the year following its publication has attracted a great deal of attention and a considerable number of papers related in one way or other to this theorem of Helson has appeared. As far as we were able to check, almost all these papers study one of the following problems: Constructing Helson sets [21,24,25,29,36,48], comparing Helson sets with other thin sets [27,28,33,41] and we were able to check, almost all these papers study one of the following problems: Constructing Helson sets [21,24,25,29,36,48], comparing Helson sets with other thin sets [27,28,33,41] and studying the stability properties of Helson sets [16,44,49,50]. These references are by no means exhaustive. We do not know any paper whose main theme is the following question.

Question: What makes that, for a Helson set $E \subseteq \widehat{G}$, the intersection $k(E)\perp \cap C_0(G)$ is trivial?

This is this question around which this paper is centred. It is clear that this question is closely related to the problem of characterizing the sets of uniqueness for the trigonometric series [20,26,51]. We prove an abstract and fairly general result (Theorem 2.4) which, when specialized to the group algebra $L^1(G)$ of a noncompact locally compact $\sigma$-compact commutative group $G$, says that the only fact that makes that the intersection $k(E)\perp \cap C_0(G)$ is trivial is the fact that the space $k(E)\perp$ (which is isomorphic to $M(E) = C_0(E)^*$) is weakly sequentially complete. Actually, if $X$ is any weakly sequentially complete norm closed $L^1(G)$-submodule of $L^\infty(G)$ (i.e. $a * f$ is in $X$ for each $a \in L^1(G)$ and $f \in X$) then $X \cap C_0(G) = \{0\}$. Our result applies, beside the group algebra $L^1(G)$, to the Fourier algebra $A(G)$ of a nondiscrete locally compact amenable first countable group $G$ [13], to the Herz–Figa–Talamanca algebra $A_p(G)$ of a nondiscrete locally compact amenable second countable group $G$ [17], to the Beurling algebra $L^1(G, \omega)$ of a noncompact locally compact $\sigma$-compact commutative group $G$ [4,40] and to several other Banach algebras. The paper also contains a certain number of applications of the main result to the study of sets of synthesis and sets of uniqueness in the setting of the group algebra $L^1(G)$ of a locally compact $\sigma$-compact commutative group $G$.

In Section 1 we have gathered the preliminary results needed in the subsequent sections. Section 2 contains the main result of the paper (Theorem 2.4) and some of its corollaries. In Section 3, as applications of our main result, we present a series of results related to the spectrum $\sigma(f)$ of the functions $f \in L^\infty(G)$ and to the question of “when is a given subset $E$ of $\widehat{G}$ a set of spectral synthesis”. Section 4 is devoted to the problem of sets of uniqueness. The main result of that section (Theorem 4.7) says that a closed subset $E$ of $\widehat{G}$ is a set of uniqueness if, for each $f \in \mathcal{L}^\infty(G)$ whose spectrum is contained in the set $E$, the space $Z_f = \{a * f: a \in L^1(G)\}$ is weakly sequentially complete.

Our proofs are essentially functional analytic. The main ingredients of the proofs are: a) The notion of narrow spectrum introduced by Beurling for the group algebra $L^1(\mathbb{R})$ [1] and extended by Domar [7] and Lindahl [32] to abstract Banach algebras; and b) a lemma (proved below) saying that, for any $\gamma \in \widehat{G}$, there is an element $e_\gamma$ in $L^1(G)^{**} \setminus \{0\}$ such that, for $a \in L^1(G)$, $ae_\gamma = \widehat{a}(\gamma)e_\gamma$. Here the product $ae_\gamma$ denotes the Arens product (defined in the next section) of the elements $a$ and $e_\gamma$ in the algebra $L^1(G)^{**}$.
1. Preliminaries and notation

Our notation and terminology are standard. Concerning harmonic analysis, they are essentially those of Rudin’s book [42]; and, concerning Banach algebras, they are those of books [3] and [23]. To be fixed, for any Banach space \( X \), we always consider \( X \) as naturally embedded into its second dual \( X^{**} \). The natural duality between \( X \) and \( X^* \) is denoted as \( \langle f, x \rangle \) (and also as \( \langle x, f \rangle \)). If we need to be more precise, instead of \( \langle x, f \rangle \), we write \( \langle x, f \rangle_{X,X^*} \).

Let now \( A \) be a commutative Banach algebra. We denote by \( \Delta(A) \) the Gelfand spectrum of \( A \) and consider each element of \( \Delta(A) \) as a multiplicative functional on \( A \). One can make the second dual \( A^{**} \) of \( A \) into a Banach algebra by equipping it with one of the two Arens multiplications. We shall equip \( A^{**} \) with the first Arens multiplication and use only this multiplication. This multiplication is defined in three steps as follows: For \( a, b \) in \( A \), \( f \) in \( A^* \) and \( n, m \) in \( A^{**} \), the functionals \( b.f, n.f \) and the product \( nm \) are defined by the equalities

\[
\langle a, b.f \rangle = \langle ab, f \rangle, \\
\langle a, n.f \rangle = \langle a, fn \rangle, \\
\langle mn, f \rangle = \langle m, n.f \rangle.
\]

Equipped with this multiplication, \( A^{**} \) is a Banach algebra and \( A \) is a subalgebra of it. In general \( A^{**} \) is not commutative but, \( A \) being commutative, for \( a \in A \) and \( m \in A^{**} \), \( am = ma \). This multiplication is in general not separately weak* to weak* continuous on \( A^{**} \) but, for \( n \in A^{**} \) fixed, the mapping \( m \mapsto mn \) is continuous in the weak* topology of \( A^{**} \). If \( (e_\alpha)_{\alpha \in I} \) is a BAI (= bounded approximate identity) in \( A \) then each weak* cluster point \( e \) of this net in \( A^{**} \) is a “right identity” in \( A^{**} \). That is, for each \( m \in A^{**}, me = m \). In general we do not have \( em = m \). Conversely any right identity in \( A^{**} \) is a cluster point in \( (A^{**}, \text{weak*}) \) of some BAI of \( A \). We remark that, as one can see easily from the definition of the Arens multiplication, for \( m, n \in A^{**} \) and \( \gamma \in \Delta(A) \), we also have \( \langle mn, \gamma \rangle = \langle n, \gamma \rangle \langle m, \gamma \rangle \). For these results we refer the reader to the book [3, Section 2.9].

For a locally compact commutative group \( G \) equipped with its Haar measure (denoted \( dt \)), let \( A = L^1(G) \) be the group algebra of \( G \). The multiplication on \( L^1(G) \) is of course the convolution. We define the duality between \( L^1(G) \) and \( L^\infty(G) \) as \( \langle a, f \rangle = \int_G a(-t) f(t) dt \). On \( L^1(G)^{**} \) we put the first Arens multiplication as defined above. For \( f \in L^\infty(G) \) and \( a \in L^1(G) \), as one can easily check, \( a.f \) is just \( a * f \).

To illustrate the abstract results and be able to give examples, we need some concrete examples of Banach algebras. A large class of Banach algebras that contain the group algebra \( L^1(G) \) of a locally compact commutative group \( G \) and also the Fourier algebra \( A(G) \) of an arbitrary locally compact group \( G \) defined by Eymard [13] as special cases is the Herz–Figa–Talamanca algebra \( A_p(G) \). To introduce this algebra, let \( G \) be an arbitrary locally compact group equipped with its left Haar measure and \( 1 < p < \infty \). The space \( A_p(G) \) is the space of the continuous functions of the form

\[
a = \sum_{n=1}^{\infty} v_n \otimes u_n^\vee,
\]

where \( v_n \in L^q(G), u_n \in L^p(G) \ (\frac{1}{p} + \frac{1}{q} = 1), \ u_n^\vee(t) = u_n(t^{-1}) \) and \( \sum_{n=1}^{\infty} \|v_n\|_q \|u_n\|_p < \infty \). The norm of \( a \) is defined as
where the infimum is taking on all the representation of $a$ as above. The space $A_p(G)$, equipped with the above norm and the pointwise multiplication, is a commutative, semisimple, regular Tauberian Banach algebra and $A_p(G) \subseteq C_0(G)$ [17]. The Gelfand spectrum of $A_p(G)$ (via Dirac measures) is $G$. The Banach algebra $A_p(G)$ has a BAI iff the group $G$ is amenable. The dual space of $A_p(G)$, which is a subalgebra of the operator algebra $B(L^p(G))$, is denoted as $PM_p(G)$. This is the space of the $p$-pseudomeasures on $G$. For $a \in L^1(G)$, let $\rho(a) : L^p(G) \rightarrow L^p(G)$ be the convolution operator defined by $\rho(a)(b) = a * b$. The norm closure of the space $\{\rho(a): a \in L^1(G)\}$ in $B(L^p(G))$ is denoted as $PF_p(G)$. This is the space of the $p$-pseudofunctions on $G$. The pointwise multiplier algebra of the algebra $A_p(G)$ is denoted as $B_p(G)$. When $G$ is amenable $PF_p(G) \subseteq PM_p(G)$ and $B_p(G) = PF_p(G)^*$. For ample information on these algebras we refer the reader to the paper [17] and the book [38].

For $p = 2$, $A_p(G) = A(G)$, the Fourier algebra of $G$; $B_p(G) = B(G)$, the Fourier–Stieltjes algebra of $G$; $PM_p(G) = VN(G)$, the von Neumann algebra of $G$ and $PF_p(G) = C^*(G)$, the group $C^*$-algebra of $G$ [13]. Finally, when $G$ is commutative, via Fourier transform, $A(G) = L^1(\hat{G})$, $B(G) = M(\hat{G})$, $VN(G) = L^\infty(\hat{G})$ and $C^*(G) = C_0(\hat{G})$ (see [13] or [38]).

We shall use these algebras only to illustrate some of the results presented in the paper. The other notions and notation will be introduced as needed.

2. An abstract form of Helson’s Theorem

In this section $A$ will be an arbitrary commutative Banach algebra. Our aim is to present an abstract version of Helson’s Theorem mentioned in the introduction, which has appeared in [14]. Our main ingredient is the fact that, when $A$ is semisimple and regular, for any $f \in A^*$, the narrow spectrum of $f$ and its Beurling (or weak*) spectrum are the same. We start by recalling these notions.

Let $f \in A^*$, $f \neq 0$, be a given functional. The following subsets $\sigma_n(f)$ and $\sigma(f)$ of $\Delta(A)$

$$\sigma_n(f) = \{a.f: a \in A\} w^* \cap \Delta(A) \quad \text{and} \quad \sigma(f) = \{a.f: a \in A\} \cap \Delta(A)$$

are called, respectively, the weak*-spectrum and the narrow spectrum of $f$. Here $A_1$ denotes the closed unit ball of $A$. The reader can find the proof of the following theorem in Lindahl’s paper [32, Theorem 4].

**Theorem 2.0.** Suppose that $A$ is semisimple and regular. Then $\sigma_n(f) = \sigma(f)$ for each $f$ in $A^*$.

Let $A_c = \{a \in A: \text{the support of } \hat{a}, \text{Supp}(\hat{a}), \text{is compact}\}$. We recall that the algebra $A$ is said to be “Tauberian” if the space $A_c$ is dense in $A$. This condition, when the algebra $A$ is semisimple and regular, guarantees that the set $\sigma_n(f)$ is not empty whenever $f \neq 0$.

Our main ingredient in this section is the following version of the above theorem. We recall that the algebra $A$ is said to be weakly compactly generated if it is generated by a weakly compact set. If this is the case (e.g. $A$ is separable) then the closed unit ball $A^*_1$ of $A^*$ under the weak* topology $\sigma(A^*, A)$ is sequentially compact [6, p. 228]. We shall need such a condition since we shall work with the weakly sequentially complete subspaces of $A^*$. 
**Theorem 2.1.** Suppose that the algebra $A$ is semisimple, regular, Tauberian and weakly compactly generated. Let $f \in A^*$ and $\gamma \in \Delta(A)$ be two functionals. Then $\gamma \in \sigma(f)$ iff there exists a sequence $(a_n)_{n \geq 0}$ in $A$ such that $\|a_n.f\| \leq 1$ for all $n \geq 0$ and $a_n.f \to \gamma$ in the weak* topology $\sigma(A^*, A)$ of $A^*$.

An essential question for us is the question when the sequence $(a_n.f)_{n \geq 0}$ in the preceding theorem is weakly Cauchy? (i.e., When does, for each $m \in A^{**}$, the sequence $(\langle m, a_n.f \rangle)_{n \geq 0}$ converge?) To explain the motivation behind the next definition, let for a moment $A$ be the group algebra $L^1(G)$ of a locally compact commutative group $G$. Let $(f_n)_{n \geq 0}$ be a sequence in $C_0(G)$ that converges in the weak* topology of the space $L^\infty(G)$ to some element $f$ of $L^\infty(G)$. Then, for any $a \in L^1(G)$, the sequence $(a * f_n)_{n \geq 0}$ is weakly Cauchy. Indeed, this sequence, which lies in the space $C_0(G)$, by the uniform boundedness principle, is uniformly bounded and it converges pointwise on $G$. To see this last point it is enough to observe that, for $t \in G$, $(a * f_n)(t) = \langle a_t, f_n \rangle \to \langle a_t, f \rangle$, where $a_t$ is the translate of $a$ by $t$. The conclusion now follows from the Lebesgue Dominated Convergence Theorem.

**Definition 2.2.** Let $Y$ be a norm closed subspace of $A^*$. We shall say that the subspace $Y$ has the wCp (= weak Cauchy property) if for any sequence $(f_n)_{n \geq 0}$ in $Y$ that converges in the weak* topology of $A^*$ to some $f \in A^*$, the sequence $(a.f_n)_{n \geq 0}$ is weakly Cauchy for each $a \in A$.

Such a subspace $Y$ of $A^*$ will play the role that is played by the space $C_0(G)$ in Helson’s Theorem. Based on the Lebesgue Dominated Convergence Theorem, we have just seen that the subspace $C_0(G)$ of $L^\infty(G)$ has the wCp. There is another more fundamental reason for this that allows itself to be generalized: The dual space of the space $C_0(G)$ is the multiplier algebra of the Banach algebra $L^1(G)$ [31]. It is this fact that is behind the examples given below.

A norm closed subspace $X$ of $A^*$ will be said “invariant” if $a.f$ is in $X$ for each $f \in X$ and $a \in A$. That is, $X$ is an $A$-module for the action $(a, f) \mapsto a.f$. This canonical module action is the only module action that we shall use on the subspaces of $A^*$, whatever the algebra $A$ is. We remark that, for any closed ideal $I$ of $A$, the subspace $X = I^\perp$ of $A^*$, the annihilator of $I$ in $A^*$, is invariant in this sense.

In the following examples, the basic pattern is this: We have a closed invariant subspace $Y$ of $A^*$ whose dual $Y^*$ identifies “naturally” with the multiplier algebra $M(A)$ of $A$. Here the term “naturally” means this: For $f \in Y$, $a \in A$ and $T \in M(A)$, we have: $\langle T, a.f \rangle_{M(A), Y} = \langle T(a), f \rangle_{A, A^*}$.

**Examples 2.3.** a) Let $A = A(G)$ be the Fourier algebra of a locally compact amenable group $G$ [13]. As recalled in the preceding section, the multiplier algebra of $A(G)$ is the Fourier–Stieltjes algebra $B(G)$ of $G$ and the dual space of $A(G)$ is the von Neumann algebra $VN(G)$ of $G$. Let $Y = C^*(G)$ be the group $C^*$-algebra of $G$. The dual space of $C^*(G)$ is the algebra $B(G)$ and, since $G$ is amenable, $C^*(G)$ is a closed subspace of $VN(G)$. Let now $(f_n)_{n \geq 0}$ be a sequence in $C^*(G)$ that converges in the weak* topology of the space $VN(G)$ to some element $f$ of the space $VN(G)$ and $a \in A(G)$ a given element. For any $u \in B(G)$, the product $ua$ is in $A(G)$ since $A(G)$ is an ideal in $B(G)$. Moreover, for $g \in C^*(G)$, the functional $a.g$ is in $C^*(G)$ and

$$\langle u, a.g \rangle_{B(G), C^*(G)} = \langle ua, g \rangle_{A(G), VN(G)}.$$
These facts combined with the Hahn Banach Theorem imply that the sequence \((a_n f_n)_{n \geq 0}\) is a weakly Cauchy sequence in the space \(VN(G)\). Hence the invariant subspace \(Y = C^*(G)\) of \(VN(G)\) has the wCp.

b) Let \(A = A_p(G)\) (\(1 < p < \infty\)) be the Herz–Figa–Talamanca algebra of a locally compact amenable group \(G\) [17]. The multiplier algebra of \(A_p(G)\) is the space \(B_p(G)\) of \(p\)-Fourier–Stieltjes algebra of \(G\). The dual space of \(A_p(G)\) is the space \(PM_p(G)\) on \(G\). The multiplier algebra \(B_p(G)\) of the algebra \(A_p(G)\) is the dual space of the space \(PF_p(G)\) of the \(p\)-pseudofunctions on \(G\), which is an invariant subspace of \(PM_p(G)\). Exactly for the same reasons as in the preceding example, the invariant subspace \(Y = PF_p(G)\) of \(PM_p(G)\) has the wCp.

c) Let \(A = L^1(G, \omega)\) be the Beurling algebra associated with a locally compact commutative group \(G\) and a continuous weight function \(\omega : G \to [0, \infty]\) such that \(\omega(t) \geq 1\) for \(t \in G\) (see [4, Chapter 7] or [40, Section 6.3]). The multiplier algebra of \(L^1(G, \omega)\) is the weighted measure algebra \(M(G, \omega)\). The subspace \(C_0(G, 1/\omega) = \{f : G \to \mathbb{C} : \text{the function } 1/\omega f \text{ is in } C^0(G)\}\) of \(L^\infty(G, 1/\omega)\) is a closed invariant subspace of the space \(M(G, \omega)\) of \(C_0(G, 1/\omega)\). Exactly for the same reasons as in a), the invariant subspace \(Y = C_0(G, 1/\omega)\) of \(L^\infty(G, 1/\omega)\) has the wCp.

d) Suppose that the algebra \(A\) is an ideal in its second dual and has a BAI. Then, since \(A\) has a BAI, the space \(AA^* = \{a.f : a \in A \text{ and } f \in A^*\}\) is a norm closed subspace of \(A^*\) [3, Corollary 2.9.26] or [18, 32.22]. The multiplier algebra of \(L^1(G, \omega)\) is the weighted measure algebra \(M(G, \omega)\). The subspace \(C_0(G, 1/\omega) = \{f : G \to \mathbb{C} : \text{the function } 1/\omega f \text{ is in } C^0(G)\}\) of \(L^\infty(G, 1/\omega)\) is a closed invariant subspace of the space \(M(G, \omega)\) of \(C_0(G, 1/\omega)\). Exactly for the same reasons as in a), the invariant subspace \(Y = C_0(G, 1/\omega)\) of \(L^\infty(G, 1/\omega)\) has the wCp.

e) Let \(B\) be another commutative Banach algebra and \(h : A \to B\) an onto bounded Banach algebra homomorphism. Then, since \(h\) is onto, as one can see it easily, for any closed invariant subspace \(Y_0\) of \(B^*\), the subspace \(Y = h^*(Y_0)\) of \(A^*\) is invariant. Moreover, since \(h^*\) is an isomorphism, \(Y\) has the wCp iff the space \(Y_0\) has the wCp.

To the above list of examples, one can also add, for instance, some of the Segal algebras, the semigroup algebras (e.g. \(L^1(\mathbb{R}_+)\)) or the hypergroup algebras etc., that they display the same pattern.

The main result of this paper is the following theorem. This theorem is an abstract form and a far reaching generalization of Helson’s Theorem mentioned in the introduction. The corollaries that follow the proof will justify, we hope, this affirmation. Compared with the known proofs of Helson’s Theorem [8,14], [18, Section 41.18.], [42, Theorem 5.6.7], as the reader will notice, our proof is much shorter and simpler, though it requires the algebra to be weakly compactly generated. This hypothesis is not an important restriction for the purpose of this paper.

**Theorem 2.4.** Suppose that the Banach algebra \(A\) is semisimple, regular, Tauberian and weakly compactly generated. Let \(X\) and \(Y\) be two norm closed invariant subspaces of \(A^*\) such that

a) \(X\) is weakly sequentially complete.
b) \(Y\) has the wCp.
c) \(X \cap Y \cap \Delta(A) = \emptyset\).

Then \(X \cap Y = \{0\}\).
Proof. For a contradiction, suppose that \( X \cap Y \neq \{0\} \). Let \( f \in X \cap Y \), \( f \neq 0 \) be a functional in this intersection. Then, since the algebra \( A \) is semisimple, regular and Tauberian, the spectrum of the functional \( f \) is not empty. Let \( \gamma \) be in the spectrum \( \sigma(f) \) of \( f \). By Theorem 2.1, there is a sequence \((a_n)_{n \geq 0}\) in \( A \) such that the sequence \((a_n f)_{n \geq 0}\) converges in the weak* topology of \( A^* \) to \( \gamma \). Choose \( a \in A \) such that \( \hat{a}(\gamma) = 1 \). Since the sequence \( f_n = a_n f \) lies in the space \( Y \) and since this space has the wCp, the sequence \((a_n f)_{n \geq 0}\) is weakly Cauchy. Since the space \( X \) is invariant and \( f \in X \) too, the sequence \((a_n f)_{n \geq 0}\) is also in the space \( X \). As this space is weakly sequentially complete, the sequence \((a_n f)_{n \geq 0}\) converges weakly to some element of the space \( X \). Since the sequence \((a_n f)_{n \geq 0}\) converges in the weak* topology of \( A^* \) to \( \gamma \) and \( a \gamma = \hat{a}(\gamma) \gamma = \gamma \), necessarily

\[
a a_n f \to \gamma
\]

in the weak topology of \( A^* \). Since the space \( Y \) is closed in \( A^* \) and the sequence \((a a_n f)_{n \geq 0}\) lies in \( Y \), \( \gamma \in Y \) too. Hence \( \gamma \in X \cap Y \cap \Delta(A) \). This contradicts hypothesis c). Thus \( X \cap Y = \{0\} \). \( \square \)

Let now \( \phi : L^1(G) \to C_0(\hat{G}) \) be the Fourier transform, \( E \subseteq \hat{G} \) a Helson set and \( \phi_E : L^1(G) \to C_0(E) \), \( \phi_E(a) = \hat{a}(\gamma) \), the restriction of the Fourier transform to \( E \). Since the homomorphism \( \phi_E \) is surjective, the Banach algebras \( C_0(E) \) and \( L^1(G) / k(E) \) are isomorphic. The adjoint \( \phi_E^* \) of \( \phi_E \) sends each measure \( \mu \in M(E) \) to its Fourier-Stieltjes transform \( \hat{\mu} : G \to \mathbb{C} \) so that \( k(E)^* \cap C_0(G) = 0 \) [14].

The next corollary is considerably stronger than Helson’s Theorem since the space \( X \) in this corollary is neither related to the Fourier transform or is assumed to be weak* closed. The group algebra \( L^1(G) \) is weakly compactly generated if the group \( G \) is \( \sigma \)-compact [5, p. 143]. Since \( G \) is not compact, \( C_0(G) \cap \hat{G} = \emptyset \) so that this result follows directly from the preceding theorem.

**Corollary 2.5.** Let \( G \) be a noncompact locally compact \( \sigma \)-compact commutative group and \( A = L^1(G) \). Then, for any weakly sequentially complete norm closed invariant subspace \( X \) of \( L^\infty(G) \), the intersection \( X \cap C_0(G) \) is trivial.

Let now \( G \) be a locally compact amenable group and \( A = A(G) \) the Fourier algebra of \( G \). Helson’s Theorem has been extended to Fourier algebra \( A(G) \) of a first countable (or compact) group \( G \) by Dunkl and Ramirez [9] by quite a complicated method. When \( G \) is first countable, the algebra \( A(G) \) is weakly compactly generated [22, Theorem 3.2] so that the compact space \( (A(G)^*, \omega^*) \) is sequentially compact. We give here, as an immediate corollary of Theorem 2.4 above, the exact analogue of the preceding corollary for the algebra \( A_p(G) \), so for the algebra \( A(G) \) as well. Since the subspace \( PF_p(G) \) of \( PM_p(G) \) has wCp (Example 2.3(b)) and \( \Delta(A_p(G)) \cap PF_p(G) = \emptyset \). The following corollary too is immediate from Theorem 2.4.

**Corollary 2.6.** Let \( G \) be a nondiscrete locally compact amenable group such that the space \( (A_p(G)^*, \omega^*) \) \( (1 < p < \infty) \) is sequentially compact. Then, for any weakly sequentially complete norm closed invariant subspace \( X \) of \( PM_p(G) \), the intersection \( X \cap PF_p(G) \) is trivial.

Now let \( G \) be a noncompact locally compact \( \sigma \)-compact commutative group and \( \omega : G \to [0, \infty[ \) is a continuous weight function chosen so that the algebra \( L^1(G, \omega) \) is regular and has a BAI (see [4, Chapter 7] and [40, Section 6.3]). We recall that this algebra is always
semisimple. In general, the Gelfand spectrum of the algebra $L^1(G, \omega)$ is quite different from that of the algebra $L^1(G)$. However, whatever the spectrum of the algebra $L^1(G, \omega)$ is, since the group $G$ is not compact, the spectrum of the algebra $L^1(G, \omega)$ and the space $C_0(G, 1/\omega)$ are disjoint. Indeed, by definition of the space $C_0(G, 1/\omega)$, for $f \in C_0(G, 1/\omega)$, the function $f/\omega$ vanishes at infinity whereas, for $\gamma \in \Delta(L^1(G, 1/\omega))$, $|\gamma/\omega| = 1$ on $G$. The following result also follows directly from Theorem 2.4.

**Corollary 2.7.** Under the above hypothesis on $G$ and $\omega$, for any weakly sequentially complete norm closed invariant subspace $X$ of $L^\infty(G, 1/\omega)$, the intersection $X \cap C_0(G, \omega)$ is trivial.

So far we have considered only weakly compactly generated Banach algebras. To a certain extent we can weaken this hypothesis. Let $X$ and $Y$ be two invariant norm closed subspaces of $A^\ast$. Suppose that $X$ is weakly sequentially complete and $Y$ has wCp. Now let $I$ be a closed ideal of the algebra $A$ and $B = A/I$. The subspace $X_0 = X \cap I^\perp$ of $B^\ast$ is weakly sequentially complete and the subspace $Y_0 = I^\perp \cap Y$ has the wCp. The proof of following proposition can easily be deduced from the proof of Theorem 2.4. For that reason we omit the proof.

**Proposition 2.8.** Let $I$ be a closed ideal of the algebra $A$ such that the quotient algebra $B = A/I$ is semisimple, regular, Tauberian and weakly compactly generated. Let $X$ and $Y$ be two invariant norm closed subspaces of $A^\ast$ such that $X$ is weakly sequentially complete and $Y$ has the wCp. If $X \cap Y \cap I^\perp \cap \Delta(B) = \emptyset$ then $X \cap Y \cap I^\perp = \{0\}$. If, in addition, $X \subseteq I^\perp$ then $X \cap Y = \{0\}$.

For instance, if $A = L^1(G)$ ($G$ is commutative) and $E \subseteq \hat{G}$ a closed set which is a set of synthesis (see the next section) then the algebra $B = L^1(G)/k(E)$ is semisimple, regular and Tauberian. So, if the algebra $B$ is also separable for instance, the preceding proposition applies to it.

**Remarks 2.9.** a) For $f \in A^\ast$, put $Z_f = \{a.f: a \in A\}$, the norm closure of the space $\{a.f: a \in A\}$ in $A^\ast$. The space $Z_f$ is an invariant subspace of $A^\ast$. Let $X$ and $Y$ be two norm closed (not necessarily invariant) subspaces of $A^\ast$. As another version of Theorem 2.4, we can state the following: The algebra $A$ being as in Theorem 2.4, suppose that

- a) For each $f \in Y$, the space $Z_f$ has the wCp.
- b) For each $g \in X$, the space $Z_g$ is weakly sequentially complete; and
- c) $Z_f \cap Z_g \cap \Delta(A) = \emptyset$ for each $f \in Y$ and $g \in X$.

Then $Z_f \cap Z_g = \{0\}$ for each $f \in Y$ and $g \in X$. Moreover, if $f \in Z_f$ and $g \in Z_g$ for each $f \in X$ and $g \in Y$, then $X \cap Y = \{0\}$.

b) A celebrated Banach space theorem due to Rosenthal [6, Chapter XI] says this: A Banach space $Y$ does not contain an isomorphic copy of the sequence space $\ell^1$ iff every bounded sequence $(y_n)_{n \geq 0}$ in $Y$ has a subsequence which is weakly Cauchy.

So, with the notation of Remark a), in Theorem 2.4, instead of assuming that the space $Y$ has the wCP, it is enough to assume that, for each $f \in Y$, the space $Z_f$ does not contains an isomorphic copy of $\ell^1$.

c) If a Banach space $Y$ does not contain an isomorphic copy of the sequence space $\ell^1$. Put $Y_0 = \{f \in A^\ast: a.f \in Y$ for each $a \in A\}$. This is also a norm closed invariant subspace of $A^\ast$ and $Y \subseteq Y_0$. It is easy to see that $Y_0$ has the wCp iff $Y$ does. If $A = L^1(G)$, $Y = C_0(G)$ and $G$ is not discrete then
Y₀ is strictly larger than C₀(G) since every bounded Borel measurable function \( f: G \to \mathbb{C} \) with compact support (and also the uniform limits of such functions) are in Y₀. In the results given above and also below where C₀(G) is involved, if one uses this space \( Y₀ = \{ f ∈ L^∞(G): a* f ∈ C₀(G), \ ∀ a ∈ L¹(G) \} \) instead of C₀(G), the results remain valid. However, for the sake of clarity and concreteness we preferred to work with the space C₀(G) rather than with the space Y₀.

In the rest of the paper we shall work with the group algebra \( L¹(G) \) of a locally compact commutative group but all the results presented below are valid, mutatis mutandis, for the Herz–Figa–Talamanca algebra \( A_p(G) \) of a locally compact group \( G \) whose discrete version \( G_d \) is amenable, for instance.

3. Applications to the study of sets of spectral synthesis

In this section \( G \) will be a noncompact locally compact commutative group and \( A = L¹(G) \) the group algebra of \( G \). On \( L¹(G)^{**} \) we put the first Arens multiplication as defined in Section 1. Theorem 2.4 above has some applications to the problem of when is a given closed subset \( E \) of \( \hat{G} \) a set of synthesis. In this section we shall present these applications. We begin by recalling the relevant definitions.

Let \( E \) be a nonempty closed subset of \( \hat{G} \). As usual, to \( E \) we associate two ideals, \( k(E) \) (already defined) and \( j(E) \):

\[
j(E) = \{ a ∈ L¹(G): \hat{a} \text{ has compact support disjoint from } E \}.
\]

Among the closed ideals of \( L¹(G) \) whose hull is \( E \), \( k(E) \) is the largest one and \( j(E) \) is the smallest one. If \( k(E) = j(E) \) then the set \( E \) is said to be a set of spectral synthesis. In the books [18, Chapter X], [23, Chapter 5], [42, Chapter 7], the reader can find ample information on this notion. The annihilator of the ideal \( k(E) \) in the space \( L^∞(G) \) is the subspace \( \text{Span}(E)^{w∗} \) of \( L^∞(G) \), the weak∗ closure of the linear span of the set \( E \) in the space \( L^∞(G) \). The annihilator of the closed ideal \( j(E) \) in \( L^∞(G) \) is the subspace

\[
L^∞_E(G) = \{ f ∈ L^∞(G): \sigma(f) ⊆ E \}.
\]

Both spaces \( \text{Span}(E)^{w∗} \) and \( L^∞_E(G) \) are invariant subspaces of \( L^∞(G) \) and \( \text{Span}(E)^{w∗} ⊆ L^∞_E(G) \). These two subspaces of \( L^∞(G) \) are equal iff the set \( E \) is a set of synthesis. We shall mostly use the following characterization of sets of synthesis. For the sake of completeness, we include a proof.

In the rest of this section, \( E \) will denote a nonempty closed subset of the dual group \( \hat{G} \) and \( e ∈ L¹(G)^{**} \) will be a fixed right identity. We also recall that, in order to have \( \langle b * a, f \rangle = \langle b, a * f \rangle \) (rather than \( \langle b * a, f \rangle = \langle b, a^˘ * f \rangle \), where \( a^˘(t) = a(-t) \)), in Section 1, we have defined the \( (L¹(G), L^∞(G)) \)-duality as

\[
\langle a, f \rangle = \int_G a(-t)f(t)dt.
\]

**Lemma 3.1.** The set \( E \) is a set of synthesis iff \( a * f = 0 \) for each \( a ∈ k(E) \) and \( f ∈ L^∞_E(G) \).

**Proof.** Suppose first that \( E \) is a set of synthesis. Then \( k(E)^⊥ = L^∞_E(G) \). Since for \( a ∈ k(E) \) and \( b ∈ L¹(G) \), the product \( b * a \) is in \( k(E) \), for \( f ∈ k(E)^⊥ \),

\[
\langle b, a * f \rangle = \langle b * a, f \rangle = 0.
\]

This being true for all \( b ∈ L¹(G) \), \( a * f = 0 \).
Conversely, suppose that for each \( a \in k(E) \) and \( f \in L^\infty_E(G) \), \( a \ast f = 0 \). Then
\[
\langle a, f \rangle = \langle ea, f \rangle = \langle e, a \ast f \rangle = 0.
\]
This being true for each \( a \in k(E) \), we conclude that \( f \in k(E)\perp \) so that \( L^\infty_E(G) = k(E)\perp \). Hence \( E \) is a set of synthesis. □

Our main tool to study the sets of synthesis is the following lemma. We recall that every maximal ideal \( \text{Ker}(\gamma) \) of \( L^1(G) \) has a bounded approximate identity. A justification (valid for all the Banach algebras considered in this paper) of this affirmation can be found in [47, Lemma 4.3]. We also recall that, for any \( \gamma \in \widehat{G} \) considered as a multiplicative functional on \( L^1(G) \) and for \( m \) and \( n \) in \( L^1(G)\ast\ast \), we have \( \langle \gamma, mn \rangle = \langle \gamma, m \rangle \langle \gamma, n \rangle \). That is, \( \gamma \) is also multiplicative on \( L^1(G)\ast\ast \). In particular the set \( \{ m \in L^1(G)\ast\ast : \langle \gamma, m \rangle = 0 \} \), which is, by bipolar theorem, the second dual of the ideal \( \text{Ker}(\gamma) \), is a maximal ideal of the Banach algebra \( L^1(G)\ast\ast \).

**Lemma 3.2.** For each \( \gamma \in \widehat{G} \), there is an element \( e_\gamma \) in \( L^1(G)\ast\ast \) such that \( \langle \gamma, e_\gamma \rangle = 1 \) and, for each \( a \in L^1(G) \), \( ae_\gamma = \widehat{a}(\gamma)e_\gamma \).

**Proof.** Let \( u_\gamma \) be a right identity in \( \text{Ker}(\gamma)\ast\ast \). Let us see that the element \( e_\gamma = e - eu_\gamma \) of \( L^1(G)\ast\ast \) will do the job. As \( u_\gamma \in \text{Ker}(\gamma)\ast\ast \), \( \langle u_\gamma, \gamma \rangle = 0 \) so that
\[
\langle \gamma, e_\gamma \rangle = \langle \gamma, e - eu_\gamma \rangle = \langle \gamma, e \rangle = 1.
\]
Let us now verify that for each \( a \in L^1(G) \), \( ae_\gamma = \widehat{a}(\gamma)e_\gamma \). Let \( a \in L^1(G) \) and decompose it in the direct sum \( L^1(G)\ast\ast = \text{Ker}(\gamma)\ast\ast \oplus \mathbb{C}e \) as \( a = n + \lambda e \). Since \( n \in \text{Ker}(\gamma)\ast\ast \) and \( \text{Ker}(\gamma)\ast\ast = \{ m \in L^1(G)\ast\ast : \langle \gamma, m \rangle = 0 \} \), one has \( \lambda = \widehat{a}(\gamma) \). Multiplying the equality \( a = n + \widehat{a}(\gamma)e \) from the right by \( u_\gamma \) and using the fact that \( ne_\gamma = n \), we get
\[
a u_\gamma = n + \widehat{a}(\gamma)e u_\gamma.
\]
Then subtracting this from \( a = n + \widehat{a}(\gamma)e \) we get
\[
a - a u_\gamma = \widehat{a}(\gamma).(e - eu_\gamma).
\]
Since \( a - a u_\gamma = a(e - eu_\gamma) \), we see that we have \( ae_\gamma = \widehat{a}(\gamma)e_\gamma \). □

As a first use of this lemma we present the following result.

**Proposition 3.3.** Let \( f \in L^\infty(G) \) and \( \gamma \in \widehat{G} \). If \( \langle e_\gamma, f \rangle \neq 0 \) then \( \gamma \in \sigma(f) \).

**Proof.** Since, for all \( a \in L^1(G) \),
\[
\langle a, e_\gamma \ast f \rangle = \langle ae_\gamma, f \rangle = \widehat{a}(\gamma) \langle e_\gamma, f \rangle,
\]
we see that \( \langle e_\gamma, f \rangle \neq 0 \) iff \( e_\gamma \ast f \neq 0 \). The preceding line also shows that
\[
e_\gamma \ast f = \langle e_\gamma, f \rangle \gamma.
\]
This being observed, suppose now that \( \langle e_\gamma, f \rangle \neq 0 \). Then \( e_\gamma.f = \langle e_\gamma, f \rangle \gamma \neq 0 \). Let \((a_i)_{i \in I}\) be a bounded net in \( L^1(G) \) that converges to \( e_\gamma \) in the weak* topology of \( L^1(G)^{**} \). Then \( a_i * f \to e_\gamma.f \) in the weak* topology of \( L^\infty(G) \). Since \( e_\gamma.f = \langle e_\gamma, f \rangle \gamma \) and \( \langle e_\gamma, f \rangle \neq 0 \), dividing \( a_i \) by \( \langle e_\gamma, f \rangle \), we see that \( \gamma \in \sigma_\ast(f) \). Since, by Theorem 2.0, \( \sigma_\ast(f) = \sigma(f), \gamma \in \sigma(f) \). \( \square \)

The family \((\langle e_\gamma, f \rangle)_{\gamma \in \hat{G}}\) can be thought as the “Fourier coefficients of \( f \)” in a generalized sense. However this family, unless \( f \) is almost periodic, does not characterize \( f \) in any way. For any \( f \in L^\infty(G) \) and \( a \in L^1(G) \), as is well known and easy to see [32, Lemma 5],

\[
\sigma(a * f) \subseteq \sigma(f) \cap \text{Supp}(\hat{a}).
\]

**Corollary 3.4.** Let \( a \in k(E) \) and \( f \in L^\infty_E(G) \). Then

a) For all \( \gamma \in \hat{G} \), \( \langle e_\gamma, a * f \rangle = 0 \).

b) Either \( a * f = 0 \) or the set \( \sigma(a * f) \) is perfect.

c) The function \( a * f \) always lies in the space \( L^\infty_\partial E(G) \), where \( \partial E \) is the topological boundary of the set \( E \) in the space \( \hat{G} \).

**Proof.** a) Let \( \gamma \in \hat{G} \). Since

\[
\langle e_\gamma, a * f \rangle = \langle e_\gamma, a \rangle = \langle a e_\gamma, f \rangle = \hat{a}(\gamma). \langle e_\gamma, f \rangle,
\]

we see that if \( \gamma \in E \), \( \langle e_\gamma, a * f \rangle = \hat{a}(\gamma). \langle e_\gamma, f \rangle = 0 \). If \( \gamma \notin E \) then \( \langle e_\gamma, f \rangle = 0 \) too. Indeed, if we had \( \langle e_\gamma, f \rangle \neq 0 \), by the preceding proposition, we would have \( \gamma \in \sigma(f) \), which is not possible since \( \sigma(f) \subseteq E \). Hence, whatever the set \( E \) is, for all \( \gamma \in \hat{G} \), \( \langle e_\gamma, a * f \rangle = 0 \).

b) Suppose that \( \gamma \) is an isolated point of the set \( \sigma(a * f) \). Then, for some open neighborhood \( V \) of \( \gamma \), \( \overline{V} \cap \sigma(a * f) = \{ \gamma \} \). Choose an element \( b \in L^1(G) \) such that \( \hat{b}(\gamma) = 1 \) and \( \text{Supp}(\hat{b}) \subseteq \overline{V} \). Since

\[
\sigma(b * a * f) \subseteq \text{Supp}(\hat{b}) \cap \sigma(a * f) \subseteq \overline{V} \cap \sigma(a * f) = \{ \gamma \},
\]

we conclude for instance by [40, Proposition 7.1.17] that \( b * a * f = \lambda \gamma \), where \( \lambda \) is a constant. The constant \( \lambda \) cannot be zero. Indeed, since \( \gamma \in \sigma(a * f) \), by definition of the set \( \sigma(a * f) \), there is a net \((b_i)_{i \in I}\) in \( L^1(G) \) such that, in the weak* topology of \( L^\infty(G) \), \( b_i * a * f \to \gamma \). Since \( \hat{b}(\gamma) = 1 \) and \( b * a * f = \lambda \gamma \), we conclude that, on the one hand, in the weak* topology of \( L^\infty(G) \),

\[
b * b_i * a * f \to \hat{b}(\gamma) \gamma = \gamma;
\]

and, on the other hand,

\[
b * b_i * a * f = \lambda \hat{b}_i(\gamma) \gamma
\]

so that \( \lambda \hat{b}_i(\gamma) \gamma \to \gamma \). Hence \( \lambda \neq 0 \). Applying now \( e_\gamma \) to the equality \( b * a * f = \lambda \gamma \) and using that \( \hat{b}(\gamma) = 1 \), we get

\[
\lambda = \langle e_\gamma, b * a * f \rangle = \langle e_\gamma a, b * f \rangle = \hat{a}(\gamma). \langle b, e_\gamma . f \rangle = \hat{a}(\gamma). \langle e_\gamma, f \rangle.
\]
As \( \lambda \neq 0 \), this is not possible by assertion a). Hence, for \( a \in k(E) \) and \( f \in L^\infty_E(G) \), whenever \( a \ast f \neq 0 \), the set \( \sigma(a \ast f) \) is perfect.

c) Since

\[ \sigma(a \ast f) \subseteq \sigma(f) \cap \text{Supp}(\widehat{a}) \subseteq E \cap \text{Supp}(\widehat{a}) \]

and \( \widehat{a} = 0 \) on \( E \), \( E \cap \text{Supp}(\widehat{a}) \subseteq \partial E \) so that \( a \ast f \in L^\infty_{\partial E}(G) \). \( \square \)

As a consequence of the preceding result we have the following result.

**Corollary 3.5.** For each \( f \in C_0(G) \) and \( \gamma \in \widehat{G} \), \( \langle e_\gamma, f \rangle = 0 \). In particular, for each \( f \in C_0(G) \), \( f \neq 0 \), the set \( \sigma(f) \) is perfect.

**Proof.** Let \( f \in C_0(G) \) and \( \gamma \in \widehat{G} \). Since \( e_\gamma \ast f = \langle e_\gamma, f \rangle \gamma \), if we had \( \langle e_\gamma, f \rangle \neq 0 \), by assertion a) of the preceding corollary, we would have

\[ e_\gamma \ast f = \langle e_\gamma, f \rangle \gamma \neq 0, \]

which is not possible since the function \( e_\gamma \ast f \) is in \( C_0(G) \) and \( \gamma \notin C_0(G) \). Indeed, the group \( G \) is not compact and for all \( m \in L^1(G) \) and \( f \in C_0(G) \), the function \( m \ast f \) is in the space \( C_0(G) \). This is because each function in \( C_0(G) \) is weakly almost periodic. Thus, \( \langle e_\gamma, f \rangle = 0 \) for all \( \gamma \in \widehat{G} \) and \( f \in C_0(G) \). If a \( \gamma \in \sigma(f) \) were an isolated point of the set \( \sigma(f) \) then, as seen in the proof of Corollary 3.4(b), we would have, for some \( b \in L^1(G) \), \( b \ast f = \gamma \). This is not possible since \( b \ast f \) is in \( C_0(G) \) and \( \gamma \notin C_0(G) \). Hence, for any \( f \in C_0(G) \), \( f \neq 0 \), the set \( \sigma(f) \) is perfect. \( \square \)

By WAP\( (G) \) and AP\( (G) \) we denote, respectively, the spaces of the weakly almost periodic and the almost periodic functions on the group \( G \) [2]. These are norm closed \( L^1(G) \)-submodules of \( L^\infty(G) \), \( C_0(G) \subseteq \text{WAP}(G) \) and \( \text{AP}(G) \cap C_0(G) = \{0\} \). We now introduce the following subset of \( L^\infty(G) \),

\[ X(G) = \{ f \in L^\infty(G) : \text{either } f = 0 \text{ or } \langle e_\gamma, f \rangle \neq 0 \text{ for at least one } \gamma \in \widehat{G} \}. \]

The next result shows that this set \( X(G) \) is fairly large.

**Proposition 3.6.**

a) The inclusion \( \text{AP}(G) \subseteq X(G) \) holds.

b) If \( f \in L^\infty(G) \) and the set \( \sigma(f) \) has an isolated point then \( f \in X(G) \).

c) If \( f \in L^\infty(G) \) and for some net \( (a_i)_{i \in I} \) in \( L^1(G) \), the net \( (a_i \ast f)_{i \in I} \) converges in the weak topology of \( L^\infty(G) \) to some \( \gamma \in \widehat{G} \) then \( f \in X(G) \).

d) \( X(G) \cap C_0(G) = \{0\} \).

**Proof.** a) Let \( f \in \text{AP}(G) \), \( f \neq 0 \), be a given almost periodic function. Then the norm spectrum of \( f \), that is, the intersection

\[ \{ a \ast f : a \in L^1(G) \} \cap \widehat{G} \]
is nonempty [12]. Let \( \gamma \) be an element in this intersection. Then, for some net \((a_i)_{i \in I}\) in \(L^1(G)\), \(a_i * f \to \gamma\) in the norm topology of \(L^\infty(G)\). Then

\[
1 = \langle e_\gamma, \gamma \rangle = \lim_i \langle e_\gamma, a_i * f \rangle = \lim_i \widehat{a_i}(\gamma). \langle e_\gamma, f \rangle.
\]

This proves that \( \langle e_\gamma, f \rangle \neq 0 \). So \( f \in X(G) \). The same argument (replacing the norm convergence by the weak convergence) also proves assertion c).

To prove assertion b), if \( \gamma \) is an isolated point of \( \sigma(f) \) then, as seen in the proof of Corollary 3.4(b), there is some \( b \in L^1(G) \) such that \( b * f = \gamma \). This implies that

\[
1 = \langle e_\gamma, \gamma \rangle = \langle e_\gamma, b * f \rangle = \widehat{b}(\gamma). \langle e_\gamma, f \rangle.
\]

This shows that \( \langle e_\gamma, f \rangle \neq 0 \). So \( f \in X(G) \).

d) Since, by the preceding corollary, for each \( \gamma \in \hat{G} \) and \( f \in C_0(G) \), \( \langle e_\gamma, f \rangle = 0 \), the intersection \( X(G) \cap C_0(G) \) is trivial.

Next we present a simple but general result. We put

\[
Z(E) = k(E) * L^\infty_E(G).
\]

That is,

\[
Z(E) = \{a * f : a \in k(E) \text{ and } f \in L^\infty_E(G)\}.
\]

We recall that, by Corollary 3.4, \( Z(E) \subseteq L^\infty_{\overline{aE}}(G) \).

**Proposition 3.7.**

a) The set \( E \) is a set of synthesis iff \( Z(E) \subseteq X(G) \).

b) The set \( E \) is a set of synthesis iff, for each \( a \in k(E) \) and \( f \in L^\infty_E(G) \), the space \( Z_{a * f} = \{b * a * f : b \in L^1(G)\} \) is reflexive.

**Proof.** a) Suppose that for each \( a \in k(E) \) and \( f \in L^\infty_E(G) \), the function \( a * f \) is in the set \( X(G) \). Then, by definition of the set \( X(G) \), either \( a * f = 0 \) or, for some \( \gamma \in \hat{G} \), \( \langle e_\gamma, a * f \rangle \neq 0 \). By Corollary 3.4(a), this latter case is not possible. Hence \( a * f = 0 \), and the set \( E \) is a set of synthesis by Lemma 3.1. The converse implication, by Lemma 3.1, is always true.

b) Suppose first that, for each \( a \in k(E) \) and \( f \in L^\infty_E(G) \), the space \( Z_{a * f} \) is reflexive. Let \( a \in k(E) \) and \( f \in L^\infty_E(G) \) and let us see that \( a * f = 0 \). If \( a * f \neq 0 \) then \( \sigma(a * f) \neq \emptyset \). Let \( \gamma \in \sigma(a * f) \). By Theorem 2.0, there is a net \((b_i)_{i \in I}\) in the space \( L^1(G) \) such that \( \|b_i * a * f\| \leq 1 \) for all \( i \in I \) and the net \((b_i * a * f)_{i \in I}\) converges in the \( \star \)-topology of the space \( L^\infty(G) \) to \( \gamma \). Since the net \((b_i * a * f)_{i \in I}\) is in the space \( Z_{a * f} \) and since this space is reflexive, the net \((b_i * a * f)_{i \in I}\) converges to \( \gamma \) in the weak topology of the space \( L^\infty(G) \). But then,

\[
1 = \langle e_\gamma, \gamma \rangle = \lim_i \langle e_\gamma, b_i * a * f \rangle = \lim_i \widehat{b_i}(\gamma). \langle e_\gamma, a * f \rangle = 0
\]

by Corollary 3.4. This contradiction proves that \( a * f = 0 \), and the set \( E \) is a set of synthesis by Lemma 3.1. The converse is trivial again by Lemma 3.1. \( \square \)
The preceding proposition shows that, in the study of sets of synthesis, the most important subsets of $L^\infty(G)$ are the set $Z(E)$ and the space $L^\infty_{\partial E}(G)$. The first main result of this section is the following theorem. As seen above (Proposition 3.6), the set $X(G) + C_0(G)$ is considerably larger than the space $AP(G) + C_0(G)$.

**Theorem 3.8.** Suppose that the group $G$ is $\sigma$-compact and that

a) The set $Z(E) = k(E) \ast L^\infty_E(G)$ is contained in the sum $X(G) + C_0(G)$.

b) For each $\gamma \in \partial E$, there is a measure $\mu \in M(G)$ such that $\hat{\mu}(\gamma) = 1$ and, with $F_\mu = \text{Supp}(\hat{\mu}) \cap \partial E$, the space $L^\infty_{F_\mu}(G)$ is weakly sequentially complete.

Then the set $E$ is a set of synthesis.

**Proof.** Let $a \in k(E)$ and $f \in L^\infty_E(G)$. We want to prove that $a \ast f = 0$. By hypothesis a), $a \ast f$ decomposes as $a \ast f = g + h$, where $g \in X(G)$ and $h \in C_0(G)$. As, for all $\gamma \in \widehat{G}$, by Corollaries 3.4 and 3.5, both $\langle e_\gamma, a \ast f \rangle = 0$ and $\langle e_\gamma, h \rangle = 0$, we have

$$0 = \langle e_\gamma, a \ast f \rangle = \langle e_\gamma, g \rangle + \langle e_\gamma, h \rangle = \langle e_\gamma, g \rangle$$

so that, for all $\gamma \in \widehat{G}$, $\langle e_\gamma, g \rangle = 0$. Hence, by definition of the set $X(G)$, $g = 0$. Thus $a \ast f = h$ so that the function $a \ast f$ is in the space $C_0(G)$. Let us see that this is possible only if $a \ast f = 0$. Indeed, for a contradiction, suppose that $a \ast f \neq 0$, and let $\gamma \in \sigma(a \ast f)$. By Corollary 3.4, $\gamma \in \partial E$. Hence, by hypothesis b), there is a measure $\mu \in M(G)$ such that $\hat{\mu}(\gamma) = 1$ and the set $F_\mu = \text{Supp}(\hat{\mu}) \cap \partial E$ is such that the space $L^\infty_{F_\mu}(G)$ is weakly sequentially complete. Next observe that the function $\mu \ast a \ast f$, which is in the space $C_0(G)$, is also in the space $L^\infty_{F_\mu}(G)$ since $\sigma(\mu \ast a \ast f) \subseteq \text{Supp}(\hat{\mu}) \cap \partial E = F_\mu$. Hence, since by Corollary 2.5, the intersection $L^\infty_{F_\mu}(G) \cap C_0(G)$ is trivial, $\mu \ast a \ast f = 0$. However this is not possible since $\gamma \in \sigma(a \ast f)$ and $\hat{\mu}(\gamma) = 1$. To explain this last point, let

$$\gamma = \lim_{n \to \infty} a_n \ast a \ast f \text{ in the weak*-topology of } L^\infty(G)$$

for some sequence $(a_n)_{n \geq 0} \in L^1(G)$. Since $L^1(G)$ is an ideal in $M(G)$,

$$\mu \ast \gamma = \lim_{n \to \infty} \mu \ast a_n \ast a \ast f \text{ in the weak*-topology of } L^\infty(G).$$

As $\mu \ast \gamma = \hat{\mu}(\gamma)\gamma$, this contradicts the equality $\mu \ast a \ast f = 0$. This contradiction proves that $\sigma(a \ast f) = \emptyset$. Hence $a \ast f = 0$, and $E$ is a set of synthesis by Lemma 3.1. $\square$

**Remark 3.9.** If, in the preceding theorem, the set $F_\mu$ is contained in some set $F$ which is a “HS-set”, that is, it is both a Helson set and a set of synthesis, then the space $L^\infty_{F_\mu}(G)$ is weakly sequentially complete. Indeed since then $L^\infty_{F_\mu}(G) \subseteq L^\infty_F(G) = k(F)\perp$ and the space $k(F)\perp$ is weakly sequentially complete. In particular, if $\partial E$ is a HS-set then condition b) holds automatically. The reader can find a characterization of compact HS-sets in Saeki [45].

The next result also gives two conditions sufficient for a set $E$ to be a set of synthesis. We recall that the space $C_0(G)$ is weak*-dense in the space $L^\infty(G)$.
Theorem 3.10. Suppose that the group $G$ is $\sigma$-compact and that

a) The intersection $C_0(G) \cap L^\infty_E(G)$ is weak* dense in the space $L^\infty_E(G)$.

b) The space $L^\infty_{\partial E}(G)$ is weakly sequentially complete.

Then $E$ is a set of synthesis.

Proof. Let $a \in k(E)$ and $f \in L^\infty_E(G)$. We want again to prove that $a \ast f = 0$. First let $g$ be in the intersection $C_0(G) \cap L^\infty_E(G)$. Then the function $a \ast g$ is in $C_0(G) \cap L^\infty_{\partial E}(G)$. This intersection being trivial by Corollary 2.5, we see that $a \ast g = 0$ for each $g$ in $C_0(G) \cap L^\infty_E(G)$. This, by a), implies that $a \ast f = 0$. Hence $E$ is a set of synthesis.

An outstanding open problem in the theory of the spectral synthesis is the problem whether the union of two sets of synthesis is a set of synthesis. The next result, which is the second main result of this section, is related to this problem. Let $F$ be another nonempty closed subset of the dual group $\hat{G}$. We put

$$D = \partial E \cap \partial F \cap \partial (E \cup F).$$

The set $X(G)$ (defined just before Proposition 3.6) has the same meaning as above. We also recall that $Z(E) = k(E) \ast L^\infty_E(G)$.

Theorem 3.11. Suppose that the group $G$ is $\sigma$-compact, both sets $E$ and $F$ are set of synthesis and that

a) $Z(E \cup F) \cap WAP(G) \subseteq X(G) + C_0(G)$.

b) For each $\gamma \in D$ there is a measure $\mu \in M(G)$ with $\hat{\mu}(\gamma) = 1$ and such that the set $F_\mu = D \cap \text{Supp}(\hat{\mu})$ is contained in some HS-set $V$.

Then $E \cup F$ is a set of synthesis.

Proof. Let $a \in k(E \cup F)$ and $f \in L^\infty_{E \cup F}(G)$. We want to prove that $a \ast f = 0$. For a contradiction, suppose that $a \ast f \neq 0$. Then $\sigma(a \ast f) \neq \emptyset$. Take $\gamma \in \sigma(a \ast f)$. One can easily see that $\gamma \in D$ (see [43, Lemma 3]). By hypothesis b), there is a measure $\mu \in M(G)$ with $\hat{\mu}(\gamma) = 1$ and such that the set $F_\mu = D \cap \text{Supp}(\hat{\mu})$ is contained in some HS-set $V$. Then the function $\mu \ast a \ast f$ is both in the spaces $Z(E \cup F)$ and $L^\infty_V(G)$. Since $V$ is a HS-set, $L^\infty_V(G) \subseteq WAP(G)$ (see Corollary 4.3 below). It follows that the function $\mu \ast a \ast f$ is in the intersection $Z(E \cup F) \cap WAP(G)$. Hence, by hypothesis a), $\mu \ast a \ast f = g + h$ for some $g \in X(G)$ and $h \in C_0(G)$. Since, by Corollaries 3.4(a) and 3.5, for all $\gamma \in G$,

$$\langle e_\gamma, \mu \ast a \ast f \rangle = 0 \quad \text{and} \quad \langle e_\gamma, h \rangle = 0,$$

we see that, for all $\gamma \in \hat{G}$, $\langle e_\gamma \ast f, g \rangle = 0$. This, by definition of the set $X(G)$, is possible only if $g = 0$ so that $\mu \ast a \ast f = h$, and the function $\mu \ast a \ast f$ is in the space $C_0(G)$. Since $\mu \ast a \ast f$ is also in the space $L^\infty_V(G)$ and since, by Corollary 2.5, $L^\infty_V(G) \cap C_0(G) = \{0\}$, necessarily $\mu \ast a \ast f = 0$. Since $\gamma \in \sigma(a \ast f)$ and $\hat{\mu}(\gamma) = 1$, the function $\mu \ast a \ast f$ cannot be the zero function. From this contradiction we deduce that the function $a \ast f$ must be the zero function.
Thus, \( Z(E \cup F) = k(E \cup F) \ast L^\infty_{E \cup F}(G) = \{0\} \), and the union \( E \cup F \) is a set of synthesis by Lemma 3.1. □

We remark that if the set \( D \) is contained in some HS-set then condition b) of the preceding theorem holds automatically and \( Z(E \cup F) \subseteq L^\infty_D(G) \subseteq \text{WAP}(G) \) (Corollary 4.3 below). If \( L^\infty_D(G) \cap \text{WAP}(G) \subseteq X(G) + C_0(G) \) then condition a) holds automatically.

4. Helson sets, Arens regularity and sets of uniqueness

As in the preceding section, in this section too, \( G \) will be a noncompact locally compact commutative group. It happens that the notion of Helson set is somewhat related to the notion of Arens regularity. In the first part of this section we shall study the connections between these two notions. In the second part, as applications of Theorem 2.4, we present some sufficient conditions for a given closed subset \( E \) of \( \hat{G} \) to be a set of uniqueness. We start by recalling Arens regularity notion.

Let \( A \) be a commutative Banach algebra. The second dual \( A^{**} \) of \( A \) equipped with the Arens multiplication as defined in Section 1 is a Banach algebra. The Banach algebra \( A^{**} \) is in general not commutative. When the algebra \( A^{**} \) is commutative the algebra \( A \) is said to be Arens regular.

a) Every commutative \( C^* \)-algebra is Arens regular. b) The closed subalgebras (so the uniform algebras) and the quotient algebras of Arens regular algebras are Arens regular. c) The group algebra \( L^1(G) \), unless \( G \) is finite, is not Arens regular. The reader can find ample information on this much studied notion in the books [3,37] and the memoir [4] among many other sources. One of the most important characterizations of Arens regularity is this:

The algebra \( A \) is Arens regular iff each functional \( f \in A^* \) is weakly almost periodic. (i.e. the set \( H(f) = \{a.f: a \in A_1\} \) is relatively weakly compact).

We denote by \( \text{WAP}(A) \) the set of the weakly almost periodic functionals on \( A \). This is a norm closed invariant subspace of \( A^* \). For the algebra \( A = L^1(G) \), one has \( \text{WAP}(A) = \text{WAP}(G) \) [10, Theorem 1.1]. We shall need the following result.

Lemma 4.1. Let \( B \) be another commutative Banach algebra and \( \phi: A \to B \) a bounded onto homomorphism. Then \( \phi^*(\text{WAP}(B)) = \text{WAP}(A) \cap \ker(\phi)^\perp \).

Proof. We first observe that, \( \phi \) being onto, by the Open Mapping Theorem, there is a constant \( c > 0 \) such that \( \phi(A_1) \supseteq c.B_1 \). Then, since \( \phi \) is a homomorphism, we observe that, for \( a \in A \) and \( g \in B^* \),

\[
\phi(a).g = a.\phi^*(g).
\]

These two observations imply that \( \phi^*(\text{WAP}(B)) \subseteq \text{WAP}(A) \). Since always \( \phi^*(\text{WAP}(B)) \subseteq \ker(\phi)^\perp \), the inclusion \( \phi^*(\text{WAP}(B)) \subseteq \text{WAP}(A) \cap \ker(\phi)^\perp \) is established. To prove the reverse inclusion, let \( f \in \text{WAP}(A) \cap \ker(\phi)^\perp \) be a given functional. Since \( \phi \) is onto and \( \ker(\phi) \subseteq \ker(f) \), by Sard’s quotient theorem [19, p. 176], there is a functional \( g \in B^* \) such that \( g \circ \phi = f \). That is, \( \phi^*(g) = f \). It remains to prove that \( g \in \text{WAP}(B) \). To see this we first recall that the set \( \{a.f: a \in A_1\} \) is relatively weakly compact since \( f \in \text{WAP}(A) \). As \( \phi^*(g) = f \),

\[
\{\phi(a).g: a \in A_1\} = \{a.\phi^*(g): a \in A_1\} = \{a.f: a \in A_1\},
\]
and the inclusion
\[ \{ b.g : a \in B_1 \} \subseteq \frac{1}{c} \{ \phi(a).g : a \in A_1 \} \]
holds, we see that the functional \( g \) is weakly almost periodic on \( B \). Hence \( \phi^*(WAP(B)) = WAP(A) \cap \ker(\phi)^\perp \). \( \square \)

As a consequence of this lemma we have the following result.

**Proposition 4.2.** Let \( I \) be a closed ideal of \( A \). Then the quotient algebra \( A/I \) is Arens regular iff \( I^\perp \subseteq WAP(G) \).

**Proof.** Let \( \pi : A \to A/I \) be the quotient homomorphism. Since \( \ker(\pi) = I \), by the above lemma,
\[ WAP(A/I) = I^\perp \cap WAP(A). \]
Hence, by what we have noted prior to Lemma 4.1, the algebra \( A/I \) is Arens regular iff \( (A/I)^* = WAP(A/I) \). That is, the algebra \( A/I \) is Arens regular iff \( I^\perp = I^\perp \cap WAP(G) \), which is possible iff \( I^\perp \subseteq WAP(A) \). \( \square \)

Since every commutative C*-algebra is Arens regular and Arens regularity is invariant under Banach algebra isomorphisms, we have the following corollary. At this point we recall that, as proved by Loomis in [34], for a compact scattered subset \( E \) of \( \hat{G} \), the inclusion \( L_E^\infty(G) \subseteq AP(G) \) holds. We also recall that every compact scattered set \( E \subseteq \hat{G} \) is a set of synthesis so that \( L_E^\infty(E) = k(E)^\perp \). The next result is an analogue of this result of Loomis for Helson sets. This result is immediate from the preceding proposition.

**Corollary 4.3.** For any closed Helson set \( E \subseteq \hat{G} \), the inclusion \( k(E)^\perp \subseteq WAP(G) \) holds.

Contrary to closed scattered sets, since not every Helson set is a set of synthesis [30,46], from the inclusion \( k(E)^\perp \subseteq WAP(G) \) we cannot conclude that the inclusion \( L_E^\infty(G) \subseteq WAP(G) \) holds, unless \( E \) is a set of synthesis.

As is well-known [2, Chapter 2], there is a unique invariant mean \( M \) on the space \( WAP(G) \). This mean induces the decomposition
\[ WAP(G) = AP(G) \oplus W_0(G), \]
where \( W_0(G) = \{ f \in WAP(G) : M(|f|) = 0 \} \). The space \( C_0(G) \) is strictly contained in the space \( W_0(G) \) [2, p. 31]. Corollary 4.3 and Corollary 2.5 imply that, for any closed Helson set \( E \), the inclusion
\[ k(E)^\perp \subseteq AP(G) + W_0(G) \setminus C_0(G) \]
holds.
Remarks 4.4. a) The converse of the preceding corollary is not true. Indeed there exist closed sets $E \subseteq \hat{G}$ such that $L^\infty_E(G) \subseteq A\mathcal{P}(G)$, in particular $k(E)^\perp \subseteq W\mathcal{A}(G)$, which are not Helson sets [39]. We remark that any closed set $E$ for which the inclusion $L^\infty_E(G) \subseteq A\mathcal{P}(G)$ holds is a set of synthesis (Proposition 3.7) so that $L^\infty_E(G) = k(E)^\perp$.

b) Since every positive definite function on the group $G$ is weakly almost periodic [11, Theorem 11.2], $B(G) = M(\hat{G})^\perp$ is contained in the space $W\mathcal{A}(G)$. From this one can deduce another proof of Corollary 4.3 but Proposition 4.2 is of independent interest.

c) Corollary 4.3 proves incidentally that the space $W_0(G)$, which is a norm closed invariant subspace of $L^\infty(G)$, does not have the wCp. Indeed, since for every nonempty closed Helson set $E \subseteq \hat{G}$, we have $k(E)^\perp \subseteq A\mathcal{P}(G) + W_0(G) \setminus C_0(G)$, the space $W_0(G)$ had the wCp, by Theorem 2.4, we would have $k(E)^\perp \cap W_0(G) = \{0\}$ so that we would always have $k(E)^\perp \subseteq A\mathcal{P}(G)$, which is not true for instance if $E$ contains a nonempty perfect subset [34].

In the rest of this section we are going to present a couple of results related to the set of uniqueness. We start by recalling the definition of this notion. Let $\mathbb{Z}$ be the group of the integer numbers and $T = \hat{\mathbb{Z}}$, the unit circle group.

A subset $E$ of $T$ is said to be a set of uniqueness for the trigonometric series if the only trigonometric series $\sum_{n=-\infty}^{\infty} c_n e^{int}$ satisfying the condition

$$\forall t \in T \setminus E, \quad \sum_{n=-\infty}^{\infty} c_n e^{int} = \lim_{n \to \infty} \sum_{k=-n}^{n} c_n e^{ikt} = 0$$

is the identically null series (i.e. $c_n = 0$ for all $n \in \mathbb{Z}$).

The reader can find in the books [20,26] and [51] ample information about historical development of this notion. We shall not consider the finer subclasses of the sets of uniqueness. For these we refer the reader to the book [26] and Lyons’ paper [35]. Some of the important results obtained in the past about sets of uniqueness are:

a) Every countable subset (closed or not) of $T$ is a set of uniqueness (Cantor, Lebesgue, W.H. Young).

b) Every closed scattered subset of $T$ is a set of uniqueness (W.H. Young).

c) The union of countably many closed sets of uniqueness is a set of uniqueness (Bary).

d) There exist perfect sets of uniqueness (Bary, Rajchman, Salem-Zygmund).

e) There exist sets of multiplicity (= nonuniqueness) of Lebesgue measure zero (Menshov).

f) There exist closed Helson sets which are not sets of uniqueness (Th.W. Körner).

For precise references about these historical results we refer the reader to the books just mentioned, especially [26].

For a general locally compact commutative group $G$, the set of uniqueness can be defined as follows. This is the approach introduced by Piatetski–Shapiro (see [15, p. 191] and [26, II.4.1 and V.4.1]).

Definition 4.5. A closed subset $E$ of $\hat{G}$ is said to be a set of uniqueness if $L^\infty_E(G) \cap C_0(G) = \{0\}$.

We are going to work with this definition. From this definition it is clear that this notion is closely related to Corollary 2.5. In the theorem given below we have collected some consequences of Corollary 2.5 related to sets of uniqueness. It follows from Helson’s Theorem that every HS-set is a set of uniqueness. Körner [29] and subsequently Kaufman [25] settling an outstanding open problem proved that there exist closed Helson sets in $T$ which are not sets of
uniqueness. For such a set $E$, the space $k(E)^\perp$ is weakly sequentially complete but the space $L^\infty_E(\mathbb{Z})$ is not.

As in Section 2, for each $f \in L^\infty(G)$, we put $Z_f = \{a * f: a \in L^1(G)\}$. The bar denotes the norm closure in $L^\infty(G)$. The space $Z_f$ is apparently a much simpler space than the space $L^\infty_E(G)$. The next lemma displays the role that the space $Z_f$ can play in the study of sets of uniqueness.

Lemma 4.6. Let $E$ be a closed subset of $\widehat{G}$. Then $E$ is a set of uniqueness iff, for each $f \in L^\infty_E(G)$, $Z_f \cap C_0(G) = \{0\}$.

Proof. Since, for each $f \in L^\infty_E(G)$, the inclusion $Z_f \subseteq L^\infty_E(G)$ holds, the necessity of the condition is obvious. To prove its sufficiency, suppose that, for each $f \in L^\infty_E(G)$, $Z_f \cap C_0(G) = \{0\}$. We have to prove that $L^\infty_E(G) \cap C_0(G) = \{0\}$. For a contradiction, suppose that there is an $f \in L^\infty_E(G) \cap C_0(G)$, $f \neq 0$. Since $f \in C_0(G)$, $f$ is weakly almost periodic. So, for any bounded approximate identity $(e_i)_{i \in I}$ in $L^1(G)$, $e_i * f \to f$ weakly. This shows that $f \in Z_f$. Since $f \in C_0(G)$ too, we see that $f \in Z_f \cap C_0(G)$. This contradiction completes the proof. \hfill $\Box$

The next theorem, which is the main result of this section, offers some conditions implying that a given set $E$ is a set of uniqueness.

Theorem 4.7. Suppose that the group $G$ is $\sigma$-compact and that $E$ is a nonempty closed subset of the dual group $\widehat{G}$. Then each of the following conditions implies that $E$ is a set of uniqueness.

a) The space $L^\infty_E(G)$ is weakly sequentially complete.

b) For each $f \in L^\infty_E(G)$, the space $Z_f$ is weakly sequentially complete.

c) For each $f \in L^\infty_E(G)$, $Z_f \cap C_0(G) = \{0\}$.

d) For each $\gamma \in E$, there is a measure $\mu \in M(G)$ such that $\widehat{\mu}(\gamma) = 1$ and, with $F_\mu = \text{Supp}(\widehat{\mu}) \cap E$, the space $L^\infty_{F_\mu}(G)$ is weakly sequentially complete.

e) For each $\gamma \in E$, there is a measure $\mu \in M(G)$ such that $\widehat{\mu}(\gamma) = 1$ and the set $E \cap \text{Supp}(\widehat{\mu})$ is contained in some HS-set $V_\mu$.

f) For each $\gamma \in E$, there is a measure $\mu \in M(G)$ such that $\widehat{\mu}(\gamma) = 1$ and the set $F_\mu = E \cap \text{Supp}(\widehat{\mu})$ is a set of uniqueness.

Proof. It is clear that condition a) is stronger than condition b). Condition b), by Corollary 2.5, implies condition c); and, by Lemma 4.6, condition f) implies that $E$ is a set of uniqueness. Hence each of the conditions a), b) and c) implies that $E$ is a set of uniqueness.

Now suppose that condition d) holds. Let, if there is any, $f \in L^\infty_E(G) \cap C_0(G)$, $f \neq 0$. Since $f \neq 0$, $\sigma(f) \neq 0$. Take a $\gamma \in \sigma(f)$. As $\sigma(f) \subseteq E$, $\gamma \in E$. So, there is a measure $\mu \in M(G)$ such that $\widehat{\mu}(\gamma) = 1$ and, with $F_\mu = \text{Supp}(\widehat{\mu}) \cap E$, the space $L^\infty_{F_\mu}(G)$ is weakly sequentially complete. As $\sigma(\mu * f) \subseteq \text{Supp}(\widehat{\mu}) \cap E = F_\mu$, the function $\mu * f$ is in the space $L^\infty_{F_\mu}(G)$. On the other hand, since $f \in C_0(G)$, the function $\mu * f$ is in $C_0(G)$. Hence the function $\mu * f$ is in the intersection $L^\infty_{F_\mu}(G) \cap C_0(G)$. This intersection, by Corollary 2.5 is trivial. So $\mu * f = 0$. However this is not possible since $\gamma \in \sigma(f)$ and $\widehat{\mu}(\gamma) = 1$. This contradiction proves that the set $E$ is a set of uniqueness.

The proof that condition e) implies that $E$ is a set of uniqueness is very similar. So we omit it. To finish the proof, suppose that condition f) holds. Let, if there is any, $f \in L^\infty_E(G) \cap C_0(G)$,
Proposition 4.8. Suppose that $G$ is $\sigma$-compact and $E \subseteq \hat{G}$ a nonempty closed set. If the intersection $L^\infty_E(G) \cap C_0(G)$ is weak*-dense in $L^\infty_E(\Sigma_f)$ and the set $\partial E$ is a set of uniqueness then $E$ is a set of synthesis.

We finish the paper with some questions and remarks.

5. Questions and remarks

Suppose that $G$ is $\sigma$-compact and $E$ a nonempty closed subset of $\hat{G}$. The results presented in the last two sections arise a certain number of questions. We present here some of them. As above, for $f \in L^\infty(G)$, we put $Z_f = \{a * f : a \in L^1(G)\}$.

Q1. Characterize those $f \in L^\infty(G)$ for which the space $Z_f$ has the wCp.

Q2. Characterize those $f \in L^\infty(G)$ for which the space $Z_f$ is weakly sequentially complete.

Q3. Characterize those $f \in L^\infty(G)$ for which one has $Z_f \cap C_0(G) = \{0\}$.

Q4. If $L^\infty_E(G) \subseteq \text{WAP}(G)$, is then $E$ a set of synthesis?

The characterizations sought after may be in terms of topological properties of the set $\sigma(f)$ or in terms of geometric properties of the Banach space $Z_f$.

R1. The weak sequential completeness is not the only geometric property on $L^\infty_E(G)$ that implies that $E$ is a set of uniqueness or set of synthesis (see the next remark). For instance, if on the unit sphere $S = \{f \in L^\infty_E(G) : \|f\| = 1\}$ of the space $L^\infty_E(G)$ the weak* and the weak topologies agree then $E$ is a set of uniqueness. To prove this, let, if there is any, $f \in L^\infty_E(G) \cap C_0(G)$, $f \neq 0$ be a function and $\gamma \in \sigma(f)$. By Theorem 2.1, there is some sequence $(\alpha_n)_{n \geq 0}$ in $L^1(G)$ such that $\|a_n * f\| \leq 1$ for all $n \geq 0$ and $a_n * f \to \gamma$ in the weak* topology of $L^\infty(G)$. Then

$$1 = \|\gamma\| \leq \liminf_n \|a_n * f\| \leq \limsup_n \|a_n * f\| \leq 1$$

so that $\|a_n * f\| \to \|\gamma\|$. Hence $a_n * f \to \gamma$ weakly. But then, since $a_n * f \in C_0(G)$ for all $n \geq 0$, $\gamma \in C_0(G)$. This contradiction proves that $L^\infty_E(G) \cap C_0(G) = \{0\}$.

R2. Let this time $S$ be the unit sphere of $L^\infty_{\partial E}(G)$. Suppose that, whenever $f_n \in S$, $f \in S$ and $f_n \to f$ in the weak* topology of $L^\infty(G)$, $a * f_n \to a * f$ weakly, for each $a \in L^1(G)$. Then, as in the preceding remark, one can see that the set $E$ is a set of synthesis.

R3. Suppose that $\hat{G}$ is separable (this is the case if $G$ is also first countable). Then, whatever the set $E$ is, one can find a function $f \in AP(G)$ such that $\sigma(f) = E$. To see this, let $(\gamma_n)_{n \geq 1}$ be a dense sequence in $E$. Let $f = \sum_{n=1}^{\infty} \frac{1}{n^2} \gamma_n$. Then $f \in AP(G)$ and $\sigma(f) = E$. Indeed, as $\langle e_{\gamma_n}, f \rangle = \frac{1}{n^2} \gamma_n$, the sequence $(\gamma_n)_{n \geq 1}$ is in $\sigma(f)$ (Proposition 3.3) so that $E \subseteq \sigma(f)$. The reverse inclusion being trivial, $\sigma(f) = E$. However, for instance, if $E$ is a perfect set of uniqueness,
it is not possible to find a function $f \in C_0(G)$ such that $\sigma(f) = E$. The perfect sets that can be realized as $\sigma(f)$ for some $f \in C_0(G)$ should have some special feature that this author does not completely comprehend.

**R4.** Let $f \in L^\infty(G)$ and put $I_f = \{a \in L^1(G): a \ast f = 0\}$. It is easy to see that $I_f \cap C_0(G) = \{0\}$ iff $I_f$ is $\sigma(M(G), C_0(G))$-dense in $M(G)$. We do not know whether there is a similar condition equivalent to the condition that $Z_f \cap C_0(G) = \{0\}$.

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**References**


