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Permanents by Möbius Inversion

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ABSTRACT. Möbius inversion techniques developed by Rota [1] are used to justify Ryser's calculation [2, 3] of the permanent of a matrix, and to establish an alternative method of calculation (Proposition 4).

Let A be a matrix with m rows, n columns, and $m \leqslant n$. For any subset E of the set C of columns of A, let $A \mid E$ be the $m \times \mid E \mid$ matrix formed by deleting all columns not in E. For any partition σ of the set R of rows of A, with rank $r(\sigma) = \mid R \mid - \mid \sigma \mid$, let A/σ be the $\mid \sigma \mid \times n$ matrix having a row for each part of σ, formed by multiplying, columnwise, all rows in that part. For example:

$$\begin{matrix} 1 & 0 & 4 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & -2 \end{matrix} \qquad \begin{matrix} 1 & 4 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -2 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 & -2 \\ 1 & 2 & 1 & 0 \\ & & & \end{matrix}$$

$$A \qquad\qquad A \mid \{c_1, c_3, c_4\} \qquad\qquad A/(r_1 r_3)(r_2)$$

Each function f from the set of rows to the set of columns of A has a *weight* $w(f) = \prod_i A(i, f(i))$. For any matrix A, let $\phi(a) = \sum_f w(f)$, summed over all functions $f \in C^R$, and let $p(A)$ be the permanent of A.

PROPOSITION 1. *$\phi(A)$ is equal to the product of the row-sums of A.*

PROOF: Each term in the expansion of the product of row-sums is the weight of a function $f \in C^R$, and conversely. Q.E.D.

PROPOSITION 2 (*Ryser*). *If $m = n$*

$$p(A) = \sum_{E \subseteq C} (-1)^{|C-E|} \phi(A \mid E).$$

PROOF: Each function $f \in C^R$ has some range $E \subseteq C$. $p(A \mid E)$ is the weight-sum for functions with range equal to E, while $\phi(A \mid E)$ is the weight-sum for functions with range contained in E. Thus

$$\phi(A \mid E) = \sum_{D;D\subseteq E} p(A \mid D)$$

and, by Möbius inversion on the Boolean algebra 2^C,

$$p(A \mid E) = \sum_{D;D\subseteq E} (-1)^{|E-D|} \phi(A \mid D).$$

Set $E = C$. Q.E.D.

If the matrix A is rectangular, with $m \leqslant n$, form a new matrix A' with $n - m$ additional rows of ones. Applying proposition 2 to A', we obtain:

PROPOSITION 3. *If* $m \leqslant n$,

$$p(A) = \frac{1}{(n-m)!} \sum_{E\subseteq C} (-1)^{|C-E|} \mid E \mid^{n-m} \phi(A \mid E).$$

If a partition σ has b_i parts of cardinality i, for $i = 1, 2, \ldots, m$, then

$$\mu(0, \sigma) = \prod_{i=1}^{m} (-1)^{b_i(i-1)} ((i-1)!)^{b_i} = (-1)^{r(\sigma)} \prod_{i=1}^{m} ((i-1)!)^{b_i}.$$

Q.E.D.

PROPOSITION 4.[1] $p(A) = \sum_\pi \mu(0, \pi) \phi(A/\pi)$, *the summation being over all partitions* π *of the set* R *of rows of* A.

PROOF: Each function from rows to columns has a partition of R as kernel. $\phi(A/\sigma)$ gives the weight-sum of functions with kernels having σ as a refinement. $p(A/\sigma)$ is the weight-sum for functions with kernel equal to σ. Thus

$$\phi(A/\sigma) = \sum_{\pi;\sigma\leqslant\pi} p(A/\pi)$$

and, by Möbius inversion on the lattice P of all partitions of the set R,

$$p(A/\sigma) = \sum_{\pi;\sigma\leqslant\pi} \mu(\sigma, \pi) \phi(A/\pi).$$

Set $\sigma = 0$, the discrete partition. Q.E.D.

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[1] In privately circulated notes, J. E. Graver has applied Proposition 4 to matrices of roots of unity, and remarks that he and W. Gustin jointly obtained the formula.

If J is the $n \times n$ matrix, each entry of which is equal to 1, and I is the $n \times n$ identity matrix, then $p(J - I)$ is the derangement number D_n. Proposition 4 gives the following identity:

$$D_n = \sum_{\sigma} (-1)^{r(\sigma)} \prod_{i=1}^{n} ((n - i)(i - 1)!)^{b_i}.$$

References

1. G.-C. Rota, On the Foundations of Combinatorial Theory, I, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **2** (1964), 340–368.
2. H. J. Ryser, *Combinatorial Mathematics*, Wiley, New York, 1963.
3. H. J. Ryser, Permanents and Systems of Distinct Representatives, Conference on Combinatorial Mathematics and Its Applications, Chapel Hill, N. C., April, 1967.