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Weighted pseudo almost periodic solutions of second-order neutral-delay differential equations with piecewise constant argument

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ABSTRACT

By introducing the method of decomposition of weighted pseudo almost periodic sequence, we present some existence theorems of weighted pseudo almost periodic solutions for second order neutral differential equations with piecewise constant argument of the form

$$\frac{d^2}{dt^2}(x(t)+px(t-1)) = qx\left(2\left[\frac{t+1}{2}\right]\right) + f(t),$$

where $|p| = 1$, $[\cdot]$ denotes the greatest integer function, q is a nonzero constant and $f(t)$ is weighted pseudo almost periodic. Our results are new and can be regarded as a complement of some known results even in the special cases of almost periodicity and pseudo almost periodicity.

1. Introduction

The differential equations with piecewise constant argument describe hybrid dynamical systems (a combination of continuous and discrete). These equations have the structure of continuous dynamical systems within intervals and the solution is continuous, and so combine properties of both differential and difference equations. They have applications in certain biomedical models and are similar in structure to those found in certain sequential continuous models of disease dynamics as treated by Busenberg and Cooke (see [1]). Therefore there are many papers concerning the differential equations with piecewise constant argument (see [2–10] and the references therein).

Meanwhile, Diagana [11] introduced the weighted pseudo almost periodic functions, which is a natural generalization of the classical pseudo almost periodic functions (see [12,13]), and has been used in the investigation of ordinary differential equations, partial differential equations and functional differential equations. For the results along this line, we refer the readers to [14–24] and the references therein.

In this paper, we consider the equation:

$$\frac{d^2}{dt^2}(x(t)+px(t-1)) = qx\left(2\left[\frac{t+1}{2}\right]\right) + f(t), \tag{1.1}$$

where $|p| = 1$, $q \neq 0$, $f : \mathbb{R} \to \mathbb{R}$, and $[\cdot]$ denotes the greatest integer function. For the case $|p| \neq 1$, some results on the existence and uniqueness of almost periodic, pseudo almost periodic or weighted pseudo almost periodic solutions for (1.1) were obtained in [2,7,10,24].

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The standard method to deal with the differential equations with piecewise constant argument such as (1.1) is always as follows. First, get the solution of the corresponding difference system which is given by a series in the form:

$$u(n) = \sum_{m\le n-1} \lambda^{n-m-1}k(m) \quad \text{or} \quad u(n) = -\sum_{m\ge n} \lambda^{n-m-1}k(m), \tag{1.2}$$

where λ is an eigenvalue of some matrix of the difference system. The convergence of the series is guaranteed by $|\lambda| \neq 1$ which was always assumed. Then construct the solutions of the differential equation inductively by

$$x(t) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} (-p)^n w(t-n), & |p| < 1, \\ \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{p^{n+1}} w(t+n+1), & |p| > 1, \end{cases} \tag{1.3}$$

where $|p| \neq 1$ and $w(t)$ is a function in term of $u(n)$ and $f(t)$ (see e.g. [2,24]). However, for the case when $|p| = 1$ and $|\lambda| = 1$, the problem becomes much different—the series in (1.2) and (1.3) may not convergent. This is the main difficult in the study of (1.1) for the case $|p| = 1$, and we have to find some other method to deal with this case.

A valid method – *decomposition of almost periodic sequence* – is introduced in [6,8] to study the following equation for the case $|p| = 1$:

$$\frac{d^2}{dt^2}(x(t) + px(t-1)) = qx([t]) + f(t).$$

Motivated by this decomposition, we introduce the *decomposition of weighted pseudo almost periodic sequence* in this paper, which is a generalization of the decomposition of almost periodic sequence. We note that the decomposition of the weighted ergodic perturbation of the weighted pseudo almost periodic sequence is "harder" than the decomposition of almost periodic sequence (see Remark 3.1(iii) and Example 6.1). By using this decomposition method, some theorems on the existence and uniqueness of weighted pseudo almost periodic solutions for (1.1) are presented (see Theorems 3.1 and 3.2), which are new and can be regarded as a complement of some known results even in the special cases of almost periodicity and pseudo almost periodicity (see Remark 3.1(i) and (ii)).

The paper is organized as follows. In Section 2, some notation and preliminary results are presented. In Section 3, we state the main results (Theorems 3.1 and 3.2), and give two auxiliary theorems (Theorems 3.3 and 3.4) which imply the main results. Then we give the proofs of these two auxiliary theorems in Sections 4 and 5 respectively. At last, an example is presented in Section 6 to illustrate our main results.

2. Preliminaries

Throughout this paper, we always assume that $|p| = 1, q \neq 0$, and denote by $\mathbb{E}^N$ the N-dimensional Euclidean space $\mathbb{R}^N$ or $\mathbb{C}^N$ endowed with Euclidean norm $|\cdot|$. Let $BC(\mathbb{R}, \mathbb{E}^N)$ be the space of bounded continuous functions $u : \mathbb{R} \to \mathbb{E}^N$. $BC(\mathbb{R}, \mathbb{E}^N)$ equipped with the sup norm defined by $\|u\| = \sup_{t\in\mathbb{R}} |u(t)|$ is a Banach space. Furthermore, $C(\mathbb{R}, \mathbb{E}^N)$ denotes the space of continuous functions from $\mathbb{R}$ to $\mathbb{E}^N$.

2.1. Weighted pseudo almost periodic function

Let U be the collection of functions (weights) $\rho : \mathbb{R} \to (0, +\infty)$, which are locally integrable over $\mathbb{R}$. If $\rho \in U$, we set

$$\mu(T, \rho) := \int_{-T}^{T} \rho(t)dt \quad \text{for } T > 0.$$

Denote

$$U_\infty := \left\{\rho \in U : \lim_{T\to\infty} \mu(T, \rho) = \infty\right\}$$

and

$$U_B := \left\{\rho \in U_\infty : \rho \text{ is bounded with } \inf_{t\in\mathbb{R}} \rho(t) > 0\right\}.$$

Let $\rho', \rho'' \in U_\infty$, ρ' is said to be equivalent to ρ'', denoting this as $\rho' \prec \rho''$, if $\rho'/\rho'' \in U_B$. Then '$\prec$' is a binary equivalence relation on U_∞ (see [11]). Let $\rho \in U_\infty,\ c \in \mathbb{R}$, define ρ_c by $\rho_c(t) = \rho(t + c)$ for $t \in \mathbb{R}$. We denote

$$U_T = \{\rho \in U_\infty : \rho \prec \rho_c \text{ for each } c \in \mathbb{R}\}.$$

It is easy to see that U_T contains plenty of weights, say, $1, e^t, 1 + 1/(1 + t^2), 1 + |t|^n$ with $n \in \mathbb{N}$, etc.

Definition 2.1 (*[25]*)**.** A set $S \subset \mathbb{R}$ is said to be relatively dense if there exists $L > 0$ such that $[a, a+L] \cap S \neq \emptyset$ for all $a \in \mathbb{R}$. A function $f \in C(\mathbb{R}, \mathbb{E}^N)$ is said to be almost periodic if the ε-translation set of f

$$T(f, \varepsilon) = \{\tau \in \mathbb{R} : |f(t+\tau) - f(t)| < \varepsilon \text{ for all } t \in \mathbb{R}\}$$

is relatively dense for each $\varepsilon > 0$. Denote by $AP(\mathbb{E}^N)$ the set of all such functions.

For $\rho \in U_\infty$, the *weighted ergodic* space $PAP_0(\mathbb{E}^N, \rho)$ is defined by

$$PAP_0(\mathbb{E}^N, \rho) := \left\{ f \in BC(\mathbb{R}, \mathbb{E}^N) : \lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} |f(t)|\rho(t)dt = 0 \right\}.$$

Definition 2.2 (*[11]*)**.** Let $\rho \in U_\infty$. A function $f \in BC(\mathbb{R}, \mathbb{E}^N)$ is called weighted pseudo almost periodic (or ρ-pseudo almost periodic) if it can be expressed as $f = f^{ap} + f^e$, where $f^{ap} \in AP(\mathbb{E}^N)$ and $f^e \in PAP_0(\mathbb{E}^N, \rho)$. Denote by $PAP(\mathbb{E}^N, \rho)$ the set of all such functions.

The functions f^{ap} and f^e in Definition 2.2 are called the *almost periodic* and the *weighted ergodic perturbation* components of f respectively. Moreover, the decomposition $f^{ap} + f^e$ of f is unique if $PAP_0(\mathbb{E}^N, \rho)$ is translation invariant (see [26]), and $PAP_0(\mathbb{E}^N, \rho)$ and $PAP(\mathbb{E}^N, \rho)$ are Banach spaces with the norm inherited from $BC(\mathbb{R}, \mathbb{E}^N)$ (see [15]).

2.2. *Weighted pseudo almost periodic sequence*

Definition 2.3 (*[25]*)**.** A sequence $x : \mathbb{Z} \to \mathbb{E}^N$ is called an almost periodic sequence if the ε-translation set of x

$$T(x, \varepsilon) = \{\tau \in \mathbb{Z} : |x(n+\tau) - x(n)| \leq \varepsilon \text{ for all } n \in \mathbb{Z}\}$$

is a relatively dense set for all $\varepsilon > 0$. τ is called the ε-period for x. Denote the set of all these sequences x by $APS(\mathbb{E}^N)$.

In the sequel, the vector $x(n) \in \mathbb{E}^N$ always means a column vector.
Let U_s denote the collection of sequences (weights) $\varrho : \mathbb{Z} \to (0, +\infty)$. For $\varrho \in U_s$ and $T \in \mathbb{Z}^+ = \{n \in \mathbb{Z} : n \geq 0\}$, set

$$\mu_s(T, \varrho) = \sum_{n=-T}^{T} \varrho(n).$$

Denote

$$U_{s\infty} := \left\{ \varrho \in U_s : \lim_{T\to\infty} \mu_s(T, \varrho) = \infty \right\},$$

and

$$U_{sB} := \left\{ \varrho \in U_{s\infty} : \varrho \text{ is bounded with } \inf_{n\in\mathbb{Z}} \varrho(n) > 0 \right\}.$$

Let $\varrho', \varrho'' \in U_{s\infty}$, ϱ' is said to be equivalent to ϱ'', denoting this as $\varrho' \prec \varrho''$, if $\{\varrho'(n)/\varrho''(n)\}_{n\in\mathbb{Z}} \in U_{sB}$. Then it is easy to see that '$\prec$' is a binary equivalence relation on $U_{s\infty}$. Let $\varrho \in U_{s\infty}$, $k \in \mathbb{Z}$, define ϱ_k by $\varrho_k(n) = \varrho(n+k)$ for $n \in \mathbb{Z}$. We denote

$$U_{sT} = \{\varrho \in U_{s\infty} : \varrho \prec \varrho_k \text{ for each } k \in \mathbb{Z}\}.$$

Definition 2.4. (i) Let $\varrho \in U_{s\infty}$. A sequence $x : \mathbb{Z} \to \mathbb{E}^N$ is said to be a $\varrho - PAP_0$ sequence if it is bounded and satisfies

$$\lim_{T\to\infty} \frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^{T} |x(n)|\varrho(n) = 0.$$

Denote the set of all such sequences x by $PAP_0S(\mathbb{E}^N, \varrho)$.

(ii) Let $\varrho \in U_{s\infty}$. A sequence $x : \mathbb{Z} \to \mathbb{E}^N$ is said to be a weighted pseudo almost periodic sequence (or a ϱ-pseudo almost periodic sequence) if x can be written as $x = x^{ap} + x^e$ with $x^{ap} \in APS(\mathbb{E}^N)$ and $x^e \in PAP_0S(\mathbb{E}^N, \varrho)$. x^{ap} and x^e are called almost periodic component and weighted ergodic perturbation, respectively, of sequence x. Denote the set of all such sequences x by $PAPS(\mathbb{E}^N, \varrho)$.

For more properties of $PAPS(\mathbb{R}^N, \varrho)$, we refer to [24], and the same properties for $PAPS(\mathbb{E}^N, \varrho)$ can be proved similarly. Notably, the decomposition $x^{ap} + x^e$ of x is unique for $\varrho \in U_{sT}$. The following two results were also given in [24], and we give the proofs here for the convenient of the readers.

Lemma 2.1. *Let $\rho \in U_T$, and denote*

$$\varrho(n) = \int_{2n-1}^{2n+1} \rho(t)dt \quad \text{for } n \in \mathbb{Z}. \tag{2.1}$$

Then $\varrho \in U_{sT}$. Moreover, given $c \in \mathbb{R}$, there exist positive constants C_1, C_2 such that, for sufficiently large T,

$$C_1\mu(T+c,\rho) \le \mu_s([T/2],\varrho) \le C_2\mu(T+c,\rho). \tag{2.2}$$

Proof. Without loss of generality, we assume that $c \ge 0$. Since $\rho \in U_T$, there exists $M > 0$ such that $\rho_{c+1}(t) \le M\rho(t)$ and $\rho_{-(c+1)}(t) \le M\rho(t)$ for $t \in \mathbb{R}$ and

$$\mu(T-1,\rho) \le \mu_s([T/2],\varrho) = \int_{-2[T/2]-1}^{2[T/2]+1} \rho(t)dt \le \mu(T+1+c,\rho). \tag{2.3}$$

For $T > c+2$, i.e., $-T+2c+3 < T-1$, we have

$$\begin{aligned}\mu(T+c,\rho) &= \int_{-T-c}^{T+c} \rho(t)dt = \int_{-T-2c-1}^{T-1} \rho_{c+1}(t)dt \\ &= \int_{-T+1}^{T-1} \rho_{c+1}(t)dt + \int_{-T-2c-1}^{-T+1} \rho_{c+1}(t)dt \\ &= \int_{-T+1}^{T-1} \rho_{c+1}(t)dt + \int_{-T+1}^{-T+2c+3} \rho_{-(c+1)}(t)dt \\ &\le \int_{-T+1}^{T-1} M\rho(t)dt + \int_{-T+1}^{T-1} M\rho(t)dt = 2M\mu(T-1,\rho).\end{aligned} \tag{2.4}$$

Similarly, we can prove that there exists $M' > 0$ such that, for T large enough,

$$\mu(T+1+c,\rho) \le M'\mu(T+c,\rho). \tag{2.5}$$

Thus by (2.3)–(2.5) we have, for T large enough,

$$\frac{1}{2M}\mu(T+c,\rho) \le \mu_s([T/2],\varrho) \le M'\mu(T+c,\rho).$$

This leads to (2.2), and from which we can get easily that $\varrho \in U_{sT}$. The proof is complete. □

Proposition 2.1. *$PAP_0S(\mathbb{E}^N,\varrho)$ with $\varrho \in U_{sT}$ is translation invariant.*

Proof. Let $x \in PAP_0S(\mathbb{E}^N,\varrho)$ and $k \in \mathbb{Z}$. Without loss of generality, we assume that $k > 0$. Then there exists $M > 0$ such that $\varrho_k(n)/\varrho(n) < M$ for $n \in \mathbb{Z}$ since $\varrho \in U_{sT}$. Let $\rho(t) = \varrho(n)/2$ for $t \in [2n-1, 2n+1)$, $n \in \mathbb{Z}$. Then $\rho \in U_T$ and $\varrho(n) = \int_{2n-1}^{2n+1} \rho(t)dt$ for $n \in \mathbb{Z}$. Now applying Lemma 2.1 we can get that

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{\mu_s(T,\varrho)} \sum_{n=-T}^{T} |x(n-k)|\varrho(n) &\le \lim_{T\to\infty} \frac{1}{\mu_s(T,\varrho)} \sum_{n=-(T+k)}^{T+k} |x(n)|\varrho_k(n) \\ &\le \lim_{T\to\infty} \frac{\mu_s(T+k,\varrho)}{\mu_s(T,\varrho)} \cdot \frac{1}{\mu_s(T+k,\varrho)} \sum_{n=-(T+k)}^{T+k} |x(n)|M\varrho(n) = 0.\end{aligned}$$

This implies that $\{x(n-k)\}_{n\in\mathbb{Z}} \in PAP_0S(\mathbb{E}^N,\varrho)$. The proof is complete. □

Let $\varrho \in U_{s\infty}$, and $2^{PAPS(\mathbb{E}^N,\varrho)} = \{U : U \subset PAPS(\mathbb{E}^N,\varrho)\}$. For the *decomposition of weighted pseudo almost periodic sequence*, we define functions $D_\gamma : PAPS(\mathbb{E}^N,\varrho) \to 2^{PAPS(\mathbb{E}^N,\varrho)}$ for $\gamma = 1$ and -1 by

$$D_\gamma\{a_n\} = \{\{b_n\} \in PAPS(\mathbb{E}^N,\varrho) : a_n = b_{n+1} + \gamma b_n,\ n \in \mathbb{Z}\}$$

for $\{a_n\} \in PAPS(\mathbb{E}^N,\varrho)$. Clearly, we have $D_\gamma\{0\} \ne \emptyset$. We note that

$$\alpha U + \beta V = \{\{c_n\} : c_n = \alpha a_n + \beta b_n,\ n \in \mathbb{Z},\ \{a_n\} \in U,\ \{b_n\} \in V\}$$

for $\alpha, \beta \in \mathbb{E}$, $U, V \in 2^{PAPS(\mathbb{E}^N,\varrho)}$. Let $\{a_n\} = \{(a_{1,n}, a_{2,n}, \ldots, a_{N,n})^T\} \in PAPS(\mathbb{E}^N,\varrho)$, then it is clear that $D_\gamma\{a_n\} \ne \emptyset$ if and only if $D_\gamma\{a_{i,n}\} \ne \emptyset$, $i = 1, 2, \ldots, N$.

Proposition 2.2. *Let $\varrho \in U_{s\infty}$, $\{a_n\}, \{b_n\} \in PAPS(\mathbb{E}^N, \varrho)$. Then the following statements hold:*

(i) $D_\gamma\{\alpha a_n\} = \alpha D_\gamma\{a_n\}$, $D_\gamma\{a_n\} + D_\gamma\{b_n\} \subset D_\gamma\{a_n + b_n\}$ *for* $\alpha \in \mathbb{E} \setminus \{0\}$.
(ii) $D_\gamma\{a_n\} \neq \emptyset$ *implies that* $D_\gamma\{Aa_n\} \neq \emptyset$ *for any matrix* $A \in \mathbb{E}^{N\times N}$.
(iii) $D_\gamma\{(-\gamma)^n c\} = \emptyset$ *for* $c \in \mathbb{E}^N \setminus \{0\}$.
(iv) *If* $\{b_n\} \in D_\gamma\{a_n\},\ k \in \mathbb{N}$,

$$D_\gamma\{a_n\} = \{\{b_n + (-\gamma)^n c\} : c \in \mathbb{E}^N\}. \tag{2.6}$$

Furthermore, there is at most one $\{b_n\} \in D_\gamma\{a_n\}$ *such that* $D_\gamma\{b_n\} \neq \emptyset$.

Proof. The statement (i) can be verified easily by the definition of D_γ, and (ii) follows from (i) directly. If $\gamma = 1$, (iii) and (iv) can be proved by an argument similar to the proof of [6, Proposition 2.1(ii), (iii)], and we omit the details. For the case $\gamma = -1$, we give the proof of (iii) and (iv) as follows.

(iii) Suppose the contrary that some $\{b_n\} \in D_{-1}\{c\}$. Then $c = b_{n+1} - b_n,\ n \in \mathbb{Z}$, and we get that $b_n = nc + b_0,\ n \in \mathbb{Z}$, which contradicts the boundedness of $\{b_n\}$. So (iii) holds.

(iv) Since $\{b_n\} \in D_{-1}\{a_n\}$, it is easy to see that $\{b_n + c\} \in D_{-1}\{a_n\}$ for any $c \in \mathbb{E}^N$. Let $\{c_n\} \in D_{-1}\{a_n\}$. Then $a_n = b_{n+1} - b_n = c_{n+1} - c_n$ for $n \in \mathbb{Z}$. This implies that $c_n = b_n + (c_0 - b_0),\ n \in \mathbb{Z}$, that is $\{c_n\} \in \{\{b_n + c\} : c \in \mathbb{E}^N\}$, and (2.6) is true. If $\{b_n\} \in D_{-1}\{a_n\}$ such that $D_{-1}\{b_n\} \neq \emptyset$, we have $D_{-1}\{a_n\} = \{\{b_n + c\} : c \in \mathbb{E}^N\}$. Suppose that $D_{-1}\{b_n + c\} \neq \emptyset$ for some $c \neq 0$, we get from (i) that $\emptyset \neq D_{-1}\{b_n + c\} - D_{-1}\{b_n\} \subset D_{-1}\{c\}$. This contradicts (iii). So (iv) holds. The proof is complete. □

3. The main results

In the sequel, we always assume that $f \in PAP(\mathbb{R}, \rho),\ \rho \in U_T$ and $\varrho(n)$ is given by (2.1). By a solution $x(t)$ of (1.1) on $\mathbb{R}$ we mean a function continuous on $\mathbb{R}$, satisfying (1.1) for all $t \in \mathbb{R},\ t \neq 2n+1$, and such that the one sided second derivatives of $x(t) + px(t-1)$ exist at $2n+1,\ n \in \mathbb{Z}$.

As in [2], let

$$\begin{cases} f_n^{(1)} = \displaystyle\int_n^{n+1}\int_n^s f(\sigma)d\sigma ds, \qquad f_n^{(2)} = \int_n^{n-1}\int_n^s f(\sigma)d\sigma ds, \qquad g_n = \int_n^{n+2} f(t)dt, \\ h_n^{(1)} = f_n^{(1)} + f_n^{(2)}, \qquad h_n^{(2)} = g_n + f_n^{(2)} - f_{n+2}^{(2)}, \end{cases} \tag{3.1}$$

and consider the following difference equations (see (11) and (12) in [2]):

$$\begin{cases} x_{2n+1} + (p-q-2)x_{2n} + (1-2p)x_{2n-1} + px_{2n-2} = h_{2n}^{(1)}, \\ \left(1 - \dfrac{q}{2}\right)x_{2n+2} + (p-1)x_{2n+1} - \left(1 + p + \dfrac{3q}{2}\right)x_{2n} - (p-1)x_{2n-1} + px_{2n-2} = h_{2n}^{(2)}. \end{cases} \tag{3.2}$$

We rewrite (3.2) as the following form:

$$y(n+1) = A_1 y(n) + l(n), \quad \text{if } q \neq 2, \tag{3.3a}$$

$$z(n+1) = A_2 z(n) + v(n), \quad \text{if } q = 2, \tag{3.3b}$$

where $y(n) = (x_{2n}, x_{2n-1}, x_{2n-2})^T$, $z(n) = (x_{2n-1}, x_{2n-2})^T$ for $n \in \mathbb{Z}$,

$$A_1 = \begin{pmatrix} \dfrac{5q - 2pq - 4p + 8}{2-q} & \dfrac{8p-8}{2-q} & \dfrac{2}{2-q} \\ q + 2 - p & 2p - 1 & -p \\ 1 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \dfrac{2p+3}{9-4p} & \dfrac{2}{4p-9} \\ \dfrac{4-4p}{9-4p} & \dfrac{2p-1}{9-4p} \end{pmatrix}, \tag{3.4}$$

$$l(n) = \begin{pmatrix} \dfrac{2}{2-q}h_{2n}^{(2)} + \dfrac{2-2p}{2-q}h_{2n}^{(1)} \\ h_{2n}^{(1)} \\ 0 \end{pmatrix}, \qquad v(n) = \begin{pmatrix} \dfrac{p+4}{9-4p}h_{2n}^{(1)} + \dfrac{4-p}{4p-9}h_{2n}^{(2)} \\ \dfrac{p-1}{9-4p}h_{2n}^{(1)} + \dfrac{1}{4p-9}h_{2n}^{(2)} \end{pmatrix}.$$

The following assumptions will be used later:

(H_1) $D_{-1}^2\{g_{2n}\} \neq \emptyset$ and $D_{-1}^2\{f_{2n}^{(i)}\} \neq \emptyset,\ i = 1, 2$.
(H_2) $D_{-1}D_1\{g_{2n}\} \neq \emptyset$ and $D_{-1}D_1\{f_{2n}^{(i)}\} \neq \emptyset,\ i = 1, 2$.
(H_3) $D_{-1}\{g_{2n}\} \neq \emptyset$ and $D_{-1}\{f_{2n}^{(i)}\} \neq \emptyset,\ i = 1, 2$.
(H_4) There exists $\bar{f} \in PAP(\mathbb{R}, \rho)$ such that $\{\bar{f}(2n+\eta)\} \in D_{-1}\{f(2n+\eta)\}$ for $\eta \in [-1, 1]$.

Now we state the main results in this paper.

Theorem 3.1. *Assume that* $p = -1$ *and* (H_4) *holds. Then the following statements are true.*

(i) *If* $q \neq -4$, (3.2) *has a unique real solution* $\{x_n\}$ *such that* $D_{-1}\{x_{2n}\} \neq \emptyset$ *and* $D_{-1}\{x_{2n-1}\} \neq \emptyset$, *and the following statement* (Q) *holds:*

(Q) *For any* $\varphi(t)$ *continuous on* $[0, 1]$ *with* $\varphi(0) = x_0$ *and* $\varphi(1) = x_1$, (1.1) *has a unique solution* $x(t) \in PAP(\mathbb{R}, \rho)$ *such that* $x(t) = \varphi(t),\ t \in [0, 1]$ *and* $x(n) = x_n,\ n \in \mathbb{Z}$.

(ii) *If* $q = -4$ *and* (H_2) *holds, then* (3.2) *has a real solution* $\{x_n\}$ *such that* $D_{-1}\{x_{2n}\} \neq \emptyset$ *and* $D_{-1}\{x_{2n-1}\} \neq \emptyset$, *and statement* (Q) *is true. Furthermore,*

$$S = \left\{\{\bar{y}(n)\} : \bar{y}(n) = (\bar{x}_{2n}, \bar{x}_{2n-1}, \bar{x}_{2n-2})^T = (x_{2n}, x_{2n-1}, x_{2n-2})^T + A_1^n C \text{ with } C = c(1, -1, -1)^T,\ c \in \mathbb{R}\right\}$$

is the set of all real solutions of (3.3a) *such that* $D_{-1}\{\bar{y}(n)\} \neq \emptyset$, *and statement* (Q) *is true for* $\bar{x}_n$ *in place of* x_n.

Theorem 3.2. *Assume that* $p = 1$, (H_1) *and* (H_4) *hold. Then statement* (i) *in Theorem* 3.1 *is true.*

Remark 3.1. (i) *Pseudo almost periodic case*: If $\rho = 1$, Theorems 3.1 and 3.2 are results for the case of pseudo almost periodicity, and can be regarded as a complement of the results in [2], where $|p| \neq 1$.

(ii) *Almost periodic case*: By an argument similar to the proof of Theorems 3.1 and 3.2, we can get similar results for the special case of almost periodicity, i.e., all the assumptions and conclusions are presented on almost periodicity, and this is a complement of the results in [7,10], where $|p| \neq 1$.

(iii) *An open question*: If conditions (H_1) and (H_2) are satisfied simultaneously, under the assumptions of Theorem 3.2, we can prove similarly that statement (ii) of Theorem 3.1 also holds for the case $p = 1$ with S replaced by

$$\bar{S} = \left\{\{\bar{y}(n)\} : \bar{y}(n) = (\bar{x}_{2n}, \bar{x}_{2n-1}, \bar{x}_{2n-2})^T = (x_{2n}, x_{2n-1}, x_{2n-2})^T + A_1^n C \text{ with } C = c(1, 1, -1)^T,\ c \in \mathbb{R}\right\}.$$

Unfortunately, it is still an open question whether there exists a function $f \in PAP(\mathbb{R}, \rho)$ with $f^e \neq 0$ such that (H_1) and (H_2) hold simultaneously. However, in the special case of almost periodicity, by [6, Proposition 2.2] and [8, Proposition 1.1], many functions satisfy (H_1) and (H_2) simultaneously (see also the almost periodic component of f in Example 6.1).

To prove Theorems 3.1 and 3.2, we first give the following lemma.

Lemma 3.1. *For* $\eta \in [-1, 1]$, *let* $\psi_n(\eta) = \int_0^\eta \int_0^s f(2n + \sigma) d\sigma ds,\ n \in \mathbb{Z}$. *Then* $\{\psi_n(\eta)\} \in PAPS(\mathbb{R}, \varrho)$.

Proof. Clearly, $\{f^{ap}(2n + \sigma)\}_n \in APS(\mathbb{R})$ and $f^{ap}(2n + \sigma)$ is uniformly continuous in $\sigma \in [-1, 1]$ uniformly in $n \in \mathbb{Z}$. So it is easy to get that $\{\{f^{ap}(2n + \sigma)\} : \sigma \in [-1, 1]\}$ is uniform almost periodic, that is

$$T(f^{ap}, \varepsilon) = \{\tau \in \mathbb{Z} : |f^{ap}(2(n + \tau) + \sigma) - f^{ap}(2n + \sigma)| \le \varepsilon,\ n \in \mathbb{Z},\ \sigma \in [-1, 1]\}$$

is relatively dense for all $\varepsilon > 0$. Then it is easy to verify that

$$\psi_n^{ap}(\eta) = \int_0^\eta \int_0^s f^{ap}(2n + \sigma) d\sigma ds, \quad n \in \mathbb{Z}$$

is almost periodic for each $\eta \in [-1, 1]$. Let $\psi_n^e = \psi_n - \psi_n^{ap},\ n \in \mathbb{Z}$. For $T \in \mathbb{Z}^+$, we get

$$\begin{aligned}
\sum_{n=-T}^{T} |\psi_n^e(\eta)|\varrho(n) &= \sum_{n=-T}^{T} \left| \int_{2n}^{2n+\eta} \int_{2n}^{s} f^e(\sigma) d\sigma ds \right| \int_{2n-1}^{2n+1} \rho(t) dt \\
&\le \sum_{n=-T}^{T} \int_{2n-1}^{2n+1} |f^e(\sigma)| d\sigma \int_{2n-1}^{2n+1} \rho(t) dt \\
&= \sum_{n=-T}^{T} \int_{2n-1}^{2n+1} \int_{2n-1-t}^{2n+1-t} |f^e(\sigma + t)| \rho(t) d\sigma dt \\
&= \sum_{n=-T}^{T} \left(\int_{-2}^{0} \int_{2n-1-\sigma}^{2n+1} + \int_{0}^{2} \int_{2n-1}^{2n+1-\sigma} \right) |f^e(\sigma + t)| \rho(t) dt d\sigma \\
&\le \sum_{n=-T}^{T} \int_{-2}^{2} \int_{2n-1}^{2n+1} |f^e(\sigma + t)| \rho(t) dt d\sigma \\
&= \int_{-2}^{2} \int_{-2T-1}^{2T+1} |f^e(\sigma + t)| \rho(t) dt d\sigma.
\end{aligned}$$

For $T \in \mathbb{Z}^+$, $\sigma \in [-2, 2]$, let

$$\Phi_T(\sigma) = \frac{1}{\mu(2T+1, \rho)} \int_{-2T-1}^{2T+1} |f^e(\sigma + t)|\rho(t)dt.$$

Then $|\Phi_T(\sigma)| \le \|f^e\|$. From Proposition 2.1, we get that $\lim_{T\to\infty} \Phi_T(\sigma) = 0$ for each $\sigma \in [-2, 2]$. Now by Lebesgue dominated convergence theorem and Lemma 2.1,

$$\lim_{T\to\infty} \frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^{T} |\psi_n^e(\eta)|\varrho(n) \le \lim_{T\to\infty} \frac{\mu(2T+1, \rho)}{\mu_s(T, \varrho)} \int_{-2}^{2} \Phi_T(\sigma)d\sigma = 0,$$

which implies that $\{\psi_n^e(\eta)\} \in PAP_0S(\mathbb{R}, \varrho)$ for each $\eta \in [-1, 1]$. This completes the proof. □

Remark 3.2. (i) Since $f_{2n}^{(1)} = \psi_n(1)$ and $f_{2n}^{(2)} = \psi_n(-1)$, $n \in \mathbb{Z}$, we have $\{f_{2n}^{(i)}\} \in PAPS(\mathbb{R}, \varrho)$, $i = 1, 2$. Moreover, we can prove that $\{g_{2n}\} \in PAPS(\mathbb{R}, \varrho)$ by an argument similar to the proof of Lemma 3.1. Then $\{h_{2n}^{(i)}\} \in PAPS(\mathbb{R}, \varrho)$, $i = 1, 2$, and consequently $\{l(n)\} \in PAPS(\mathbb{R}^3, \varrho)$, $\{v(n)\} \in PAPS(\mathbb{R}^2, \varrho)$.

(ii) We note that (H_4) implies (H_3). In fact, let $\bar{\psi}_n(\eta) = \int_0^\eta \int_0^s \bar{f}(2n + \sigma)d\sigma ds$, $n \in \mathbb{Z}$, $\eta \in [-1, 1]$. Then $\{\bar{\psi}_n(\eta)\} \in PAPS(\mathbb{R}, \varrho)$ by (H_4) and Lemma 3.1. Moreover, it is easy to verify that $\bar{\psi}_{n+1}(\eta) - \bar{\psi}_n(\eta) = \psi_n(\eta)$ for $n \in \mathbb{Z}$ by (H_4). This means that $\{\bar{\psi}_n(\eta)\} \in D_{-1}\{\psi_n(\eta)\}$, and consequently, $D_{-1}\{f_{2n}^{(i)}\} \neq \emptyset$, $i = 1, 2$. Similarly, we can prove that $D_{-1}\{g_{2n}\} \neq \emptyset$. That is ($H_3$) holds.

For the relation between the difference system (3.2) and the differential equation (1.1), we have the following result.

Theorem 3.3. *Assume that* (H_4) *holds and* $\{x_n\}$ *is a real solution of* (3.2) *such that* $D_{-1}\{x_{2n}\} \neq \emptyset$ *and* $D_{-1}\{x_{2n-1}\} \neq \emptyset$. *Then statement* (Q) *in Theorem 3.1 is true.*

For the difference system (3.2) or equivalently, (3.3a) and (3.3b), we have the following result.

Theorem 3.4. (i) *Assume that* $p = -1$ *and* (H_3) *holds. If* $q \neq -4$, (3.2) *has a unique real solution* $\{x_n\}$ *such that* $D_{-1}\{x_{2n}\} \neq \emptyset$ *and* $D_{-1}\{x_{2n-1}\} \neq \emptyset$. *If* $q = -4$ *and* (H_2) *holds,* (3.3a) *has a real solution* $\{y(n)\}$ *such that* $D_{-1}\{y(n)\} \neq \emptyset$. *Furthermore, the set* S *given in Theorem 3.1*(ii) *is the set of all real solutions of* (3.3a) *such that* $D_{-1}\{\bar{y}(n)\} \neq \emptyset$.

(ii) *Assume that* $p = 1, q \neq -4$ *and* (H_1) *holds. Then* (3.2) *has a unique real solution* $\{x_n\}$ *such that* $D_{-1}\{x_{2n}\} \neq \emptyset$ *and* $D_{-1}\{x_{2n-1}\} \neq \emptyset$.

It is clear that the main results Theorems 3.1 and 3.2 follow from Theorems 3.3 and 3.4 directly by the fact that assumption (H_4) implies (H_3). So we need only to prove Theorems 3.3 and 3.4, and this will be done in the next two sections.

4. Proof of Theorem 3.3

Proof of Theorem 3.3. Let $\{x_n\}$ be as in Theorem 3.3. Let

$$w(t) = x_{2n} + px_{2n-1} + A_{2n}(t - 2n) + \frac{q}{2}x_{2n}(t - 2n)^2 + \psi_n(t - 2n), \tag{4.1}$$

for $t \in [2n - 1, 2n + 1)$, $n \in \mathbb{Z}$, with $A_n = (1 + q/2)x_n + (p - 1)x_{n-1} - px_{n-2} + f_n^{(2)}$ and ψ_n given by Lemma 3.1. Then by the same way as the proof of [2, Lemma 14], we can get that $w(t)$ is continuously differentiable in $\mathbb{R}$. Let $x(t) = \varphi(t)$ for $t \in [0, 1]$, and define $x(t)$ inductively in $\mathbb{R}$ by

$$x(t) = \begin{cases} (w(t+1) - x(t+1))/p, & t \in [-n, -n+1),\ n = 1, 2, \ldots, \\ w(t) - px(t-1), & t \in (n, n+1],\ n = 1, 2, \ldots. \end{cases}$$

Then $x(t)$ is a solution of (1.1) with $x(n) = x_n$ and

$$w(t) = x(t) + px(t - 1) \quad \text{for } t \in \mathbb{R}. \tag{4.2}$$

Now we need only to prove that $x \in PAP(\mathbb{R}, \rho)$. Let $\bar{\psi}_n(\eta) = \int_0^\eta \int_0^s \bar{f}(2n+\sigma)d\sigma$, $\bar{g}_{2n} = \int_{2n}^{2n+2} \bar{f}(t)dt$ for each $\eta \in [-1, 1]$ with $\bar{f}$ given in (H_4), and $\bar{f}_{2n}^{(1)} = \bar{\psi}_n(1)$, $\bar{f}_{2n}^{(2)} = \bar{\psi}_n(-1)$. Then $\{\bar{f}_{2n}^{(i)}\} \in D_{-1}\{f_{2n}^{(i)}\}$, $i = 1, 2$, $\bar{g}_{2n} \in D_{-1}\{g_{2n}\}$ and $\bar{\psi}_n(\eta) \in D_{-1}\{\psi_n(\eta)\}$ for $\eta \in [-1, 1]$, and by Proposition 2.2(i) we have $\{\bar{h}_{2n}^{(1)}\} = \{\bar{f}_{2n}^{(1)} + \bar{f}_{2n}^{(2)}\} \in D_{-1}\{h_{2n}^{(1)}\}$ and $\{\bar{h}_{2n}^{(2)}\} = \{\bar{g}_{2n} + \bar{f}_{2n}^{(2)} + \bar{f}_{2n+2}^{(2)}\} \in D_{-1}\{h_{2n}^{(2)}\}$. Moreover, again by Proposition 2.2(i), we can choose $\{\bar{x}_{2n}\} \in D_{-1}\{x_{2n}\}$, $\{\bar{x}_{2n-1}\} \in D_{-1}\{x_{2n-1}\}$, $\bar{A}_{2n} \in D_{-1}\{A_{2n}\}$ such that (3.2) holds with x_n and $h_{2n}^{(i)}$, $i = 1, 2$ replaced by $\bar{x}_n$ and $\bar{h}_{2n}^{(i)}$, $i = 1, 2$. Let

$$\bar{w}(t) = \bar{x}_{2n} + p\bar{x}_{2n-1} + \bar{A}_{2n}(t - 2n) + \frac{q}{2}\bar{x}_{2n}(t - 2n)^2 + \bar{\psi}_n(t - 2n), \tag{4.3}$$

$$\bar{w}^{ap}(t) = \bar{x}_{2n}^{ap} + p\bar{x}_{2n-1}^{ap} + \bar{A}_{2n}^{ap}(t - 2n) + \frac{q}{2}\bar{x}_{2n}^{ap}(t - 2n)^2 + \bar{\psi}_n^{ap}(t - 2n) \tag{4.4}$$

for $t \in [2n-1, 2n+1),\ n \in \mathbb{Z}$, where $\bar{\psi}_n^{ap}(t-2n) = \int_0^{t-2n} \int_0^s \bar{f}^{ap}(2n+\sigma)d\sigma$. Then by an argument the same as that to prove $w \in BC(\mathbb{R}, \mathbb{R})$ (see [2, Lemma 14]), we can prove that

$$\bar{w}, \bar{w}^{ap} \in BC(\mathbb{R}, \mathbb{R}). \tag{4.5}$$

Meanwhile, by (4.3) and Proposition 2.2(i),

$$\{\bar{w}(2n-1+\delta)\} \in D_{-1}\{w(2n-1+\delta)\}, \qquad \{\bar{w}(2n+\delta)\} \in D_{-1}\{w(2n+\delta)\}. \tag{4.6}$$

Now we complete the proof by the following 3 steps.

Step 1. Let $\delta \in [0, 1)$, we prove that

$$\{x(2n-1+\delta)\}, \qquad \{x(2n+\delta)\} \in PAPS(\mathbb{R}, \varrho). \tag{4.7}$$

From (4.2) we get $w(t) - pw(t-1) = x(t) - x(t-2)$ for $t \in \mathbb{R}$. Then

$$\begin{aligned} x(2n-1+\delta) - x(2n-3+\delta) &= w(2n-1+\delta) - pw(2n-2+\delta) \\ &= \bar{w}(2(n+1)-1+\delta) - p\bar{w}(2n+\delta) - (\bar{w}(2n-1+\delta) - p\bar{w}(2(n-1)+\delta)) \end{aligned}$$

for $n \in \mathbb{Z}$. This implies that

$$\begin{aligned} &x(2n-1+\delta) = \bar{w}(2(n+1)-1+\delta) - p\bar{w}(2n+\delta) + C_1 \text{ with} \\ &x^{ap}(2n-1+\delta) = \bar{w}^{ap}(2(n+1)-1+\delta) - p\bar{w}^{ap}(2n+\delta) + C_1, \quad n \in \mathbb{Z}, \end{aligned} \tag{4.8}$$

where $C_1 = x(\delta+1) - \bar{w}(3+\delta) + p\bar{w}(2+\delta)$. This together with (4.6) follows that $\{x(2n-1+\delta)\} \in PAPS(\mathbb{R}, \varrho)$. Similarly, we can get

$$\begin{aligned} &x(2n+\delta) = \bar{w}(2(n+1)+\delta) - p\bar{w}(2(n+1)-1+\delta) + C_2 \text{ with} \\ &x^{ap}(2n+\delta) = \bar{w}^{ap}(2(n+1)+\delta) - p\bar{w}^{ap}(2(n+1)-1+\delta) + C_2, \quad n \in \mathbb{Z}, \end{aligned} \tag{4.9}$$

where $C_2 = \varphi(\delta) - \bar{w}(2+\delta) + p\bar{w}(1+\delta)$. This together with (4.6) yields that $\{x(2n+\delta)\} \in PAPS(\mathbb{R}, \varrho)$. Then (4.7) is true.

Step 2. For $\delta \in [0, 1)$, let $x_1 : \mathbb{R} \to \mathbb{R}$ be given by

$$x_1(t) = \begin{cases} x^{ap}(2n-1+\delta), & t = 2n-1+\delta, \\ x^{ap}(2n+\delta), & t = 2n+\delta \end{cases}$$

for $n \in \mathbb{Z}$. We prove that $x_1 \in AP(\mathbb{R})$. By (4.4) we can get easily that $\bar{w}^{ap}(2n-1+\delta)$, $\bar{w}^{ap}(2n+\delta)$ are uniformly continuous in $\delta \in [0, 1)$ uniformly in $n \in \mathbb{Z}$. Then it follows from (4.5), (4.8) and (4.9), that $x_1 \in BC(\mathbb{R}, \mathbb{R})$ and $x^{ap}(2n-1+\delta)$, $x^{ap}(2n+\delta)$ are uniformly continuous in $\delta \in [0, 1)$ uniformly in $n \in \mathbb{Z}$. Thus given $\varepsilon > 0$, we can choose $\delta_1, \delta_2, \ldots, \delta_m \in [0, 1)$ satisfying that, for each $\delta \in [0, 1)$, there exist some $1 \le i, j \le m$, such that

$$|x^{ap}(2n-1+\delta) - x^{ap}(2n-1+\delta_i)| < \frac{\varepsilon}{3}, \quad n \in \mathbb{Z}, \tag{4.10}$$

$$|x^{ap}(2n+\delta) - x^{ap}(2n+\delta_j)| < \frac{\varepsilon}{3}, \quad n \in \mathbb{Z}. \tag{4.11}$$

Let $G = \{\{x^{ap}(2n-1+\delta_i)\}_n, \{x^{ap}(2n+\delta_i)\}_n : 1 \le i \le m\}$. Clearly, G is uniformly almost periodic since it is a finite set. Let $2T(G, \varepsilon/3) = \{2\tau : \tau \in T(G, \varepsilon/3)\}$. Then $2T(G, \varepsilon/3)$ is relatively dense. For $\tau \in 2T(G, \varepsilon/3),\ t \in \mathbb{R}$. If $[t] = 2n-1$ for some $n \in \mathbb{Z}$, let $\delta = t - [t]$, and by (4.10),

$$\begin{aligned} |x_1(t+\tau) - x_1(t)| &= |x^{ap}(2n-1+\delta+\tau) - x^{ap}(2n-1+\delta)| \\ &\le |x^{ap}(2(n+\tau/2)-1+\delta) - x^{ap}(2(n+\tau/2)-1+\delta_i)| + |x^{ap}(2(n+\tau/2)-1+\delta_i) \\ &\quad - x^{ap}(2n-1+\delta_i)| + |x^{ap}(2n-1+\delta_i) - x^{ap}(2n-1+\delta)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Similarly, if $[t] = 2n$ for some $n \in \mathbb{Z}$, by (4.11) we can get

$$|x_1(t+\tau) - x_1(t)| = |x^{ap}(2n+\delta+\tau) - x^{ap}(2n+\delta)| < \varepsilon.$$

Hence $2T(G, \varepsilon/3) \subset T(x^{ap}, \varepsilon)$. So $T(x^{ap}, \varepsilon)$ is relatively dense, and $x_1 \in AP(\mathbb{R})$.

Step 3. For $\delta \in [0, 1),\ n \in \mathbb{Z}$, let

$$x_2(t) = x(t) - x_1(t) = \begin{cases} x^e(2n-1+\delta), & t = 2n-1+\delta, \\ x^e(2n+\delta), & t = 2n+\delta \end{cases}$$

for $n \in \mathbb{Z}$. Then $x_2 \in BC(\mathbb{R}, \mathbb{R})$. We prove that $x_2 \in PAP_0(\mathbb{R}, \rho)$. By the same argument in the proof of (4.10) and (4.11), we can choose $\delta_1, \delta_2, \ldots, \delta_m \in [0, 1)$ satisfying that, for each $t \in [0, 1)$, there exist some $1 \le i, j \le m$, such that

$$|x^e(2n-1+t) - x^e(2n-1+\delta_i)| < \frac{\varepsilon}{3}, \quad n \in \mathbb{Z}, \tag{4.12}$$

$$|x^e(2n+t) - x^e(2n+\delta_j)| < \frac{\varepsilon}{3}, \quad n \in \mathbb{Z}. \tag{4.13}$$

Set

$$\mathcal{O}_i^1 = \{t \in [0, 1) : |x^e(2n-1+t) - x^e(2n-1+\delta_i)| < \varepsilon/3\},$$
$$\mathcal{O}_i^2 = \{t \in [0, 1) : |x^e(2n+t) - x^e(2n+\delta_i)| < \varepsilon/3\},$$

for $i = 1, 2, \ldots, m$, we have $[0, 1) = \bigcup_{i=1}^m \mathcal{O}_i^j,\ j = 1, 2$. Let

$$\mathcal{B}_1^j = \mathcal{O}_1^j, \qquad \mathcal{B}_i^j = \mathcal{O}_i^j \setminus \bigcup_{k=1}^{i-1} \mathcal{O}_k^j, \quad j = 1, 2,\ i = 2, 3, \ldots, m.$$

Then

$$[0, 1) = \bigcup_{i=1}^m \mathcal{B}_i^j, \qquad \mathcal{B}_i^j \bigcap \mathcal{B}_k^j = \emptyset \quad \text{for } j = 1, 2,\ i \ne k. \tag{4.14}$$

Moreover, by (2.1), (4.12) and (4.14) we have, for $n \in \mathbb{Z}$,

$$\begin{aligned}
\sum_{i=1}^m \int_{\mathcal{B}_i^1} |x^e(2n-1+t)|\rho(2n-1+t)dt &\le \sum_{i=1}^m \int_{\mathcal{B}_i^1} \left(|x^e(2n-1+\delta_i)| + \frac{\varepsilon}{3}\right) \rho(2n-1+t)dt \\
&= \sum_{i=1}^m |x^e(2n-1+\delta_i)| \int_{\mathcal{B}_i^1} \rho(2n-1+t)dt + \frac{\varepsilon}{3} \int_0^1 \rho(2n-1+t)dt \\
&\le \left(\sum_{i=1}^m |x^e(2n-1+\delta_i)| + \frac{\varepsilon}{3}\right) \varrho(n).
\end{aligned} \tag{4.15}$$

Similarly, by (2.1), (4.13) and (4.14) we have, for $n \in \mathbb{Z}$,

$$\sum_{i=1}^m \int_{\mathcal{B}_i^2} |x^e(2n+t)|\rho(2n+t)dt \le \left(\sum_{i=1}^m |x^e(2n+\delta_i)| + \frac{\varepsilon}{3}\right) \varrho(n). \tag{4.16}$$

Denote $Q(n) = \sum_{i=1}^m (|x^e(2n-1+\delta_i)| + |x^e(2n+\delta_i)|)$. Then $\{Q(n)\} \in PAP_0S(\mathbb{R}, \varrho)$ since $\{x^e(2n-1+\delta_i)\}, \{x^e(2n+\delta_i)\} \in PAP_0S(\mathbb{R}, \varrho),\ i = 1, 2, \ldots, m$, and there exists $T_0 > 0$ such that, for $T > T_0$,

$$\frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^T Q(n)\varrho(n) < \frac{\varepsilon}{3}. \tag{4.17}$$

Now by (4.14)–(4.16) we have

$$\begin{aligned}
\int_{2n-1}^{2n+1} |x^e(t)|\rho(t)dt &= \int_{2n-1}^{2n} |x^e(t)|\rho(t)dt + \int_{2n}^{2n+1} |x^e(t)|\rho(t)dt \\
&= \int_0^1 |x^e(2n-1+t)|\rho(2n-1+t)dt + \int_0^1 |x^e(2n+t)|\rho(2n+t)dt \\
&= \sum_{i=1}^m \int_{B_i^1} |x^e(2n-1+t)|\rho(2n-1+t)dt + \sum_{i=1}^m \int_{B_i^2} |x^e(2n+t)|\rho(2n+t)dt \\
&\le \left(\sum_{i=1}^m |x^e(2n-1+\delta_i)| + \frac{\varepsilon}{3}\right) \varrho(n) + \left(\sum_{i=1}^m |x^e(2n+\delta_i)| + \frac{\varepsilon}{3}\right) \varrho(n) \\
&= \left(Q(n) + \frac{2\varepsilon}{3}\right) \varrho(n).
\end{aligned} \tag{4.18}$$

Meanwhile, by Lemma 2.1, there exist $M > 0$ and $T_0' > 2T_0 + 1$ such that $\mu_s\left(\left[\frac{T}{2}\right]+1, \varrho\right) \le M\mu(T, \rho)$ for $T > T_0'$. Then by (4.17) and (4.18), for $T > T_0'$,

$$\begin{aligned}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}|x^e(t)|\rho(t) &\le \frac{1}{\mu(T,\rho)}\sum_{n=-\left[\frac{T}{2}\right]-1}^{\left[\frac{T}{2}\right]+1}\int_{2n-1}^{2n+1}|x^e(t)|\rho(t)dt\\ &\le \frac{\mu_s\left(\left[\frac{T}{2}\right]+1,\varrho\right)}{\mu(T,\rho)}\frac{1}{\mu_s\left(\left[\frac{T}{2}\right]+1,\varrho\right)}\sum_{n=-\left[\frac{T}{2}\right]-1}^{\left[\frac{T}{2}\right]+1}\left(Q(n)+\frac{2\varepsilon}{3}\right)\varrho(n)\\ &< M\left(\frac{\varepsilon}{3}+\frac{2\varepsilon}{3}\right) = M\varepsilon.\end{aligned}$$

This implies that $x^e \in PAP_0(\mathbb{R}, \rho)$, and the proof is complete.

5. Proof of Theorem 3.4

Denote by $\lambda_i,\ i = 1, 2, 3$ the three eigenvalues of A_1, we may assume that A_1 does not have triple eigenvalues (in fact, this can be guaranteed by Lemma 5.1). Then there exists a nonsingular matrix $P = (p_{ij})_{3\times 3} \in \mathbb{C}^{3\times 3}$ such that

$$PA_1P^{-1} = \Lambda, \tag{5.1}$$

where

$$\Lambda = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix} \quad \text{or} \quad \Lambda = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 1 & \lambda_3\end{pmatrix}$$

with $\lambda_2 \ne \lambda_3$ or $\lambda_2 = \lambda_3$, and (3.3a) can be rewritten as the following form

$$u(n+1) = \Lambda u(n) + k(n), \tag{5.2}$$

where $u(n) = (u_1(n), u_2(n), u_3(n))^T = Py(n)$ and $k(n) = (k_1(n), k_2(n), k_3(n))^T = Pl(n)$ for $n \in \mathbb{Z}$. Then $\{k(n)\} \in PAPS(\mathbb{C}^3, \varrho)$ by Remark 3.2(i) and Proposition 2.2(ii).

By (3.4), it is easy to verify that the eigenvalues of A_2 are 1 and 1/5 if $p = 1$, and are $(-1\pm 2\sqrt{3}i)/13$ if $p = -1$. Moreover, we can get the characteristic equation of A_1 if $p = 1$:

$$\lambda^3 + \frac{2q+6}{q-2}\lambda^2 + \frac{3q+6}{2-q}\lambda + \frac{2}{q-2} = 0. \tag{5.3}$$

Clearly, 1 is a solution of (5.3), say, $\lambda_1 = 1$. Meanwhile, if $p = -1$, the characteristic equation of A_1 is

$$\lambda^3 + \frac{10q+6}{q-2}\lambda^2 + \frac{5q-6}{q-2}\lambda + \frac{2}{q-2} = 0. \tag{5.4}$$

For the eigenvalues of A_1, we have the following lemma.

Lemma 5.1. (i) *Assume $p = 1, q \ne 2$. Then $q \ne -4$ if and only if $|\lambda_i| \ne 1,\ i = 2, 3$.*
(ii) *If $p = -1$, A_1 has no triple eigenvalue and the following statements are equivalent:*
 (a) *one of the eigenvalues of A_1 has absolute value* 1.
 (b) *-1 is an eigenvalue of A_1 and the absolute values of the other two eigenvalues of A_1 are different from* 1.
 (c) *$q = -4$.*

Proof. (i) Assume that $q = -4$. Then it follows from (5.3) that -1 is an eigenvalue of A_1. On the other hand, without loss of generality, suppose that $|\lambda_2| = 1$. Then $\lambda_2 = e^{i\theta}$ for $0 \le \theta \le \pi$. If $\theta = 0$, that is $\lambda_2 = 1$. By (5.3), we know

$$\lambda_1\lambda_2\lambda_3 = \lambda_3 = \frac{2}{2-q}, \qquad \lambda_1 + \lambda_2 + \lambda_3 = 2 + \lambda_3 = \frac{2q+6}{2-q}.$$

Thus $4q + 2 = 2$, and $q = 0$, which contradicts the assumption $q \ne 0$. So $\theta \ne 0$. If $0 < \theta < \pi$, then λ_3 must be $e^{-i\theta}$. By (5.3),

$$\lambda_1\lambda_2\lambda_3 = 1 = \frac{2}{2-q}.$$

Thus $q = 0$, which contradicts $q \neq 0$. So $\theta \bar{\in} (0, \pi)$, and then $\theta = \pi$, i.e. $\lambda_2 = -1$. This together with (5.3) implies that $q = -4$. The proof is complete.

(ii) If $\lambda_1 = \lambda_2 = \lambda_3$, from (5.4) we have

$$3\lambda_1^2 = \frac{5q-6}{q-2}, \qquad 3\lambda_1 = -\frac{10q+6}{q-2}, \qquad \lambda_1^3 = -\frac{2}{q-2}. \tag{5.5}$$

By the first two equations of (5.5) we get that $q = -168/85$, $\lambda_1 = -195/169$. Then by the third equation of (5.5) we have $(-195/169)^3 = 85/169$, which is impossible. Thus A_1 has no triple eigenvalue.

(a) $\Rightarrow$ (b) Assume that $|\lambda_1| = 1$. Then $\lambda_1 = e^{i\theta}$ for $0 \le \theta \le \pi$. If $\theta = 0$, that is $\lambda_1 = 1$, by (5.4) we get $16q = 0$, which contradicts $q \neq 0$. So $\theta \neq 0$. If $0 < \theta < \pi$, another eigenvalue of A_1, say, λ_2, must be $e^{-i\theta}$. By (5.4),

$$\begin{cases} \lambda_1\lambda_2\lambda_3 = \lambda_3 = -\dfrac{2}{q-2}, \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 1 + 2\lambda_3\cos\theta = \dfrac{5q-6}{q-2}, \\ \lambda_1 + \lambda_2 + \lambda_3 = 2\cos\theta + \lambda_3 = -\dfrac{10q+6}{q-2}. \end{cases}$$

This implies

$$2\cos\theta = 2 - 2q = \frac{10q+4}{2-q}.$$

Then $\cos\theta = -7$, which is impossible. So $\theta \notin (0, \pi)$. If $\theta = \pi$, $\lambda_1 = -1$. By (5.4) we have $q = -4$, and then it follows from (5.4) that $\lambda_2 = \frac{-7+2\sqrt{13}}{3}$, $\lambda_3 = \frac{-7-2\sqrt{13}}{3}$. This means that (b) is true.

(b) $\Rightarrow$ (c) Let $\lambda_1 = -1$. Then by (5.4) we have $q = -4$, and (c) is true.

(c) $\Rightarrow$ (a) Since $q = -4$, it is easy to verify that -1 is a solution of (5.4). Then (a) holds. $\square$

Proof of Theorem 3.4. The proof of (i) is completed in the following three cases:

Case I. Suppose that $q \neq -4$ and $q \neq 2$. We may assume that $\lambda_1 \neq \lambda_j$, $j = 2, 3$ and $|\lambda_i| \neq 1$, $i = 1, 2, 3$ by Lemma 5.1. Let α be a constant defined as that, $\alpha = 0$ if $\lambda_2 \neq \lambda_3$; $\alpha = 1$ if $\lambda_2 = \lambda_3$. Define $u(n) = (u_1(n), u_2(n), u_3(n))^T$ by

$$\begin{cases} u_i(n) = \begin{cases} \displaystyle\sum_{m \le n-1} \lambda_i^{n-m-1} k_i(m), & |\lambda_i| < 1, \\ \displaystyle -\sum_{m \ge n} \lambda_i^{n-m-1} k_i(m), & |\lambda_i| > 1, \end{cases} \quad i = 1, 2, \\ u_3(n) = \begin{cases} \displaystyle\sum_{m \le n-1} \lambda_3^{n-m-1}(k_3(m) + \alpha u_2(m)), & |\lambda_3| < 1, \\ \displaystyle -\sum_{m \ge n} \lambda_3^{n-m-1}(k_3(m) + \alpha u_2(m)), & |\lambda_3| > 1 \end{cases} \end{cases} \tag{5.6}$$

for $n \in \mathbb{Z}$. It is clear that $u(n)$ is the unique bounded solution of (5.2). Next we prove that $\{u_i(n)\} \in PAPS(\mathbb{C}, \varrho)$, $i = 1, 2, 3$. Suppose that $|\lambda_1| < 1$. Let

$$u_1^{ap}(n) = \sum_{m \le n-1} \lambda_1^{n-m-1} k_1^{ap}(m),$$
$$u_1^{e}(n) = u_1(n) - u_1^{ap}(n) = \sum_{m \le n-1} \lambda_1^{n-m-1} k_1^{e}(m).$$

It is not difficult for us to check that $\{u_1^{ap}(n)\}_{n\in\mathbb{Z}} \in APS(\mathbb{C})$. Meanwhile, for $T \in \mathbb{Z}^+$,

$$\begin{aligned} \frac{1}{\mu_s(T,\varrho)} \sum_{n=-T}^{T} |u_1^e(n)|\varrho(n) &= \frac{1}{\mu_s(T,\varrho)} \sum_{n=-T}^{T} \sum_{m=-\infty}^{n-1} |\lambda_1|^{n-m-1} |k_1^e(m)|\varrho(n) \\ &= \frac{1}{\mu_s(T,\varrho)} \sum_{n=-T}^{T} \sum_{m=0}^{\infty} |\lambda_1|^{m} |k_1^e(n-1-m)|\varrho(n) \\ &= \sum_{m=0}^{\infty} |\lambda_1|^m \frac{1}{\mu_s(T,\varrho)} \sum_{n=-T}^{T} |k_1^e(n-1-m)|\varrho(n). \end{aligned} \tag{5.7}$$

For $m \in \mathbb{Z}^+$, let

$$\Phi_T(m) = \frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^{T} |k_1^e(n-1-m)|\varrho(n).$$

From Proposition 2.1, we get

$$\lim_{T\to\infty} \Phi_T(m) = 0, \quad \Phi_T(m) \le \sup_{n\in\mathbb{Z}} |k_1^e(n)| \triangleq M_1 \text{ for } m \in \mathbb{Z}^+. \tag{5.8}$$

Given $\varepsilon > 0$, it is clear that there exists an integer $K > 0$ such that

$$\sum_{m=K+1}^{\infty} |\lambda_1|^m < \varepsilon. \tag{5.9}$$

Then by (5.8), there exists $T_0 > 0$ such that for $T > T_0$,

$$\Phi_T(m) < \frac{\varepsilon}{K+1} \quad \text{for } 0 \le m \le K. \tag{5.10}$$

Now by (5.7)–(5.10), for $T > T_0$ we have

$$\begin{aligned}\frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^{T} |u_1^e(n)|\varrho(n) &= \sum_{m=0}^{K} |\lambda_2|^m \Phi_T(m) + \sum_{m=K+1}^{\infty} |\lambda_1|^m \Phi_T(m) \\ &\le (K+1)\frac{\varepsilon}{K+1} + M_1\varepsilon = (1+M_1)\varepsilon.\end{aligned}$$

This implies that $\{u_1^e(n)\} \in PAP_0S(\mathbb{C}, \varrho)$ for $|\lambda_1| < 1$. Similarly, we can get that $\{u_1^e(n)\} \in PAP_0S(\mathbb{C}, \varrho)$ for $|\lambda_1| > 1$. Thus $\{u_1(n)\} \in PAPS(\mathbb{C}, \varrho)$. Moreover, we can prove similarly that $\{u_i(n)\} \in PAPS(\mathbb{C}, \varrho)$ for $i = 2, 3$. Let $y(n) = P^{-1}u(n)$, $n \in \mathbb{Z}$. Then $\{y(n)\}$ is the unique bounded solution of (3.3a) and $\{y(n)\} \in PAPS(\mathbb{C}^3, \varrho)$.

Next we prove that $\{y(n)\} \in PAPS(\mathbb{R}^3, \varrho)$ such that $D_{-1}\{y(n)\} \neq \emptyset$. In fact, by (5.1) and a fundamental calculation, we can see that the entries of P and P^{-1} can be get from the rational operations of q and λ_i, $i = 1, 2, 3$. Therefore, if all the eigenvalues of A_1 are real, by (5.6) we have $u(n) \in \mathbb{R}^3$, $n \in \mathbb{Z}$, and then $y(n) \in \mathbb{R}^3$, $n \in \mathbb{Z}$. If one of the eigenvalues of A_1, say, λ_2 is complex, then $\lambda_3 = \overline{\lambda}_2$. Consequently, $\alpha = 0$, and by (5.1) and a fundamental calculation we can get:

$$P = (p_{ij}) = d_1^{-1} \begin{pmatrix} P_1(\lambda_1, \lambda_2, \lambda_3) & P_2(\lambda_1, \lambda_2, \lambda_3) & P_3(\lambda_1, \lambda_2, \lambda_3) \\ P_1(\lambda_2, \lambda_3, \lambda_1) & P_2(\lambda_2, \lambda_3, \lambda_1) & P_3(\lambda_2, \lambda_3, \lambda_1) \\ P_1(\lambda_3, \lambda_1, \lambda_2) & P_2(\lambda_3, \lambda_1, \lambda_2) & P_3(\lambda_3, \lambda_1, \lambda_2) \end{pmatrix},$$

$$P^{-1} = (q_{ij}) = \begin{pmatrix} \lambda_1(\lambda_1+3) & \lambda_2(\lambda_2+3) & \lambda_3(\lambda_3+3) \\ (q+3)\lambda_1+1 & (q+3)\lambda_2+1 & (q+3)\lambda_3+1 \\ \lambda_1+3 & \lambda_2+3 & \lambda_3+3 \end{pmatrix},$$

where $d_1 = (3q+8)(\lambda_1^2+\lambda_2\lambda_3-\lambda_1\lambda_2-\lambda_1\lambda_3)(\lambda_2-\lambda_3)$, $P_1(a, b, c) = (3q+8)(b-c)$, $P_2(a, b, c) = (bc+3b+3c+9)(c-b)$ and $P_3(a, b, c) = ((q+3)bc + b + c + 3)(b-c)$. It is easy to see that $p_{1i}, q_{i1} \in \mathbb{R}$ and $p_{2i} = \overline{p_{3i}}$, $q_{i2} = \overline{q_{i3}}$, $i = 1, 2, 3$. Meanwhile, $u_1(n) \in \mathbb{R}$ and $u_2(n) = \overline{u_3(n)}$ for $n \in \mathbb{Z}$ by (5.6). Then we can verify easily that $y(n) = P^{-1}u(n) \in \mathbb{R}^3$ for $n \in \mathbb{R}$, i.e. $\{y(n)\} \in PAPS(\mathbb{R}^3, \varrho)$. By (H$_3$) and Proposition 2.2 we can get that $D_{-1}\{k(n)\} \neq \emptyset$. Then it is easy to get from (5.6) that $D_{-1}\{u(n)\} \neq \emptyset$, and we have $D_{-1}\{y(n)\} \neq \emptyset$ by Proposition 2.2(ii). This completes the proof of (i) in the case $q \neq 2$ and $q \neq -4$.

Case II. Suppose that $q = 2$. Then the eigenvalues of A_2 are $\left(-1 \pm 2\sqrt{3}i\right)/13$. By an argument similar to the proof of *Case* I, we can prove that (i) holds for this simpler case.

Case III. Suppose that $q = -4$. Then the eigenvalues of A_1 are $\lambda_1 = -1$, $\lambda_2 = \left(-7 + 2\sqrt{13}\right)/3$, $\lambda_3 = \left(-7 - 2\sqrt{13}\right)/3$. By (5.1) and a fundamental calculation, P can be chosen as

$$P = \begin{pmatrix} 3 & -4 & 1 \\ 38 - 10\sqrt{13} & 56 - 16\sqrt{13} & -18 + 6\sqrt{13} \\ 38 + 10\sqrt{13} & 56 + 16\sqrt{13} & -18 - 6\sqrt{13} \end{pmatrix}.$$

Moreover, it is easy to get that $D_{-1}D_1\{k(n)\} \neq \emptyset$ and $D_{-1}\{k(n)\} \neq \emptyset$ by (H$_2$), (H$_3$) and Proposition 2.2(i), (ii). Choose $\{r(n)\} \in D_1\{k(n)\}$ with $r(n) = (r_1(n), r_2(n), r_3(n))^T$, $n \in \mathbb{Z}$ such that $D_{-1}\{r(n)\} \neq \emptyset$, and define

$$\begin{aligned}u(n) &= (u_1(n), u_2(n), u_3(n))^T \\ &= \left(r_1(n), \sum_{m\le n-1} \lambda_2^{n-m-1}k_2(m), -\sum_{m\ge n} \lambda_3^{n-m-1}k_3(m)\right)^T\end{aligned}$$

for $n \in \mathbb{Z}$. Then it is easy to see that $\{u(n)\}$ is a solution of (5.2), and by an argument similar to that of *Case* I we can get that $\{u(n)\} \in PAPS(\mathbb{R}^3, \varrho)$ with $D_{-1}\{u(n)\} \neq \emptyset$. Let $y(n) = P^{-1}u(n)$ for $n \in \mathbb{Z}$. Then $\{y(n)\} \in PAPS(\mathbb{R}^3, \varrho)$ is a solution of (3.3a), and $D_{-1}\{y(n)\} \neq \emptyset$ by Proposition 2.2(ii).

Now we show that S is the set of all real solutions of (3.3a) such that $D_{-1}\{\bar{y}(n)\} \neq \emptyset$. Let $\{\bar{y}(n)\} = \{(\bar{x}_{2n}, \bar{x}_{2n-1}, \bar{x}_{2n-2})^T\} = \{y(n) + A_1^n C\} \in S$ with $C = c(1, -1, -1)^T,\ c \in \mathbb{R}$. Then it is easy to verify that $\{\bar{y}(n)\}$ is a solution of (3.3a) and $PA_1^n C = \Lambda^n PC = 6c((-1)^n, 0, 0)^T,\ n \in \mathbb{Z}$. Then $D_{-1}\{\Lambda^n PC\} \neq \emptyset$ by Proposition 2.2(iii), and $D_{-1}\{A_1^n C\} = D_{-1}\{P^{-1}\Lambda^n PC\} \neq \emptyset$ by Proposition 2.2(ii). Hence $D_{-1}\{\bar{y}(n)\} = D_{-1}\{y(n) + A_1^n C\} \neq \emptyset$ by Proposition 2.2(i).

On the other hand, assume that $\{\bar{y}(n)\} = \{(\bar{x}_{2n}, \bar{x}_{2n-1}, \bar{x}_{2n-2})^T\}$ is a solution of (3.3a) such that $D_{-1}\{\bar{y}(n)\} \neq \emptyset$. By (3.3a) we have that $\bar{y}(n) - y(n) = A_1^n C,\ n \in \mathbb{Z}$, with $C = (c_1, c_2, c_3)^T = \bar{y}(0) - y(0)$. Then $D_{-1}\{A_1^n C\} = D_{-1}\{\bar{y}(n) - y(n)\} \neq \emptyset$ by Proposition 2.2(i), and thus, $D_{-1}\{\Lambda^n PC\} = D_{-1}\{PA_1^n C\} \neq \emptyset$ by Proposition 2.2(ii). Meanwhile,

$$\Lambda^n PC = \begin{pmatrix} (3c_1 - 4c_2 + c_3)(-1)^n \\ \left((38 - 10\sqrt{13})c_1 + (56 - 16\sqrt{13})c_2 + (-18 + 6\sqrt{13})c_3\right)\lambda_2^n \\ \left((38 + 10\sqrt{13})c_1 + (56 + 16\sqrt{13})c_2 + (-18 - 6\sqrt{13})c_3\right)\lambda_3^n \end{pmatrix}$$

for $n \in \mathbb{Z}$. It follows from Proposition 2.2(iii) that

$$\begin{cases} (38 - 10\sqrt{13})c_1 + (56 - 16\sqrt{13})c_2 + (-18 + 6\sqrt{13})c_3 = 0, \\ (38 + 10\sqrt{13})c_1 + (56 + 16\sqrt{13})c_2 + (-18 - 6\sqrt{13})c_3 = 0. \end{cases}$$

This implies that $c_1 = -c_2 = -c_3$. Therefore $\{\bar{y}(n)\} \in S$, and (i) is true for the case $p = -4$.

(ii) From (H_1), Remark 3.2 and Proposition 2.2, it is easy to see that $D_{-1}^2\{k(n)\} \neq \emptyset$. Let $\{r(n)\} \in D_{-1}\{k(n)\}$ with $r(n) = (r_1(n), r_2(n), r_3(n))^T$ such that $D_{-1}\{r(n)\} \neq \emptyset$. By Lemma 5.1(i), $\lambda_1 = 1$ and $|\lambda_i| \neq 1,\ i = 1, 2$. Let α be a constant defined as that, $\alpha = 0$ if $\lambda_2 \neq \lambda_3$; $\alpha = 1$ if $\lambda_2 = \lambda_3$. Define $u(n) = (u_1(n), u_2(n), u_3(n))^T$ by

$$\begin{cases} u_1(n) = r_1(n), \\ u_2(n) = \begin{cases} \displaystyle\sum_{m \le n-1} \lambda_2^{n-m-1} k_2(m), & |\lambda_2| < 1, \\ \displaystyle -\sum_{m \ge n} \lambda_2^{n-m-1} k_2(m), & |\lambda_2| > 1, \end{cases} \\ u_3(n) = \begin{cases} \displaystyle\sum_{m \le n-1} \lambda_3^{n-m-1} (k_3(m) + \alpha u_2(m)), & |\lambda_3| < 1, \\ \displaystyle -\sum_{m \ge n} \lambda_3^{n-m-1} (k_3(m) + \alpha u_2(m)), & |\lambda_3| > 1 \end{cases} \end{cases}$$

for $n \in \mathbb{Z}$. Let $y(n) = P^{-1}u(n)$ for $n \in \mathbb{Z}$. By the similar argument of (i), we can get that $\{y(n)\} \in PAPS(\mathbb{R}^3, \varrho)$ is a solution of (3.3a) such that $D_{-1}\{y(n)\} \neq \emptyset$.

To prove the uniqueness of $\{y(n)\} \in PAPS(\mathbb{R}^3, \varrho)$ as a solution of (3.3a) such that $D_{-1}\{y(n)\} \neq \emptyset$, it is sufficient to prove that $\{u(n)\} = \{Py(n)\} \in PAPS(\mathbb{C}^3, \varrho)$ is a unique solution of (5.2) such that $D_{-1}\{u(n)\} \neq \emptyset$. Assume that $\{\bar{u}(n)\}$ is a solution of (5.2) such that $D_{-1}\{\bar{u}(n)\} \neq \emptyset$. By (5.2), we get $\bar{u}_1(n) - r_1(n) = \bar{u}_1(0) - r_1(0)$ for $n \in \mathbb{Z}$. Noticing that $D_{-1}\{\bar{u}_1(n)\} \neq \emptyset$ and $D_{-1}\{r_1(n)\} \neq \emptyset$, we have $D_{-1}\{\bar{u}_1(0) - r_1(0)\} \neq \emptyset$ by Proposition 2.2(i), and this implies that $\bar{u}_1(0) = r_1(0)$ by Proposition 2.2(iii). So $\bar{u}_1(n) = u_1(n)$ for $n \in \mathbb{Z}$. If $|\lambda_2| < 1$, since $\{\bar{u}_2(n)\}$ is bounded, by (5.2) we have, for $n \in \mathbb{Z}$,

$$\bar{u}_2(n) = \lambda_2^l \bar{u}_2(n-l) + \sum_{m=n-l}^{n-1} \lambda_2^{n-m-1} k_2(m) \to \sum_{m \le n-1} \lambda_2^{n-m-1} k_2(m) \quad \text{as } l \to \infty.$$

So $\bar{u}_2(n) = u_2(n),\ n \in \mathbb{Z}$. Similarly, we can prove that $\bar{u}_2(n) = u_2(n)$ if $|\lambda_2| > 1$. Moreover, we can also get similarly that $\bar{u}_3(n) = u_3(n)$ for $n \in \mathbb{Z}$. Thus $\bar{u}(n) = u(n)$. The proof is complete.

6. Example

At last, we give the following example to illustrate our main results Theorems 3.1 and 3.2.

Example 6.1. Let $\rho = 1 + t^2$, for $i = 1, 2,\ t \in \mathbb{R}$,

$$f_i(t) = \sum_{j=1}^{k} \left(\cos \gamma_j(\alpha_j t + \beta_j) + \gamma_j' \sin(\alpha_j' t + \beta_j')\right) + \phi_i(t) \sin \pi t$$

with $\alpha_j, \beta_j, \gamma_j, \alpha_j', \beta_j', \gamma_j' \in \mathbb{R}, \alpha_j, \alpha_j' \neq k\pi, k \in \mathbb{Z}, j = 1, 2, \ldots, k$ and

$$\phi_i(t) = \begin{cases} e^{-n}, & t \in [2n-1, 2n+1),\ n \geq 0, \\ c_i, & t \in [-3, -1), \\ (-1)^i e^{-|n|+2}, & t \in [2n-1, 2n+1),\ n < -1, \end{cases} \qquad i = 1, 2,$$

where $c_1 = 2/(e^{-1} - 1)$ and $c_2 = 0$. It is clear that $\rho \in U_T, \varrho(n) = 2 + (2/3)(12n^2 + 1) \in U_{sT}, f_i \in PAP(\mathbb{R}, \rho)$, and for each $\eta \in [-1, 1]$, $\{f_i(2n + \eta)\} \in PAPS(\mathbb{R}, \varrho)$ with

$$f_i^{ap}(2n + \eta) = \sum_{j=1}^{k} \left(\cos \gamma_j(\alpha_j t + \beta_j) + \gamma_j' \sin(\alpha_j' t + \beta_j')\right),$$

$f_i^e(2n + \eta) = \phi_i(2n + \eta) \sin \pi\eta$ for $i = 1, 2$. Let $\bar{f}_i(t) = \bar{f}_i^{ap}(t) + \bar{\varphi}_i(t) \sin \pi t,\ i = 1, 2$ with

$$\bar{f}_i^{ap}(t) = \sum_{j=1}^{k} \left(\frac{\gamma_j \sin\left(\alpha_j t + \beta_j - \frac{\alpha}{2}\right)}{2 \sin \frac{\alpha}{2}} - \frac{\gamma_j' \cos\left(\alpha_j' t + \beta_j' - \frac{\alpha'}{2}\right)}{2 \sin \frac{\alpha'}{2}} \right),$$

$$\bar{\varphi}_i(t) = \begin{cases} \dfrac{e^{-n}}{e^{-1} - 1}, & t \in [2n-1, 2n+1),\ n \geq 0, \\ (-1)^i \dfrac{e^{-|n|+1}}{e^{-1} - 1}, & t \in [2n-1, 2n+1),\ n < 0. \end{cases}$$

Then $\bar{f}_i^e(t) = \bar{\varphi}_i(t) \sin \pi t$, and it is easy to verify that $\bar{f}_i \in PAP(\mathbb{R}, \rho)$ and $\{\bar{f}_i(2n + \eta)\} \in D_{-1}\{f_i(2n + \eta)\}$ for $\eta \in [-1, 1]$, $i = 1, 2$. That is f_1 and f_2 satisfy (H_4). Moreover, by [6, Proposition 2.2] and [8, Proposition 1.1], we can see easily that

$$D_{-1}^2\{f_1^{ap}(2n + \eta)\} \neq \emptyset, \qquad D_{-1}D_1\{f_2^{ap}(2n + \eta)\} \neq \emptyset, \quad \eta \in [-1, 1]. \tag{6.1}$$

Let

$$\bar{\phi}_i(t) = \begin{cases} \dfrac{e^{-n}}{e^{-1} + (-1)^i}, & t \in [2n-1, 2n+1),\ n \geq 0, \\ -\dfrac{e^{-|n|+1}}{e^{-1} + (-1)^i}, & t \in [2n-1, 2n+1),\ n < 0, \end{cases} \qquad i = 1, 2,$$

$$\bar{\bar{\phi}}_i(t) = \frac{e^{-|n|}}{(e^{-1} - 1)(e^{-1} + (-1)^i)}, \quad t \in [2n-1, 2n+1),\ n \in \mathbb{Z},\ i = 1, 2.$$

Then it is not hard for us to verify that, for $\eta \in [-1, 1],\ i = 1, 2, \bar{\phi}_i, \bar{\bar{\phi}}_i \in PAP_0(\mathbb{R}, \rho)$ and

$$\{\bar{\phi}_1(2n + \eta) \sin \pi\eta\} \in D_{-1}\{f_1^e(2n + \eta)\},$$

$$\{\bar{\phi}_2(2n + \eta) \sin \pi\eta\} \in D_1\{f_2^e(2n + \eta)\}, \qquad \{\bar{\bar{\phi}}_i(2n + \eta)\} \in D_{-1}\{\bar{\phi}_i(2n + \eta)\}.$$

This together with (6.1) implies that $D_{-1}^2\{f_1(2n + \eta)\} \neq \emptyset$ and $D_{-1}D_1\{f_2(2n + \eta)\} \neq \emptyset$ for each $\eta \in [-1, 1]$. Now by an argument similar to Remark 3.2(ii), it is easy to verify that f_i satisfies (H_i), $i = 1, 2$. So for $f = f_2$, all the conditions of Theorem 3.1 are satisfied, and then all the conclusions of Theorem 3.1 are true. Similarly, we get that all the conclusions of Theorem 3.2 hold for $f = f_1$.

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