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Constructions for mutually orthogonal frequency hyperrectangles with a prescribed type

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Abstract

In this paper, we give two different ways to construct mutually orthogonal frequency hyperrectangles (MOFHR). Firstly, we exhibit sets of linear polynomials over finite fields that represent complete sets of MOFHR of prime power order, which generalize Mullen's method in (G.L. Mullen, Discrete Math. 69 (1988) 79–84). Secondly, a recursive algorithm is given to construct $(d+1)$-dimensional MOFHR of type $t+1$ from d-dimensional MOFHR of type t, which generalizes a recursive procedure described in (Laywine et al., Monatsch Math. 119 (1995) 223–238). © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

Frequency squares and hyperrectangles have numerous statistical properties and as a result, there has been considerable interest in various aspects of the theory and construction of such objects. In this paper, we provide two different ways of constructing mutually orthogonal frequency hyperrectangles of a prescribed type.

Firstly, we exhibit sets of linear polynomials over finite fields that represent complete sets of mutually orthogonal frequency hyperrectangles (MOFHR) of a prescribed type and of prime power order, which generalize Mullen's method in [3].

Secondly, we give a recursive algorithm to construct $(d+1)$-dimensional MOFHR of type $t+1$ from d-dimensional MOFHR of type t, which generalizes a recursive procedure in [2].

We begin with some notation.

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For a natural number n, we use $\underline{n}$ for the set $\{1,2,\ldots,n\}$, and we let $P_k(S)$ denote the set consisting of all k-subsets of the set S. When $k=0$, we define $P_k(S)$ as $\{\phi\}$, where ϕ is the empty set.

Definition 1.1. We coordinatize the $\prod_{i=1}^{d} n_i$ cells of a d-dimensional hyperrectangle of size $n_1\times\cdots\times n_d$ by the d-tuple of integers $(j_1,\ldots,j_d)$ where $0\leqslant j_i\leqslant n_i-1$. A *frequency hyperrectangle* (F-hyperrectangle) of size $n_1\times\cdots\times n_d$ and type t, $0\leqslant t\leqslant d-1$, denoted by $\mathrm{FHR}(n_1,\ldots,n_d;t;m)$, where $m\,|\,n_i$ for $1\leqslant i\leqslant d$, is an $n_1\times\cdots\times n_d$ array consisting of $m\geqslant 2$ symbols, say $0,1,\ldots,m-1$, with the property that whenever any t of the coordinates are fixed, all m symbols occur equally often in that subarray.

Definition 1.2. Two F-hyperrectangles $\mathrm{FHR}(n_1,\ldots,n_d;t;m)$ are *orthogonal* if upon superposition, each ordered pair (i,j), $0\leqslant i,j\leqslant m-1$, appears equally often, i.e., $\prod_{i=1}^{d} n_i/m^2$ times. A set of F-hyperrectangles is called *mutually orthogonal* if every pair of F-hyperrectangles is orthogonal.

The following upper bound on the maximum number of mutually orthogonal F-hyperrectangles with a prescribed type is given in [1]. This result generalizes Theorem 3.1 of [2].

Theorem 1.3 (Cheng [1]). *The maximal number of MOFHR of size $n_1\times\cdots\times n_d$ and type t, based on m symbols, is bounded above by*

$$r\frac{1}{m-1}\left(\prod_{i=1}^{d} n_i-\sum_{k=1}^{t}\sum_{\{i_1,\ldots,i_k\}\in P_k(\underline{d})}\prod_{j=1}^{k}(n_{i_j}-1)-1\right).$$

Definition 1.4. A set of r MOFHR of size $n_1\times\cdots\times n_d$ and type t, based on m symbols, is called *complete* if r equals the bound from Theorem 1.3.

2. Polynomial representation of orthogonal F-hyperrectangles

Let F_q denote the finite field of order q, where q is a prime power. Following Niederreiter in [4], we say that a polynomial $f(x_1,\ldots,x_n)$ with coefficients in F_q is a *permutation polynomial* in n variables over F_q if the equation $f(x_1,\ldots,x_n)=\alpha$ has exactly q^{n-1} solutions in F_q^n for each $\alpha\in F_q$. More generally, we say that a system $f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n)$ of polynomials with $1\leqslant m\leqslant n$ is *orthogonal* in F_q if the system of equations $f_i(x_1,\ldots,x_n)=\alpha_i$ $(i=1,\ldots,m)$ has exactly q^{n-m} solutions in F_q^n for each $(\alpha_1,\ldots,\alpha_m)\in F_q^m$.

As indicated by Niederreiter in [4], the system $f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n)$ is orthogonal if and only if for all $(b_1,\ldots,b_m)\in F_q^m$ with $(b_1,\ldots,b_m)\neq(0,\ldots,0)$, the polynomial $b_1f_1(x_1,\ldots,x_n)+\cdots+b_mf_m(x_1,\ldots,x_n)$ is a permutation polynomial in n variables over F_q.

Let $m=q$, a prime power, and let $n_i=q^{s_i}$, where $s_i\geqslant 1$ is an integer. Now we have the following theorem.

Theorem 2.1. *The* $(1/q-1)(q^{\sum_{i=1}^{d} s_i} - \sum_{k=1}^{t}\sum_{\{i_1,\dots,i_k\}\in P_k(\underline{d})}\prod_{j=1}^{k}(q^{s_{i_j}}-1)-1)$ *polynomials*

$$f_{(a_{11},\dots,a_{1s_1},\dots,a_{d1},\dots a_{ds_d})}(x_{11},\dots,x_{1s_1},\dots,x_{d1},\dots,x_{ds_d}) = \sum_{i=1}^{d}\sum_{j=1}^{s_i} a_{ij}x_{ij} \tag{1}$$

over F_q, *where*

(a) *at least* $t+1$ *of the subvectors* $(a_{11},\dots,a_{1s_1}),\dots,(a_{d1},\dots,a_{ds_d})$ *are nonzero*;
(b) *no two sets of a's are nonzero* F_q *multiples of each other*, *i.e.*,

$$(a'_{11},\dots,a'_{1s_1},\dots,a'_{d1},\dots,a'_{ds_d}) \neq e(a_{11},\dots,a_{1s_1},\dots,a_{d1},\dots,a_{ds_d})$$

for any $e\neq 0\in F_q$, *represent a complete set of* $MOFHR(q^{s_1},\dots,q^{s_d};t;q)$ *of dimension* d *and type* t.

Proof. There are

$$\frac{1}{q-1}\left(q^{\sum_{i=1}^{d} s_i} - \sum_{k=1}^{t}\sum_{\{i_1,\dots,i_k\}\in P_k(\underline{d})}\prod_{j=1}^{k}(q^{s_{i_j}}-1)-1\right)$$

polynomials over F_q defined by (1) and conditions (a) and (b).

Label the ith coordinate with all s_i-tuples $(j_{i1},\dots,j_{is_i})$ over F_q, for $1\leqslant i\leqslant d$. Now, we may view an FHR$(n_1,\dots,n_d;t;m)$ as a function $f: F_q^{\sum_{s=1}^{d} s_i}\to F_q$, where the element $(j_{11},\dots,j_{1s_1},\dots,j_{d1},\dots,j_{ds_d})$ becomes the element

$$f(j_{11},\dots,j_{1s_1},\dots,j_{d1},\dots,j_{ds_d})\in F_q.$$

If $(j_{i_k,1},\dots j_{i_k,s_{i_k}})$, for $k=1,\dots,t$, is fixed, then

$$f_{(a)}(x_{11},\dots,x_{1s_1},\dots,x_{d1},\dots,x_{ds_d})|_{(x_{i_k,1},\dots,x_{i_k,s_{i_k}})=(j_{i_k,1},\dots,j_{i_k,s_{i_k}}),\ k=1,\dots,t} = \alpha$$

has the same number of solutions in $F_q^{\sum_{w\neq i_1,\dots,i_k} s_w}$ for each $\alpha\in F_q$, so that in the subarray obtained by fixing the i_1th, $\dots, i_t$th coordinates, each element of F_q is picked up equally often. Hence $f_{(a)}(x_{11},\dots,x_{1s_1},\dots,x_{d1},\dots,x_{ds_d})$ represents an FHR$(n_1,\dots,n_d;t;m)$.

Clearly, the F-hyperrectangles represented by $f_1=f_{(a)}(x_{11},\dots,x_{1s_1},\dots,x_{d1},\dots,x_{ds_d})$ and $f_2=f_{(a')}(x_{11},\dots,x_{1s_1},\dots,x_{d1},\dots,x_{ds_d})$ are orthogonal if and only if f_1 and f_2 form an orthogonal system of polynomials in $\sum_{s=1}^{d} s_i$ variables over F_q. By the Corollary of [4, p. 417], f_1 and f_2 form an orthogonal system over F_q if and only if for all $(b_1,b_2)\neq(0,0)\in F_q^2$, the polynomial $b_1f_1+b_2f_2$ is a permutation polynomial in $\sum_{s=1}^{d} s_i$ variables over F_q. Any linear polynomial of the form $\sum_{j=1}^{r} c_jx_j$ is a permutation polynomial in r variables provided at least one $c_j\neq 0$.

Let $(b_1,b_2)\neq(0,0)\in F_q^2$. If $b_1=0$, then b_2f_2 is a permutation polynomial since $b_2\neq 0$ while if $b_2=0$, then b_1f_1 is a permutation polynomial. Suppose $b_1b_2\neq 0$, so $b_1f_1+b_2f_2$ is a permutation polynomial unless all $\sum_{s=1}^{d} s_i$ coefficients vanish, in which case $b_1a'_j = -b_2a_j$ for $j = 1,\dots,\sum_{s=1}^{d} s_i$, i.e., unless $a'_j = -b_2a_j/b_1$ for $j = 1,\dots,\sum_{s=1}^{d} s_i$, a contradiction of condition (b). Hence f_1 and f_2 form an orthogonal system and the proof is complete. □

For example, Theorem 2.1 gives the complete sets Ξ_1 of 9 MOFHR(4,4;1;2); Ξ_2 of 3 MOFHR(4,2;1;2); and Ξ_3 of 3 MOFHR(4;0;2).

Consider the 9 polynomials over GF(2) given by

$$f_1(x_1,x_2,x_3,x_4) = x_1 + x_3,$$
$$f_2(x_1,x_2,x_3,x_4) = x_2 + x_3,$$
$$f_3(x_1,x_2,x_3,x_4) = x_2 + x_4,$$
$$f_4(x_1,x_2,x_3,x_4) = x_1 + x_4,$$
$$f_5(x_1,x_2,x_3,x_4) = x_2 + x_3 + x_4,$$
$$f_6(x_1,x_2,x_3,x_4) = x_1 + x_2 + x_3,$$
$$f_7(x_1,x_2,x_3,x_4) = x_1 + x_3 + x_4,$$
$$f_8(x_1,x_2,x_3,x_4) = x_1 + x_2 + x_4,$$
$$f_9(x_1,x_2,x_3,x_4) = x_1 + x_2 + x_3 + x_4.$$

These 9 polynomials represent the complete set of 9 MOFHR(4,4;1;2):

$$\Xi_1 : H_1 = \begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix} \quad H_2 = \begin{matrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{matrix} \quad H_3 = \begin{matrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}$$

$$H_4 = \begin{matrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{matrix} \quad H_5 = \begin{matrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{matrix} \quad H_6 = \begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{matrix}$$

$$H_7 = \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{matrix} \quad H_8 = \begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{matrix} \quad H_9 = \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}.$$

Consider the 3 polynomials over GF(2) given by

$$f_1(x_1,x_2,x_3) = x_1 + x_3,$$
$$f_2(x_1,x_2,x_3) = x_1 + x_2 + x_3,$$
$$f_3(x_1,x_2,x_3) = x_2 + x_3.$$

These 3 polynomials represent the complete set of 3 MOFHR(4,2;1;2):

$$\Xi_2 : Q_1 = \begin{matrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{matrix} \quad Q_2 = \begin{matrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix} \quad Q_3 = \begin{matrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{matrix}.$$

Consider the 3 polynomials over GF(2) given by

$$f_1(x_1,x_2)=x_1,$$
$$f_2(x_1,x_2)=x_2,$$
$$f_3(x_1,x_2)=x_1+x_2.$$

These 3 polynomials represent the complete set of 3 MOFHR$(4;0;2)$:

$$\Xi_3:\ 0\ 0\ 1\ 1 \quad 0\ 1\ 0\ 1 \quad 0\ 1\ 1\ 0.$$

3. Type 0 canonical F-hyperrectangles

The following construction gives type 0 canonical FHR$(n_1,\ldots,n_d;0;m)$ from MOFHR$(n_1;0;m),\ldots,$MOFHR$(n_d;0;m)$. Furthermore, adding a set of MOFHR of type 1, we will have an enlarged set of MOFHR of type 0. If the set of MOFHR of type 1 and the sets of MOFHR$(n_1;0;m),\ldots,$MOFHR$(n_d;0;m)$ are both complete, then the enlarged set of MOFHR of type 0 is also complete.

Suppose X is an FHR$(n_1,\ldots,n_d;t;m)$. Let $X(x_1,x_2,\ldots,x_d)$ denote the entry in position $(x_1,x_2,\ldots,x_d)$. The subarray obtained by assigning some fixed values $a_1,\ldots,a_t$ to the i_1th$,\ldots,i_t$th coordinates, where $0\leqslant a_j\leqslant n_{i_j}-1$ for $1\leqslant j\leqslant t$, will be called a *hyperplane* and denoted by $X(x_{i_j}=a_j,\ j=1,\ldots,t)$. The class of hyperplanes into which X is partitioned by coordinates $x_{i_1},\ldots,x_{i_t}$ is denoted by $\{X(x_{i_1}=a_1,\ldots,x_{i_t}=a_t)\mid 0\leqslant a_j\leqslant n_{i_j}-1\}$.

A set Ψ_i of type 0 canonical FHR can be constructed from a set Δ of MOFHR$(n_i;0;m)$. Suppose $L\in\Delta$, then define a size $n_1\times\cdots\times n_d$ FHR, L^* as follows:

$$L^*(x_1,x_2,\ldots,x_d)=L(x_i).$$

It is clear that L^* is a type 0, d-dimensional FHR. The set $\bigcup_{i=1}^d \Psi_i$ is called the set of type 0 *canonical* FHR.

For example, from the set Ξ_3 of MOFHR$(4;0;2)$ in Section 2, we can construct two sets Ψ_1 and Ψ_2. The set $\Psi_1\cup\Psi_2$ is the set of type 0 cononical FHR$(4,4;0;2)$. Furthermore, $(\Psi_1\cup\Psi_2)\cup\Xi_1$ is a complete set of MOFHR$(4,4;0;2)$.

$$\Psi_1:C_1=\begin{matrix}0&0&0&0\\0&0&0&0\\1&1&1&1\\1&1&1&1\end{matrix}\quad C_2=\begin{matrix}0&0&0&0\\1&1&1&1\\0&0&0&0\\1&1&1&1\end{matrix}\quad C_3=\begin{matrix}0&0&0&0\\1&1&1&1\\1&1&1&1\\0&0&0&0\end{matrix}$$

$$\Psi_2:C_4=\begin{matrix}0&0&1&1\\0&0&1&1\\0&0&1&1\\0&0&1&1\end{matrix}\quad C_5=\begin{matrix}0&1&0&1\\0&1&0&1\\0&1&0&1\\0&1&0&1\end{matrix}\quad C_6=\begin{matrix}0&1&1&0\\0&1&1&0\\0&1&1&0\\0&1&1&0\end{matrix}.$$

Theorem 3.1. *Given a set of l_1 MOFHR$(n_1;0;m),\ldots,$ a set of l_d MOFHR$(n_d;0;m)$, there exists a set Ψ of $\sum_{i=1}^d l_i$ type 0 canonical FHR$(n_1,\ldots,n_d;0;m)$. Furthermore,* *adding a set Λ of h MOFHR$(n_1,\dots,n_d;1;m)$, the enlarged set $\Psi\cup\Lambda$ is a set of $\sum_{i=1}^d l_i+h$ MOFHR$(n_1,\dots,n_d;0;m)$.*

Proof. We only need to show that the members of $\Psi\cup\Lambda$ are orthogonal.

Let X and Y be members of $\Psi\cup\Lambda$. If $X,Y\in\Lambda$, then X and Y are orthogonal.

Otherwise, we may assume $X\in\Psi$, which implies that there exists k, where $1\leqslant k\leqslant d$, such that $X\in\Psi_k$.

If $Y\in\Psi_k$, we assume that X is constructed from X', Y is constructed from Y', where X', Y' are MOFHR$(n_k;0;m)$. By the fact that each ordered pair occurs n_k/m^2 times in (X',Y'), where (X',Y') denotes the F-hyperrectangle obtained by superimposing X' and Y', we have that each ordered pair occurs

$$\frac{n_k}{m^2}\frac{\prod_{i=1}^d n_i}{n_k}=\frac{\prod_{i=1}^d n_i}{m^2}$$

times in (X,Y). Hence X and Y are orthogonal.

If $Y\notin\Psi_k$, then each element occurs $\prod_{i=1}^d n_i/(m\times n_k)$ times in each hyperplane $Y(x_k=a)$, $0\leqslant a\leqslant n_k-1$. Hence each ordered pair occurs exactly $(n_k/m)\prod_{i=1}^d n_i/(m\times n_k)=\prod_{i=1}^d n_i/m^2$ times in (X,Y). Therefore X and Y are orthogonal. □

Corollary 3.2. *If the initial sets of MOFHR$(n_1;0;m),\dots,$MOFHR$(n_d;0;m)$ and the set of MOFHR$(n_1,\dots,n_d;1;m)$ are both complete, so is the enlarged set of MOFHR$(n_1,\dots,n_d;0;m)$.*

Proof. If $l_i=(n_i-1)/(m-1)$ and

$$h=\frac{1}{m-1}\left(\prod_{i=1}^d n_i-\sum_{i=1}^d (n_i-1)-1\right),$$

then

$$\sum_{i=1}^d l_i+h=\frac{1}{m-1}\left(\prod_{i=1}^d n_i-1\right).\qquad\square$$

4. A recursive procedure

The following procedure constructs MOFHR$(n_1,\dots,n_d,n_{d+1};t+1;m)$ from MOFHR$(n_1,\dots,n_d;t;m)$ and MOFHR$(n_{d+1},m;1;m)$.

Given a set Ω of h MOFHR$(n_1,\dots,n_d;t;m)$, we can divide the set into two classes, Ω_1 and Ω_2. The class Ω_1 consists of all FHR$(n_1,\dots,n_d;t+1;m)$, and Ω_2 consists of the rest. Let h_1 be the cardinality of Ω_1.

Given a set Γ of l MOFHR$(n_{d+1},m;1;m)$, we append the following $n_{d+1}\times m$ rectangle

$$R=\begin{matrix} 0 & 1 & \dots & m-1 \\ 0 & 1 & \dots & m-1 \\ \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & m-1 \end{matrix} \quad \text{to } \Gamma.$$

We denote this new set as Γ^+. Using this we now construct $hl+h_1$ MOFHR$(n_1,\dots,n_d,n_{d+1};t+1;m)$. Suppose $X\in\Omega$, and $L\in\Gamma^+$, then define the $(d+1)$-dimensional hyperrectangle X^L as follows:

$X^L(x_1,x_2,\dots,x_d,x_{d+1}) = L(X(x_1,x_2,\dots,x_d),x_{d+1})$. The expression is interpreted to mean that X^L is partitioned into the hyperplanes $\{X^L(x_{d+1}=0),X^L(x_{d+1}=1),\dots,X^L(x_{d+1}=n_{d+1}-1)\}$, where $X^L(x_{d+1}=i)$ is X with a permutation applied to its symbols as defined by the ith row of L. We can view each row of L as the image of a permutation from S_m, the symmetric group on m letters, $0,1,\dots,m-1$.

The construction gives a new set

$$\Phi=\{X^L\colon X\in\Omega_1,L\in\Gamma^+\}\cup\{X^L\colon X\in\Omega_2,\ L\in\Gamma\}.$$

Note that

$$|\Phi|=|\Omega_1|\times(|\Gamma|+1)+|\Omega_2|\times|\Gamma|=h_1(l+1)+(h-h_1)l=hl+h_1,$$

as earlier claimed.

Let us look at an example. Set $\Omega=\{H_1,\dots,H_9,C_1,\dots,C_6\}$, and $\Gamma=\{Q_1,Q_2,Q_3\}$. From the above procedure we can construct a complete set of 54 MOFHR(4,4,4;1;2). The following is $H_5^{Q_2}$.

	$x_3=0$				$x_3=1$				$x_3=2$				$x_3=3$			
$x_1=0$	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
$x_1=1$	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
$x_1=2$	1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1
$x_1=3$	1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1
$x_2=0,1,2,3$																

Before showing that Φ is a set of MOFHR$(n_1,\dots,n_d,n_{d+1};t+1;m)$, we make some observations about the hyperplanes of members of Φ.

Let $(i_1,\dots,i_t,i_{t+1})$ be an arbitrary element in $P_{t+1}(\underline{d+1})$. Now we consider two classes of hyperplanes in X^L, a typical member of Φ.

Class 1 hyperplanes are of the form $X^L(x_{i_1}=a_1,\dots,x_{i_t}=a_t,x_{i_{t+1}}=a_{t+1})$, where $i_{t+1}=d+1$. By definition this is $X(x_{i_1}=a_1,\dots,x_{i_t}=a_t)$ with the permutation determined by row a_{t+1} of L applied to the symbols.

Class 2 hyperplanes are of the form $X^L(x_{i_1}=a_1,\dots,x_{i_t}=a_t,x_{i_{t+1}}=a_{t+1})$, where $i_{t+1}<d+1$. Say $P=X^L(x_{i_1}=a_1,\dots,x_{i_t}=a_t,x_{i_{t+1}}=a_{t+1})$. Partition P into $\{P(x_{d+1}=0),\dots,P(x_{d+1}=n_{d+1}-1)\}$. Further $P(x_{d+1}=k)$ is obtained from $X(x_{i_1}=a_1,\dots,x_{i_t}=$

$a_t, x_{i_{t+1}} = a_{t+1}$) by permuting the symbols in $X(x_{i_1} = a_1, \ldots, x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$ according to the permutation defined by the kth row of L.

Lemma 4.1. *The members of Φ are of type $t+1$.*

Proof. Suppose $X^L \in \Phi$, and let $(i_1, \ldots, i_t, i_{t+1})$ be an arbitrary element in $P_{t+1}(\underline{d+1})$. We have to show that each symbol occurs an equal number of times in the hyperplane $X^L(x_{i_1} = a_1, \ldots, x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$, where $0 \leqslant a_k \leqslant n_{i_k} - 1$, $1 \leqslant k \leqslant t+1$.

This is obvious if the hyperplane is in class 1.

Consider a hyperplane P in class 2. Then P consists of n_{d+1} copies of hyperplane $X(x_{i_1} = a_1, \ldots, x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$ with the kth copy having the symbols permuted by the kth row of L.

If $X \in \Omega_1$, then each symbol occurs equally often in $X(x_{i_1} = a_1, \ldots, x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$ and therefore each symbol occurs equally often in P since permutations of the symbols of $X(x_{i_1} = a_1, \ldots, x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$ leave the number of occurrences of each symbol unchanged. So if $X \in \Omega_1$, then X^L is an FHR$(n_1, \ldots, n_d, n_{d+1}; t+1; m)$.

If $X \notin \Omega_1$, then $X \in \Omega_2$. Hence $L \in \Gamma$. Hyperplane P has partition $\{P(x_{d+1} = 0), P(x_{d+1} = 1), \ldots, P(x_{d+1} = n_{d+1} - 1)\}$, where $P(x_{d+1} = k)$ is $X(x_{i_1} = a_1, \ldots, x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$ with the symbols replaced according to row k of L. For any element e, $0 \leqslant e \leqslant m-1$, the first row of L permutes e_1 to $e, \ldots,$ the n_{d+1} row of L permutes $e_{n_{d+1}}$ to e, where $0 \leqslant e_1, \ldots, e_{n_{d+1}} \leqslant m-1$.

By the fact that $L \in \Gamma$, we see that the multi-set $\{e_1, \ldots, e_{n_{d+1}}\} = \{0, \ldots, 0, 1, \ldots, 1, \ldots, m-1, \ldots, m-1\}$ (each element with multiplicity n_{d+1}/m). Hence the number of times that symbol e appears in P is $\sum_{k=1}^{n_{d+1}}$ (the number of times that symbol e_k appears in $X(x_{i_1} = a_1, \ldots, x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$), which is $n_{d+1}m \times \sum_{j=0}^{m-1}$ (the number of times that symbol j appears in $X(x_{i_1} = a_1, \ldots, x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$), which equals

$$\frac{n_{d+1}}{m} \times \frac{\prod_{k=1}^{d} n_k}{\prod_{k=1}^{t+1} n_{i_k}}.$$

Thus each symbol occurs equally often in P. □

Lemma 4.2. *The members of Φ are mutually orthogonal.*

Proof. Let X^L and Y^M be members of Φ, and assume $X \neq Y$. Then X and Y are orthogonal, and X^L and Y^M, respectively, have partitions $\{X^L(x_{d+1} = 0), \ldots, X^L(x_{d+1} = n_{d+1} - 1)\}$ and $\{Y^M(x_{d+1} = 0), \ldots, Y^M(x_{d+1} = n_{d+1} - 1)\}$. Each member of these partitions is obtained from X or Y by a permutation of the symbols, an operation that does not affect orthogonality. Hence $X^L(x_{d+1} = k)$ is orthogonal to $Y^M(x_{d+1} = k)$ since X is orthogonal to Y. Therefore X^L is orthogonal to Y^M.

Assume $X = Y$. Then L is orthogonal to M. Let (L, M) denote the F-hyperrectangle obtained by superimposing L and M.

If the ordered pair (α,β) appears in the position (i,j) of (L,M), then (α,β) appears in $(X^L,X^M)(x_{d+1}=i)$, $\prod_{k=1}^{d} n_k/m$ times by the fact that element j appears $\prod_{k=1}^{d} n_k/m$ times in X.

Since L and M are orthogonal F-hyperrectangles, we see that each ordered pair occurs exactly $(n_{d+1}\times m/m^2)=n_{d+1}/m$ times in (L,M). Hence each pair occurs exactly $\prod_{k=1}^{d+1} n_k/m^2$ times in (X^L,X^M). □

The following theorem follows from Lemmas 4.1 and 4.2.

Theorem 4.1. *Given a set of h MOFHR$(n_1,\ldots,n_d;t;m)$, which consists of h_1 MOFHR$(n_1,\ldots,n_d;t+1;m)$, and a set of l MOFHR$(n_{d+1},m;1;m)$, there exists a set of $hl+h_1$ MOFHR$(n_1,\ldots,n_d,n_{d+1};t+1;m)$.*

We note that Theorem 4.1 provides a generalization of Theorem 3.6 of [2].

Given a complete set Φ of *MOFHR*$(n_1,\ldots,n_d;t;m)$, where $0\leqslant t\leqslant d-2$, if the subset of Φ, consisting of all type $(t+1)$ F-hyperrectangles, is also complete, then we call the set Φ *strongly complete*.

Corollary 4.2. (1) *Given a complete set of MOFHR$(n_1,\ldots,n_d;d-1;m)$ and a complete set of MOFHR$(n_{d+1},m;1;m)$, then the above recursive algorithm gives a complete set of MOFHR$(n_1,\ldots,n_d,n_{d+1};d;m)$.*

(2) *Given a strongly complete set of MOFHR$(n_1,\ldots,n_d;t;m)$, where $0\leqslant t\leqslant d-2$, and a complete set of MOFHR$(n_{d+1},m;1;m)$, then the above recursive algorithm gives a complete set of MOFHR$(n_1,\ldots,n_d,n_{d+1};t+1;m)$.*

Proof. (1) If

$$h=\frac{1}{m-1}\left(\prod_{i=1}^{d} n_i-\sum_{k=1}^{d-1}\sum_{\{i_1,\ldots,i_k\}\in P_k(\underline{d})}\prod_{j=1}^{k}(n_{i_j}-1)-1\right)=\frac{1}{m-1}\left(\prod_{i=1}^{d}(n_i-1)\right),$$

$h_1=0$ and $l=n_{d+1}-1$, then

$$\begin{aligned}hl+h_1&=\frac{1}{m-1}\left(\prod_{i=1}^{d+1}(n_i-1)\right)\\&=\frac{1}{m-1}\left(\prod_{i=1}^{d+1} n_i-\sum_{k=1}^{d}\sum_{\{i_1,\ldots,i_k\}\in P_k(\underline{d+1})}\prod_{j=1}^{k}(n_{i_j}-1)-1\right).\end{aligned}$$

(2) If

$$h=\frac{1}{m-1}\left(\prod_{i=1}^{d} n_i-\sum_{k=1}^{t}\sum_{\{i_1,\ldots,i_k\}\in P_k(\underline{d})}\prod_{j=1}^{k}(n_{i_j}-1)-1\right),$$

$$h_1 = \frac{1}{m-1}\left(\prod_{i=1}^{d} n_i - \sum_{k=1}^{t+1} \sum_{\{i_1,\ldots,i_k\}\in P_k(\underline{d})} \prod_{j=1}^{k}(n_{i_j}-1) - 1\right)$$

and $l = n_{d+1} - 1$, then

$$hl + h_1 = \frac{1}{m-1}\left(\prod_{i=1}^{d+1} n_i - \sum_{k=1}^{t+1} \sum_{\{i_1,\ldots,i_k\}\in P_k(\underline{d+1})} \prod_{j=1}^{k}(n_{i_j}-1) - 1\right). \qquad \square$$

It is easily seen that the complete set of MOFHR($q^{s_1},\ldots,q^{s_d};t;m$) constructed by Theorem 2.1 is strongly complete and so this construction (Theorem 2.1) gives a strongly complete set of MOFHR of prime power order.

We conclude this paper with some conjectures.

Conjecture 1. If there exists a complete (or strongly complete) set of MOFHR($n_1,\ldots,n_d;t;m$), then m is a prime power.

Conjecture 2. A complete (or strongly complete) set of MOFHR($n_1,\ldots,n_d;t;m$) exists if and only if m is a prime power and $n_1,\ldots,n_d$ are all powers of m.

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We refer to [5] and [6] for related material.